Invariant cones and convex sets for bilinear control systems and parabolic type of semigroups.

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Abstract

This paper studies invariant cones for bilinear control systems, and relates them to the controllability of the system. The full picture is provided by the parabolic type of a semigroup.

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1 Introduction

Let

$$\dot{x} = Ax + uBx, x \in \mathbb{R}^n \setminus \{0\}, u \in \mathbb{R}, \tag{1}$$

be a bilinear control system where A and B are $n \times n$-matrices. The problem we address in this note concerns the relation between complete controllability of (1) and the (non) existence of proper convex cones and convex sets in $\mathbb{R}^n$ which are invariant under the system. This is a natural question since the most direct way of checking that a control system is not controllable is by showing the existence of

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some invariant proper subset of the state space. For bilinear control systems one should search invariant sets among the convex sets and in particular among the convex cones, because if a set C is invariant under (1) then the convex closure of C is also invariant.

In this perspective we can ask whether the condition of do not leaving a proper cone $W \subset \mathbb{R}^n$ invariant is necessary and sufficient for the bilinear control system to be completely controllable. This question was made in Sachkov [4] where it is conjectured that a control systems with $B = \mathrm{diag}(b_1, \ldots, b_n)$ regular, that is, $b_i \neq b_j$ if $i \neq j$ is controllable in $\mathbb{R}^n \backslash \{0\}$ if and only if there are no positive and negative invariant orthants and the system satisfies the Lie algebra rank condition.

In this article we analyze this question under the light of the classification of semigroups into their parabolic types (see [7], [11] and [12]). We consider mainly the case where the group of the system is the special linear group $\mathrm{Sl}(n, \mathbb{R})$.

We explain briefly the idea. Let $S \subset \mathrm{Sl}(n, \mathbb{R})$ be a semigroup with $\mathrm{int} S \neq \emptyset$. Any $g \in S$ maps lines of $\mathbb{R}^n$ into lines, giving rise to an action of S in the projective space $\mathbb{P}^{n-1}$. It can be proved that there exists a unique invariant control set $C \subset \mathbb{P}^{n-1}$ for S in $\mathbb{P}^{n-1}$ (see [5]). Let $D \subset \mathbb{R}^n$ be the union of the lines belonging to C. The invariance of C implies that D is S-invariant. Now, if C is nicely included in $\mathbb{P}^{n-1}$ then the subset D is a union of a double cone in $\mathbb{R}^n$. In this case (and only in this case) S leaves invariant a proper cone in $\mathbb{R}^n$.

The description of the cases where C is nicely included in $\mathbb{P}^{n-1}$ is given by the parabolic type of S. This is a partition of n, that is, a finite sequence of integers $r_1, \ldots, r_k$ with $r_1 + \cdots + r_k = n$. In case $r_1 = 1$, and only in this case the semigroup S leaves invariant a proper cone in $\mathbb{R}^n$. Taking in particular the semigroup generated by the control system, the classification of the semigroups gives the complete picture of the existence of invariant cones. In conclusion it shows that there are several cases where the system is not controllable but does not have invariant a proper cone. Guided by this classification we work out concrete examples of this phenomenon.

In concluding this introduction we mention that the concept of parabolic type of a semigroup is applicable to subsemigroups of general noncompact semi-simple Lie groups. Although we work here only with $\mathrm{Sl}(n, \mathbb{R})$ the methods should, in principle work for other semi-simple Lie groups acting transitively in $\mathbb{R}^n$, appearing in the Boothby [1] classification.

2 Controllability on different spaces

Given the bilinear control system (1) we denote by $\mathfrak{g}_\Sigma$ the Lie algebra of matrices generated by A and B and by G_Σ the group of the system, which is the connected Lie subgroup whose Lie algebra is $\mathfrak{g}_\Sigma$. Also, we let $S_\Sigma \subset G_\Sigma$ stand for the semigroup of the system:

$$S_\Sigma = \{e^{X_1} \cdots e^{X_l} : X_i = A + u_i B,\ u_i \in \mathbb{R}\}.$$

It is well known that S_Σ has nonempty interior in G_Σ.

By definition the system (1) is completely controllable in $\mathbb{R}^n \setminus \{0\}$ if S_Σ is transitive in $\mathbb{R}^n \setminus \{0\}$, that is, $S_\Sigma x = \mathbb{R}^n \setminus \{0\}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. A necessary condition for complete controllability is that G_Σ acts transitively in $\mathbb{R}^n \setminus \{0\}$. The connected linear groups transitive on $\mathbb{R}^n \setminus \{0\}$ where classified long ago by Boothby [1]. Such a group G has reductive Lie algebra $\mathfrak{g}$ (that is, $\mathfrak{g}$ is the direct sum of its center by a semi-simple Lie algebra). One of these transitive Lie groups is the identity component $\mathrm{Gl}_0(n, \mathbb{R})$ of the general linear group $\mathrm{Gl}(n, \mathbb{R})$, whose Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ decomposes as

$$\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}\mathrm{id} \oplus \mathfrak{sl}(n, \mathbb{R}),$$

where $\mathfrak{sl}(n, \mathbb{R})$ are the zero trace matrices, which is the Lie algebra of $\mathrm{Sl}(n, \mathbb{R})$, the unimodular group. At the group level we have $\mathrm{Gl}_0(n, \mathbb{R}) = (\mathbb{R}^+\mathrm{id}) \times \mathrm{Sl}(n, \mathbb{R})$.

We recall some controllability properties in the case when $\mathfrak{g}_\Sigma$ is $\mathfrak{sl}(n, \mathbb{R})$. In this case the semigroup S_Σ of the system has nonempty interior in $\mathrm{Sl}(n, \mathbb{R})$. This semigroup is the accessible set from the identity of the right invariant system defined by the matrices of the bilinear system. The following result shows that controllability in $\mathbb{R}^n \setminus \{0\}$ is equivalent to controllability in $\mathrm{Sl}(n, \mathbb{R})$.

Proposition 2.1 *Let $S \subset \mathrm{Sl}(n, \mathbb{R})$ be a semigroup with $\mathrm{int}S \neq \emptyset$. Then S is transitive on $\mathbb{R}^n \setminus \{0\}$ if and only if $S = \mathrm{Sl}(n, \mathbb{R})$.*

Proof: The action of S on $\mathbb{R}^n \setminus \{0\}$ induces an action of S in the projective space $\mathbb{P}^{n-1}$ given by $g[v] = [gv]$, where $[u]$ denote the class of $u \in \mathbb{R}^n \setminus \{0\}$ in $\mathbb{P}^{n-1}$. By $g[v] = [gv]$ it follows that S is transitive on $\mathbb{P}^{n-1}$ if it is transitive on $\mathbb{R}^n \setminus \{0\}$. But if S is transitive on $\mathbb{P}^{n-1}$ then $S = \mathrm{Sl}(n, \mathbb{R})$, by [12], Theorem 6.2 (see also [9]). □

In case $\mathfrak{g}_\Sigma = \mathfrak{gl}(n, \mathbb{R})$ we can associate to (1) a new system evolving on $\mathrm{Sl}(n, \mathbb{R})$ as follows. Take $A \in \mathfrak{gl}(n, \mathbb{R})$ and consider $\delta(A) = \frac{\mathrm{tr}(A)}{n}$ and $\widetilde{A} = A - \delta(A)\mathrm{id}$, where id denote the identity $n \times n$ matrix. It follows that $\widetilde{A} \in \mathfrak{sl}(n, \mathbb{R})$ and $A = \widetilde{A} + \delta(A)\mathrm{id}$. By this decomposition, associated to a system $\Sigma \subset \mathfrak{gl}(n, \mathbb{R})$, we have the system $\widetilde{\Sigma} = \{\widetilde{A} \text{ such that } A \in \Sigma\} \subset \mathfrak{sl}(n, \mathbb{R})$.

Proposition 2.2 *Suppose that* Σ *generates* $\mathfrak{gl}(n,\mathbb{R})$. *Then* $\tilde{\Sigma}$ *is controllable if* Σ *is controllable*

Proof: Note that an element in S_Σ can be represented by products of exponentials like

$$\exp X = \exp(\delta(X))\exp(\tilde{X}). \tag{2}$$

Therefore the elements of S_Σ are of the form λg with $\lambda > 0$ and $g \in S_{\tilde{\Sigma}}$. Since the action of λg on $\mathbb{P}^{n-1}$ coincides with the action of g, it follows that $S_{\tilde{\Sigma}}$ is transitive on $\mathbb{R}^n \backslash \{0\}$ if S_Σ is transitive. Therefore, by Proposition 4.3 in [5], $\tilde{\Sigma}$ is controllable if Σ is controllable. □

This result shows that the analysis of controllability for zero trace matrices is essential in the study of controllability of bilinear systems. And in this way, we also have the following result that generalizes the Proposition 6.3 of [2].

Proposition 2.3 *Suppose that* $\tilde{\Sigma}$ *is controllable and assume that* S_Σ *is transitive on a ray* r *starting in the origin of* $\mathbb{R}^n$*, that is, for every pair* $x, y \in r$ *there exists* $g \in S_\Sigma$ *such that* $gx = y$. *Then* Σ *is controllable on* $\mathbb{R}^n \backslash \{0\}$.

Proof: Since $S_{\tilde{\Sigma}}$ is transitive on $\mathbb{R}^n \backslash \{0\}$ it is transitive on the rays, i. e., given two rays r_1 and r_2 there exists $g \in S_{\tilde{\Sigma}}$ with $gr_1 = r_2$. Since S_Σ acts on the rays as $S_{\tilde{\Sigma}}$ does, we have that S_Σ is also transitive on the rays. Take $x \in r$ and let $y \in \mathbb{R}^n \backslash \{0\}$ be an arbitrarily element. Let r_1 a rays which contains y. Then there is $g \in S_\Sigma$ such that $gr_1 = r$ so that $gy \in r$. Hence there exists $h \in S_\Sigma$ such that $ghy = x$. Analogously, there is $g_1 \in S_\Sigma$ with $g_1 r = r_1$. Then $g_1^{-1} y \in r$ and there is $h_1 \in S_\Sigma$ such that $h_1 x = g_1^{-1} y$, i.e., $g_1 h_1 x = y$. Therefore S_Σ is transitive on $\mathbb{R}^n \backslash \{0\}$. □

Considering invariant cones note that by decomposition (2) a cone W is invariant under Σ if and only if it is invariant under $\tilde{\Sigma}$. On the other hand there may have convex sets invariant under Σ but not invariant under $\tilde{\Sigma}$ as show the 2-dimensional system where

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then any ball $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq \rho\}$, $\rho \geq 0$, is Σ-invariant but not $\tilde{\Sigma}$-invariant.

3 Parabolic type of a semigroup

Regarding the existence of invariant cones the full picture is given by the concept of parabolic type of a semigroup (cf. [7], [8] and [12]). This concept describes several properties of semigroups in general semi-simple Lie groups. In this section we make an exposition specific for semigroups in $\mathrm{Sl}(n,\mathbb{R})$.

The parabolic type emerges with the study of the action of semigroups in the flag manifolds. Thus take a finite sequence of integers $\Theta = \{r_1,\ldots,r_k\}$ with $0 < r_1 \leq \cdots \leq r_k < n$ and denote by $\mathbb{F}_\Theta$ the flag manifold made of all flags

$$(V_1 \subset \cdots \subset V_k)$$

of subspaces $V_i \subset \mathbb{R}^n$ with $\dim V_i = r_i$, $i = 1,\ldots,k$. Each $\mathbb{F}_\Theta$ is a compact manifold where $\mathrm{Sl}(n,\mathbb{R})$ acts transitively.

We say that the sequence $\Theta_1 = \{s_1,\ldots,s_m\}$ is refined by $\Theta = \{r_1,\ldots,r_k\}$ if $\Theta_1 \subset \Theta$, that is, $m \leq k$ and for every $j = 1,\ldots,m$ there exists i such that $s_j = r_i$. In case Θ_1 refines Θ there exists a map $\pi^{\Theta}_{\Theta_1} : \mathbb{F}_\Theta \to \mathbb{F}_{\Theta_1}$ which is given by forgetting the subspaces whose dimensions appear in Θ but not in Θ_1. Of particular importance is the full flag manifold, which is given by the sequence $\Theta_M = \{1,2,\ldots,n-1\}$ and denoted simply by $\mathbb{F}$. The sequence Θ_M refines any sequence Θ. Hence for every Θ we have a map $\pi_\Theta : \mathbb{F} \to \mathbb{F}_\Theta$.

Let $S \subset \mathrm{Sl}(n,\mathbb{R})$ be a semigroup with $\mathrm{int} S \neq \emptyset$. Then S also acts in $\mathbb{F}_\Theta$. Recall that an invariant control set for the action of S on $\mathbb{F}_\Theta$ is a subset $C \subset \mathbb{F}_\Theta$ such that $\mathrm{cl}\,(Sx) = D$ for all $x \in C$ and C is maximal with this property. It is known that an invariant control set is closed and its interior $\mathrm{int} D$ is dense in D. Also, in each flag manifold $\mathbb{F}_\Theta$ there exists just one invariant control set for the action of S. In the sequel we denote this unique invariant control set by C_Θ. We denote simply by C the invariant control set in the full flag manifold $\mathbb{F}$.

As a consequence of the results of [12] we have the following fact.

Theorem 3.1 *Suppose that $S \neq \mathrm{Sl}\,(n,\mathbb{R})$. Then there are flag manifolds $\mathbb{F}_\Theta$ such that $C = \pi_\Theta^{-1}(C_\Theta)$. Among these flag manifolds there is exactly one, say $\mathbb{F}_{\Theta(S)}$, which is minimal, that is, if $C = \pi_\Theta^{-1}(C_\Theta) = \pi_{\Theta(S)}^{-1}\left(C_{\Theta(S)}\right)$ then Θ is a refinement of $\Theta\,(S)$.*

Proof: For a proof see Theorem 4.3 in [12]. □

Definition 3.2 *The flag manifold $\mathbb{F}_{\Theta(S)}$ of the above theorem is called the* parabolic type *or* flag type *of S. We also say that $\Theta\,(S)$ is the parabolic type of S.*

The invariant control set $C_{\Theta(S)}$ in the flag type $\mathbb{F}_{\Theta(S)}$ of S has a nice contractibility property. To state this property we introduce the following notations: Let us say that an element $h \in \mathrm{Sl}(n, \mathbb{R})$ is regular if it has real distinct eigenvalues $\lambda_1 > \cdots > \lambda_n > 0$. Given a regular element h let $\beta(h) = \{e_1(h), \ldots, e_n(h)\}$ be a basis of eigenvectors of h ordered by decreasing ordering of the eigenvalues, and denote by $N(h)$ the subgroup of $\mathrm{Sl}(n, \mathbb{R})$ given by lower triangular matrices with respect to $\beta(h)$. Note that for any $n \in N(h)$ the product $h^{-k} n h^k$ converges to the identity matrix when $k \to +\infty$.

For a sequence $\Theta = \{r_1, \ldots, r_k\}$ let $b_\Theta(h) \in \mathbb{F}_\Theta$ be the flag

$$b_\Theta(h) = (\mathrm{span}\{e_1(h), \ldots, e_{r_1}(h)\} \subset \mathrm{span}\{e_1(h), \ldots, e_{r_2}(h)\} \subset \cdots) \qquad (3)$$

spanned by the vectors of $\beta(h)$. Then the orbit $N(h) \cdot b_\Theta(h)$ is open and dense in the flag manifold $\mathbb{F}_\Theta$.

Now let $S \subset \mathrm{Sl}(n, \mathbb{R})$ be a semigroup with $\mathrm{int} S \neq \emptyset$ and denote by $\mathrm{reg}(S)$ the set of regular elements in $\mathrm{int} S$. It is known that $\mathrm{reg}(S) \neq \emptyset$ and the following property in the flag type $\mathbb{F}_{\Theta(S)}$ of S holds.

Proposition 3.3 *Let $C_{\Theta(S)} \subset \mathbb{F}_{\Theta(S)}$ be the invariant control set in the $\mathbb{F}_{\Theta(S)}$, the flag type of S. Then for any $h \in \mathrm{reg}(S)$ we have $C_{\Theta(S)} \subset N(h) \cdot b_{\Theta(S)}(h)$.*

Proof: For a proof see Proposition 4.8 in [12]. □

Corollary 3.4 *For any Θ contained in $\Theta(S)$ we have $C_\Theta \subset N(h) \cdot b_\Theta(h)$.*

Proof: In fact, if $\pi : \mathbb{F}_{\Theta(S)} \to \mathbb{F}_\Theta$ is the natural projection then $\pi\left(C_{\Theta(S)}\right) = C_\Theta$ and $\pi\left(N(h) \cdot b_{\Theta(S)}\right) = N(h) \cdot b_\Theta(h)$. □

We note that once we know the invariant control set $C_{\Theta(S)}$ in the parabolic type $\mathbb{F}_{\Theta(S)}$ then every invariant control set is known because for any Θ we have $C_\Theta = \pi_\Theta(C)$ and $C = \pi_{\Theta(S)}^{-1}\left(C_{\Theta(S)}\right)$.

To see the parabolic type of the inverse semigroup S^{-1} we must introduce the notion of dual flag manifold. Given $\Theta = \{r_1, \ldots, r_k\}$ with $0 < r_1 \leq \cdots \leq r_k < n$ define $\Theta^* = \{n - r_k, \ldots, n - r_1\}$. The flag manifold $\mathbb{F}_{\Theta^*}$ is said to be dual of $\mathbb{F}_\Theta$.

The following statement was proved in [7].

Proposition 3.5 *The parabolic type of S^{-1} is given by the flag manifold $\mathbb{F}_{\Theta(S)^*}$ dual to the parabolic type $\mathbb{F}_{\Theta(S)}$ of S.*

4 Invariant cones

Let $S \subset \mathrm{Sl}(n,\mathbb{R})$ be a connected, proper semigroup with non-empty interior and containing the identity in its closure. Clearly these assumptions are satisfied by the semigroup of a control system.

We look at the existence of S-invariant cones in $\mathbb{R}^n$ via the invariant control set of S in the projective space $\mathbb{P}^{n-1}$. For simplicity we write a cone to mean a closed convex cone in $\mathbb{R}^n$.

Of course, $\mathbb{P}^{n-1}$ is the flag manifold corresponding to the sequence $\Theta_{\mathbb{P}} = \{1\}$. The invariant control set in $\mathbb{P}^{n-1}$ is given by $C_{\Theta_{\mathbb{P}}} = \pi_{\Theta_{\mathbb{P}}}(C)$ where $C = \pi^{-1}_{\Theta(S)}\left(C_{\Theta(S)}\right)$ is the invariant control set on the full flag manifold and where $\mathbb{F}_{\Theta(S)}$ is the parabolic type of S.

We start by noting that invariant cones are necessarily pointed and generating.

Lemma 4.1 *Let $S \subset \mathrm{Sl}(n,\mathbb{R})$ be a semigroup with non-empty interior. If $\{0\} \neq W \subset \mathbb{R}^n$ is a proper S-invariant cone then W is pointed (that is, $W \cap -W = \{0\}$) and generating (that is,* $\mathrm{int}W \neq \emptyset$*).*

Proof: Put $H = W \cap -W$. Then H is an S-invariant vector subspace. We have also that H is S^{-1}-invariant. In fact, if $g \in S$ then $gH \subset H$. But g is invertible so that gH is a subspace of H with $\dim gH = H$, that is, $gH = H$. Consequently $H = g^{-1}H$. But since $\mathrm{int}S \neq \emptyset$ the group $\mathrm{Sl}(n,\mathbb{R})$ is generated by $S \cup S^{-1}$. This implies that H is $\mathrm{Sl}(n,\mathbb{R})$-invariant and hence $H = \{0\}$ because $\mathrm{Sl}(n,\mathbb{R})$ is irreducible and W is assumed to be proper. This shows that W is pointed.

To see that W is generating note that for any $0 \neq x \in W$ the set $Sx \subset W$ and it has nonempty interior because the map $g \in \mathrm{Sl}(n,\mathbb{R}) \mapsto gx \in \mathbb{R}$ is an open map. $\square$

In the sequel we let $p : \mathbb{R}^n \backslash \{0\} \to \mathbb{P}^{n-1}$ be the natural projection and write also $[x] = p(x)$ for the class of $x \neq 0$. For a subset $U \subset \mathbb{R}^n \backslash \{0\}$, we write $[U] \subset \mathbb{P}^{n-1}$ for its projection on $\mathbb{P}^{n-1}$, that is,

$$[U] = \{[(x_1, \ldots, x_n)] \in \mathbb{P}^{n-1} \text{ such that } (x_1, \ldots, x_n) \in U\}.$$

Theorem 4.2 *Let $S \subset \mathrm{Sl}(n,\mathbb{R})$ be a connected, proper semigroup with non-empty interior and containing 1. Denote $\Theta(S) = \{r_1, \ldots, r_k\}$ its parabolic type. If $r_1 = 1$ (that is, if $\Theta_{\mathbb{P}} \subset \Theta(S)$) then there exists an S-invariant proper cone $W \subset \mathbb{R}^n$.*

Proof: We begin by choosing a good coordinates for $\mathbb{R}^n$. Take $h \in \mathrm{reg}(S)$. As before, let $\beta(h) = \{e_1(h), \ldots, e_n(h)\}$ be a basis of eigenvectors of h ordered by decreasing ordering of the eigenvalues. In this case $b_{\Theta_{\mathbb{P}}}(h) = [e_1(h)]$ and it is easy to see that the orbit $N(h)\, b_{\Theta_{\mathbb{P}}}(h)$ is the subset $[M]$, where M is the affine subspace

$$M = \{(x_1, \ldots, x_n) : x_1 = 1\}.$$

Also, the restriction of p to M is a diffeomorphism.

By Proposition 3.4 we have $C_{\mathbb{P}} \subset [M]$. Define $M_1 = p^{-1}(C_{\mathbb{P}}) \cap M$, so that $C_{\mathbb{P}} = [M_1]$. Now, $C_{\mathbb{P}}$ is compact, hence M_1 is compact as well.

Now let $D \subset M$ be the convex closure of M_1 and W the cone generated by M_1. We have $W = \mathbb{R}^+ D$ and D is a cone basis of W. Since D is the convex closure of the compact set M_1, it is also compact. This implies that W is a pointed cone. It is also easy to see that this cone is generating because $\mathrm{int} C_{\mathbb{P}} \neq \emptyset$, hence $\mathrm{int} D \neq \emptyset$.

It remains to check that W is S-invariant. Since $C_{\mathbb{P}}$ is invariant and $p(M_1) = C_{\mathbb{P}}$ we conclude that the subset $(\mathbb{R} \setminus \{0\})\, M_1$ is also invariant. Now, S is assumed to be connected. This implies that $C_{\mathbb{P}}$ and hence M_1 are connected. Furthermore, since for every $x \in \mathbb{R}^n \setminus \{0\}$ the map $g \in S \mapsto gx \in \mathbb{R}^n$ is continuous we conclude that S leaves invariant the connected components of $(\mathbb{R} \setminus \{0\})\, M_1$. The subset $\mathbb{R}^+ M_1$ is one of these components. Hence $\mathbb{R}^+ M_1$ is invariant, implying that its convex closure, that is, W is S-invariant. □

For the converse to the above theorem we note the following facts.

Lemma 4.3 *Let $W \subset \mathbb{R}^n$ be a cone which is S-invariant. Then $C_{\mathbb{P}} \subset [W]$.*

Proof: In fact, $[W] \subset \mathbb{P}^{n-1}$ is closed (and hence compact) and S-invariant. Then $[W]$ contains an invariant control set. But in $\mathbb{P}^{n-1}$ there is exactly one invariant control set, showing the lemma. □

Lemma 4.4 *Let $W \subset \mathbb{R}^n$ be a pointed cone and suppose that $V \subset \mathbb{R}^n$ is a vector subspace with $[V] \subset [W]$. Then* $\dim V = 1$.

Proof: Suppose that $\dim V \geq 2$. Then there exist $u, v \in V$, linearly independent, with $u \in W$ and $v \in -W$. Consider the convex function $f(t) = tu + (1-t)v$ and note that $f(t) \in V \subset W \cup -W$ for $0 \leq t \leq 1$. Take $t_0 = \sup\{t : tu + (1-t)v \in -W\}$. As W and $-W$ is closed it follows that $t_0 u + (1-t_0)v \in W \cap -W$ which contradicts the hypothesis that $W \subset \mathbb{R}^n$ is a pointed cone. □

Theorem 4.5 *Let $S \subset \mathrm{Sl}(n,\mathbb{R})$ be a semigroup satisfying the same assumptions as before and write its parabolic type as $\Theta(S) = \{r_1, \ldots, r_k\}$. Assume that $W \subset \mathbb{R}^n$ is a proper cone S-invariant. Then $r_1 = 1$, that is, $\Theta_{\mathbb{P}}$ refines $\Theta(S)$.*

Proof: Consider the natural fibrations

$$\pi_{\Theta(S)} : \mathbb{F} \to \mathbb{F}_{\Theta(S)} \text{ and } \pi_1 : \mathbb{F} \to \mathbb{P}^{n-1}.$$

We have $C_{\mathbb{P}} = \pi_1(C)$ and $C = \pi^{-1}_{\Theta(S)}\left(C_{\Theta(S)}\right)$, where C, $C_{\mathbb{P}}$ and $C_{\Theta(S)}$ are the invariant control sets in the full flag manifold $\mathbb{F}$, in the projective space and in $\mathbb{F}_{\Theta(S)}$, respectively. Take $f \in C_{\Theta(S)}$. It is a flag of the type

$$(V_1 \subset \cdots \subset V_k)$$

with $\dim V_1 = r_1$. The fiber $\pi^{-1}_{\Theta(S)}\{f\}$ above is entirely contained in C. This fiber contains the complete flags $(W_1 \subset \cdots \subset W_{n-1})$ such that $\dim W_i = i$ and $W_{r_j} = V_j$, $j = 1, \ldots, k$. This implies that the projection $C_{\mathbb{P}} = \pi_1(C)$ contains the k_1-dimensional subspace $[V_1]$.

Now, let W be an S-invariant proper cone. Then W is a pointed cone and $C_{\mathbb{P}} \subset [W]$ by Lemma 4.3. Hence $[V_1] \subset [W]$, so that by Lemma 4.4, we have $k_1 = \dim V_1 = 1$, concluding the proof. □

Combining the above theorems with Proposition 3.5 we arrive immediately at the following characterization of the semigroups having backward invariant cones.

Corollary 4.6 *Let S be as above with $\Theta(S) = \{r_1, \ldots, r_k\}$. Then there exists a S^{-1}-invariant proper cone if and only if $r_k = n - 1$.*

5 Invariant convex sets

We improve here the above results by showing that there are no S-invariant convex sets in case $r_1 > 1$ is the parabolic type of S. This will require another general result on semigroups. To state it let $\Theta(S) = \{r_1, \ldots, r_k\}$ be the parabolic type of S. Take $h \in \mathrm{reg}(S)$ and let $\beta(h) = \{e_1(h), \ldots, e_n(h)\}$ be the basis which diagonalizes h with decreasing eigenvalues. Denote by $P_{\Theta(S)}(h)$ the subgroup of those elements in $\mathrm{Sl}(n,\mathbb{R})$ whose matrices with respect to $\beta(h)$ are block upper triangular matrices of the form

$$\begin{pmatrix} A_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_k \end{pmatrix}$$

with diagonal blocks of sizes $r_1, r_2 - r_1, \ldots, n - r_k$. Clearly, $h \in P_{\Theta(S)}(h) \cap \mathrm{int}S$, so that $S_{\Theta(S)}(h) = P_{\Theta(S)}(h) \cap \mathrm{int}S$ is a non void open semigroup of $P_{\Theta(S)}(h)$.

Before proceeding we discuss briefly the well known structure of $P_{\Theta(S)}(h)$. Consider the following subgroups of $P_{\Theta(S)}(h)$:

- $M_{\Theta(S)}(h)$ the subgroup of unimodular block diagonal matrices
$$\begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_k \end{pmatrix} \tag{4}$$
with $\det A_i = 1$.

- $Q_{\Theta(S)}(h)$ the subgroup of upper triangular matrices having scalar matrices on the blocks
$$\begin{pmatrix} a_1\mathrm{id} & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_k\mathrm{id} \end{pmatrix}. \tag{5}$$

Then there is the decomposition $P_{\Theta(S)}(h) = M_{\Theta(S)}(h)Q_{\Theta(S)}(h)$. Also, let $Q^0_{\Theta(S)}(h) \subset Q_{\Theta(S)}(h)$ be the subgroup formed by those matrices in (5) with $a_j > 0$. Then $Q^0_{\Theta(S)}(h)$ is the identity component of $Q_{\Theta(S)}(h)$ and $P^0_{\Theta(S)}(h) = M_{\Theta(S)}(h)Q^0_{\Theta(S)}(h)$ is the identity component of $P^0_{\Theta(S)}(h)$. Furthermore, $P_{\Theta(S)}(h)$ has a finite number of connected components.

Now, we have the following result proved in [12] (see Theorem 4.1 and the discussion before it; see also [3], Lemma 5.3 and Corollary 5.4).

Proposition 5.1 *Let the notations be as above with $h \in \mathrm{reg}(S)$. Then for every $g \in M_{\Theta(S)}(h)$ there exists*
$$q = \begin{pmatrix} a_1\mathrm{id} & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_k\mathrm{id} \end{pmatrix} \in Q^0_{\Theta(S)}(h)$$
such that $gq \in S_{\Theta(S)}(h) \subset \mathrm{int}S$. Furthermore, we can chose q so that $a_1 > \cdots > a_k > 0$ (in particular $a_1 > 1$, since $\det q = 1$).

As a consequence we have the transitivity result.

Proposition 5.2 *Let the notations be as above with* $h \in \text{reg}(S)$ *and* $S_{\Theta(S)}(h) = P_{\Theta(S)}(h) \cap \text{int}S$. *Let* $b_\Theta(h)$ *be the flag spanned by* $\beta(h)$ *(see (3)) and write* $V_1(h) = \text{span}\{e_1(h), \ldots, e_{k_1}(h)\}$ *for the first vector subspace in* $b_\Theta(h)$. *Suppose that* $r_1 > 1$. *Let* $\{0\} \neq K \subset V_1(h)$ *be a convex set invariant by* $S_{\Theta(S)}(h)$. *Then* $K = V_1(h)$.

Proof: Take $0 \neq x \in K$. Then for any $y \in V_1(h) \setminus \{0\}$, there are linear maps $A_1, A_2 \in V_1(h) \to V_1(h)$ with determinant 1 such that $A_1 x = y$ and $A_2 x = -y$. Then

$$g_i = \begin{pmatrix} A_i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \text{id} \end{pmatrix} \in M_{\Theta(S)}(h) \qquad i = 1, 2.$$

Hence by Proposition 5.1 there are $q_1, q_2 \in Q^0_{\Theta(S)}(h)$ such that $g_i q_i \in \text{int}S$, $i = 1, 2$. Note that for any $z \in V_1(h)$ and $q \in Q^0_{\Theta(S)}(h)$ we have $qz = az$, $a \in \mathbb{R}$, and we can choose q_i such that $q_1 z = a_1 z$ and $q_2 z = a_2 z$ with $a_1, a_2 > 1$. Therefore, $g_1 x = a_1 A_1 x = a_1 y$ and $g_2 x = -a_2 y$. By invariance of K we have that $a_1 y$ and $-a_2 y$ belong to K, and since $a_1, a_2 > 1$ we conclude that the segment from $-y$ to y is contained in K. This holds for arbitrary $y \in V_1(h) \setminus \{0\}$, showing that $K = V_1(h)$. □

Theorem 5.3 *Let* $S \subset \text{Sl}(n, \mathbb{R})$ *be a semigroup satisfying the same assumptions as before and write its parabolic type as* $\Theta(S) = \{r_1, \ldots, r_k\}$. *Assume that* S *leaves invariant a proper convex set* $\{0\} \neq K \subset \mathbb{R}^n$. *Then* $r_1 = 1$.

Proof: Let $K \neq \{0\}$ be an invariant convex set and assume that $r_1 > 1$. First we claim that $V_1(h) \subset K$ for every $h \in \text{reg}S$. In fact, if $h \in \text{reg}S$ then $b_{\Theta(S)}(h)$ belongs to the set of transitivity $C^0_{\Theta(S)}$ of the invariant control set $C_{\Theta(S)}$. Hence by Theorem 3.1 the whole subspace $[V_1(h)]$ is contained in $C^0_{\mathbb{P}}$. Now the uniqueness of the invariant control set implies S is backwards controllable from every $y \in C^0_{\mathbb{P}}$, that is, for every $x \in \mathbb{P}^{n-1}$ there exists $g \in S$ such that $g \cdot x = y$. Hence if we take $v \in K$ there exists $g \in S$ such that $g \cdot [v] \in [V_1(h)]$ or equivalently, $gv \in V_1(h)$. By invariance of K it follows that the convex set $V_1(h) \cap K$ is not empty. Hence by Proposition 5.2 we conclude that $V_1(h) \subset K$, as claimed.

Now, let $F = \bigcup V_1(h)$ with h running through $\text{reg}S$. By the above, $F \subset K$. Since $\text{reg}S$ is an open subset of $\text{Sl}(n, \mathbb{R})$, it follows that F generates $\mathbb{R}^n$. Also, $-F = F$, hence $\mathbb{R}^n$ is the convex closure of F, because every linear combination of F is already a convex combination. Thus $K = \mathbb{R}^n$, concluding the proof. □

The above theorem has the following immediate consequences.

Corollary 5.4 *A semigroup* $S \subset \mathrm{Sl}(n, \mathbb{R})$ *with* $\mathrm{int} S \neq \emptyset$ *admits invariant proper convex sets if and only if it leaves invariant a proper convex cone.*

Corollary 5.5 *Let* S *be as above with* $\Theta(S) = \{r_1, \ldots, r_k\}$. *Then there exists a* S^{-1}*-invariant proper convex set if and only if* $r_k = n - 1$.

6 An example

In this section we work out an example of a noncontrollable system which does not admit proper invariant cones nor proper convex sets forward and backward in time. Our example is in dimension 4. This is the smallest possible dimension in view of Proposition 7.1 below.

We denote by $\{e_1, \ldots, e_4\}$ the standard basis of $\mathbb{R}^4$ and by E_{ij} the basic matrix having entry 1 at the i, j-position and 0 elsewhere.

Consider the control system $A + uB$ with

$$A = \begin{pmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -2 & 0 & 1 & 0 \end{pmatrix}$$

and $B = \mathrm{diag}\{b_1, b_2, b_3, b_4\}$ with $\mathrm{tr} B = 0$ and such that $b_2 > b_1, b_4 > b_3$ and $b_i - b_j \neq b_r - b_s$ if $\{i, j\} \neq \{r, s\}$.

This system satisfies the Lie algebra rank condition. In fact, the last condition on B implies that the brackets $\mathrm{ad}\,(B)^k A$, $k \geq 0$ span the subspace of those matrices having zero entries at the positions of the zero entries of A. Hence the basic matrices E_{ij}, $|i - j| = 1$ belong to the Lie algebra generated by A and B. But it is easy to check that these basic matrices generate $\mathfrak{sl}(4, \mathbb{R})$, which shows that this is the Lie algebra generated by A and B. Hence, it follows that the semigroup S of the system has nonempty interior in $\mathrm{Sl}(4, \mathbb{R})$.

For a direct computations, using mathematical softwares, we can take for instance $B = \mathrm{diag}\{1, 4, -2, -3\}$, the matrix A as above and to verify that A, B, $\mathrm{ad}\,(B)^k A$ with $1 \leq k \leq 7$, $\mathrm{ad}\,(A)^l B$ with $l = 2, 4, 6, 8$, $\mathrm{ad}\,(B)\,(\mathrm{ad}\,(A)^2 B)$ and $\mathrm{ad}\,(A)^2\,(\mathrm{ad}\,(B)\,(\mathrm{ad}\,(A)^2 B))$ form a basis of $\mathfrak{sl}(4, \mathbb{R})$.

On the other hand by Proposition 2 of [10] we have that both matrices A and B belong to the Lie wedge of a proper semigroup of $\mathrm{Sl}(4, \mathbb{R})$, so that $S \neq \mathrm{Sl}(4, \mathbb{R})$, implying that the system is not controllable.

It is easy to check that neither A nor $-A$ leave invariant an orthant of $\mathbb{R}^4$. In fact, by [4], Lemma 1, a matrix $X = (x_{ij})$ leaves invariant the orthant with signs $(\varepsilon_1, \ldots, \varepsilon_n)$ if and only if $\varepsilon_i \varepsilon_j x_{ij} > 0$. Applying this condition to $\pm A$ we arrive at the contradiction that $\varepsilon_1 \varepsilon_4$ must be simultaneously $+1$ and -1, so that there are no invariant orthants.

The verification that there are no invariant cones is more involved. To this end we start by noting that an easy computation shows that the eigenvalues of A are

$$\frac{1}{2}+\frac{1}{2}\sqrt{13}, \frac{1}{2}-\frac{1}{2}\sqrt{13}, -\frac{1}{2}+\frac{1}{2}\sqrt{13}, -\frac{1}{2}-\frac{1}{2}\sqrt{13}$$

with $\frac{1}{2}+\frac{1}{2}\sqrt{13}$ the largest one and $-\frac{1}{2}-\frac{1}{2}\sqrt{13}$ the smallest. The eigenspace of $\frac{1}{2}+\frac{1}{2}\sqrt{13}$ is spanned by

$$v_{\max} = \left(1, -\frac{3}{2}+\frac{1}{2}\sqrt{13}, \frac{3}{2}-\frac{1}{2}\sqrt{13}, -1\right)$$

which belongs to the orthant $(+,+,-,-)$.

Now, suppose that W is a proper cone invariant by forward trajectories of the system. By Lemma 4.1 W is a generating cone. Then W must intercept the principal eigenspace of A, that is, one of the rays $\mathbb{R}_+ v_{\max}$ or $-\mathbb{R}_+ v_{\max}$ must be contained in W. To fix ideas let us suppose that $\mathbb{R}_+ v_{\max} \subset W$. Let $l_{\max}$ be the ray spanned by $v_{\max}$. By the choice of the ordering of the eigenvalues of B we have that $(\exp tB)\, l_{\max}$ converges to $\mathbb{R}_+ e_2$ when $t \to +\infty$ and converges to $-\mathbb{R}_+ e_3$ as $t \to -\infty$. Since $\exp tB$ belongs to the closure of the control semigroup for all $t \in \mathbb{R}$ we conclude that the rays $\mathbb{R}_+ e_2$ and $-\mathbb{R}_+ e_3$ are contained in W.

On the other hand define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = \langle (\exp tA)\, e_2, e_3 \rangle$. Then $f'(0) = \langle Ae_2, e_3 \rangle = 1$. So that there exists small $t_0 > 0$ such that $f(t_0) > 0$, that is, $(\exp t_0 A)\, e_2$ belongs to the half-space $\langle x, e_3 \rangle > 0$. Clearly, $(\exp t_0 A)\, e_2 \in W$ since W is forward invariant and $e_2 \in W$. Now, let l_{t_0} be the ray spanned by $(\exp t_0 A)\, e_2$. Then $(\exp tB)\, l_{t_0}$ converges to the ray $\mathbb{R}_+ e_3$ as $t \to -\infty$, so that $\mathbb{R}_+ e_3$ is also contained in W. But this contradicts Lemma 4.1 which ensures that the proper invariant cone W is pointed.

The proof that there are no backwards invariant cone is analogous. We argue with the eigenspace of the smallest eigenvalue $-\frac{1}{2}-\frac{1}{2}\sqrt{13}$ which is spanned by

$$v_{\min} = \left(1, \frac{3}{2}-\frac{1}{2}\sqrt{13}, \frac{3}{2}-\frac{1}{2}\sqrt{13}, 1\right).$$

This eigenvector belongs to the orthant $(+,-,-,+)$. So that if we assume without loss of generality that $v_{\min}$ belongs to an invariant cone we conclude that $-e_2$ and

$-e_3$ are also in the invariant cone. Now, $\langle -A(-e_2), e_3\rangle = 1$ so that the invariant cone enters the half-space $\langle x, e_3\rangle > 0$, which permits to conclude that e_3 also belongs to the cone which is a contradiction.

This finishes the proof that our system do not have forward or backward invariant cones. As a consequence we have that the parabolic type of the semigroup S of the system is $\{2\}$, which is the only possible parabolic type in dimension 4 which does not start with 1 nor ends with 3.

7 Concluding remarks

In general the existence of an invariant cone in $\mathbb{R}^n$ is a sufficient but not necessary condition for a bilinear control system to be noncontrollable. However, in low dimensions we can say more about this question. For example in dimension 2 the only possibility for the parabolic type of a semigroup is $\Theta(S) = \{1\}$, hence by the above results a control system is not controllable if and only if it leaves invariant a cone in $\mathbb{R}^2$ (See [2] for a detailed analysis of the 2-dimensional case).

In dimension 3 we have three possibilities for the parabolic type, namely $\Theta(S) = \{1\}$ (where $\mathbb{F}_{\{1\}} = \mathbb{P}^2$), $\Theta(S) = \{2\}$ (where $\mathbb{F}_{\{2\}}$ is the Grassmannian $\mathrm{Gr}_2(3)$) and $\Theta(S) = \{1,2\}$ (where $\mathbb{F}_{\{1,2\}}$ is the full flag manifold). There are invariant cones only in the cases $\Theta(S) = \{1\}$ and $\Theta(S) = \{1,2\}$. In the case $\Theta(S) = \{2\}$ we look at the inverse semigroups S^{-1}. By the results of [7], Section 6, it follows that the parabolic type of S^{-1} is the flag manifold dual to $\mathbb{F}_{\Theta(S)}$. In our case the projective space $\mathbb{P}^2$ and the Grassmannian $\mathrm{Gr}_2(3)$ are dual to each other, while $\mathbb{F}_{\{1,2\}}$ is self-dual (see [7], Section 3). Thus $\Theta(S^{-1}) = \{1\}$ if $\Theta(S) = \{2\}$ and S^{-1} leaves invariant a cone in $\mathbb{R}^3$. Thus in dimension 3 we have the following necessary and sufficient condition for controllability.

Proposition 7.1 *Suppose that $S \subset \mathrm{Sl}(3,\mathbb{R})$ is a connected semigroup with nonempty interior. Then S is controllable if and only if neither S nor S^{-1} leave invariant a proper cone $W \subset \mathbb{R}^3$.*

Clearly, if S is the semigroup of the control system $\dot{x} = Ax + uBx$ then S^{-1} is the semigroup of the control system $\dot{x} = -Ax + uBx$.

In general we must look at invariant cones in the Grassmann spaces $\bigwedge^r \mathbb{R}^n$ of exterior products of $\mathbb{R}^n$. The idea is that the parabolic type $\Theta(S) = \{r_1, \ldots, r_k\}$ contains $\{r_1\}$ and $\mathbb{F}_{\{r_1\}}$ is the Grassmannian $\mathrm{Gr}_{r_1}(n)$ of r_1-dimensional subspaces of $\mathbb{R}^n$. By the projection $\mathbb{F}_{\Theta(S)} \to \mathbb{F}_{\{r_1\}}$ the invariant control set in the Grassmannian is contractible in the sense of Corollary 3.4. Now the $\mathrm{Gr}_{r_1}(n)$ embeds into the projective space of $\bigwedge^{r_1} \mathbb{R}^n$. Using this embedding and arguing as in the proof of Theorem 4.2 the existence of an invariant cone in $\bigwedge^{r_1} \mathbb{R}^n$ is obtained. We refer to [6] for the details.

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