DECOMPOSITION OF FUZZY SETS AND MULTILINEARIZATION FOR THE EXTENSION PRINCIPLE

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ABSTRACT

In this article we present a proposal for the descomposition of large uncertainties associated with a fuzzy set u and the multilinearization of a function f to be used in the approximation of the fuzzy set $\hat{f}(u)$ obtained by extension principle. Also, we present conditions on f assuring that $\hat{f}$ preserves differentiability.

KEYWORDS: Fuzzy sets, extension principle, differentiable functions.

1 INTRODUCTION

It is well known that Zadeh's extension principle play an important role in the fuzzy set theory and has been studied and applied by many authors, including Barros (Barros and Tonelli, 2000), Román&et.(Román-Flores and Bassanezi, 2001) in the analysis of discrete fuzzy dynamical systems and continuity of $\hat{f}$, Cabrelli et al. (Cabrelli and Vrscay, 1992) in the study of fuzzy fractals, B$\check{e}$lohlávek (lohlávek, 2000) in the study of similarity, Liu (Shiang-Tai Liu, 2004) in fuzzy transportation problems, among others.

In general, the calculus of $\hat{f}(u)$ is a complex problem. If f is linear there are not bigger difficulties to visualize $\hat{f}(u)$, what doesn't happen if f is not linear. Therefore, it would be interesting to approach $\hat{f}(u)$ for well-known things.

Handling large uncertainties in non-linear problems presents difficulties that can compromise the evaluation and significance of the obtain result. Classical approaches use linearized model and result have their validation dependent on the linearization quality. In most cases, the problem of non-linearity is severe and linearization is valid only on a small region around the linearization point. Recently, in (Saavedra and Mangueira, n.d.) is presented a way of obtaining solution of non linear fuzzy systems using decomposition of incremental fuzzy numbers.

In this work, we state a form that allows the decomposition of a fuzzy set into varius fuzzy set with degree of incertain smaller than the original one. Consequently, it is possible to state a multilinearization process for obtain a approximation of the fuzzy set $\hat{f}(u)$.

On the other hand, differentiable fuzzy sets are an important tool for the implementation of fuzzy expert systems and its applications. For example, in neuro-fuzzy learning models, which are based on a gradient descent strategy, it is necessary to have differentiable membership functions, that is to say, differentiable input fuzzy sets (see Castellano et al.(Castellano and Mencar, 2004)). Also, there are defuzzification methods that keep all features regarding differentiabilty, which says us that the construction of differentiable membership functions plays a relevant role in the modelling and resolution of real problems (see Grauel&Ludwig (Grauel and L. Ludwig, 1999)).

Consequently, in the Section 4 of this work, we establish conditions assuring that $\hat{f}$ preserves differentiability.

2 PRELIMINARIES AND DESCOMPOSITION OF FUZZY SETS

A fuzzy set in a universe set X is a mapping $u : X \rightarrow [0, 1]$. We think of u as assigning to each element $x \in X$ a degree of membership, $0 \le u(x) \le 1$. Let u be a fuzzy set in $\mathbb{R}$. We define $[u]^\alpha = \{x \in \mathbb{R}/\ u(x) \le \alpha\}$ the α-level of u, with $0 < \alpha \le 1$. For $\alpha = 0$ we have $[u]^0 = \overline{\{x \in \mathbb{R}/\ u(x) > 0\}}$, the support of u. Note that $[u]^0$ indicate the degree of uncertain of the fuzzy set u (also is called uncertainty interval, confidence interval). We denote by $\mathcal{F}(\mathbb{R})$ the family of compact fuzzy sets in $\mathbb{R}$, i.e. , the family of all fuzzy sets u such that $[u]^\alpha$ is compact for all $\alpha \in [0, 1]$. Also, we denote by $\mathcal{F}_N(\mathbb{R})$ the family of fuzzy numbers, i.e., the family of compact fuzzy set $u \in \mathcal{F}(\mathbb{R})$ such that
(i) u is convex, i.e., $[u]^\alpha$ is convex for all $\alpha \in [0, 1]$;
(ii) u is normal, i.e., there exist x in $\mathbb{R}$ such that $u(x) = 1$.
With the conditions above, $\mathcal{F}_N(\mathbb{R})$ is the family of the fuzzy sets u such that $[u]^\alpha$ is a interval for all $\alpha \in [0, 1]$.

Now, we consider a distance between two compact fuzzy sets u and v by

$$D(u, v) = \sup_{\alpha \in [0,1]} H([u]^\alpha, [v]^\alpha),$$

where H is the well-known Hausdorff metric defined by

$$H([u]^\alpha, [v]^\alpha) = \max \{d([u]^\alpha, [v]^\alpha), d([v]^\alpha), [u]^\alpha)\}$$

with $d(A, B) = \sup_{a \in A} d(a, B)$ and $d(a, B) = \inf_{b \in B} d(a, b)$.

Let u, v be any two fuzzy sets in $\mathbb{R}$. Then $u \vee v$, the union of u and v, is a fuzzy set defined by

$$(u \vee v)(x) = u(x) \vee v(x) \quad \text{for all } x \in \mathbb{R};$$

where $\vee$ denote the supremum. Note that in general the union of two fuzzy number is not a fuzzy number.

2.1 Decomposition of fuzzy numbers

Given a fuzzy number u with support or uncertainty interval $[a, b]$, we consider a particular value $x_d \in [a, b]$. Then, is possible to descompose u into two compact fuzzy sets u_l and u_r, with support $[a, x_d]$ and $[x_d, b]$ respectively, with complementary uncertainties such as

$$u(x) = u_l(x) \vee u_r(x).$$

Clearly, these new two fuzzy set have smaller associated uncertainty than u, that in numerical terms means that we can work with smaller desviations or increment (half the original). On the other hand, the uncertainty intervals are complementary: one representing the incertainty of be smaller than x_d, the other considering the incertainty of be bigger than x_d, see Figure 1.

In general, we can decompose a fuzzy number into n compact fuzzy sets u_1, u_2 ...,u_n in the following way:
(i) We consider a regular partition of the uncertainty interval $[a, b]$, i.e., we divide $[a, b]$ in n subintervals $[x_i, x_{i+1}]$;
(ii) We consider the fuzzy sets u_i defined by $u_i(x) = u(x)$ for all $x \in [x_i, x_{i+1}]$ and $u_i(x) = 0$ another case, and consequently we have

$$u(x) = \bigvee_{i=1,\ldots,n} u_i(x),$$

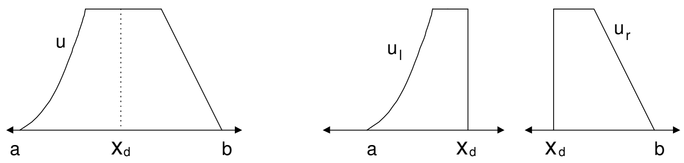

Figure 1: Decomposition of a fuzzy number u

for all $x \in \mathbb{R}$. Note that the uncertainty interval of each fuzzy set u_i is $[x_i, x_{i+1}]$, which has smaller degree uncertainty than the original u.

It is important remark that descomposition of uncertainty is not a novel topic. In fact, in the last decade several authors have been dealing with this issue, see (Saavedra and Mangueira, n.d.), (Hofer, 1996).

2.2 Extension principle

In (Zadeh, 1975), Zadeh proposed a so called extension principle which became an important tool in fuzzy set theory and its applications. The idea is that each function $f : X \rightarrow Y$ induces a corresponding function $\hat{f} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ (i.e., $\hat{f}$ is a function mappings fuzzy sets in X to fuzzy sets in Y) defined for each fuzzy sets u in X by

$$\hat{f}(u)(y) = \bigvee_{x \in X,\ f(x) = y} u(x).$$

The function $\hat{f}$ is said to be obtained from f by the extension principle.

If we consider $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = ax + b$, we obtain $\hat{f} : \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})$ without difficulty, being

$$\hat{f}(u)(y) = u\left(\frac{y - b}{a}\right),$$

for all $u \in \mathcal{F}(\mathbb{R})$. In general, if f is biyective we have that

$$\hat{f}(u)(y) = u\left(f^{-1}(y)\right).$$

On the other hand, it is interesting to know that if $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, then $\hat{f} : \mathcal{F}(\mathbb{R}^n) \rightarrow \mathcal{F}(\mathbb{R}^n)$ is a well defined function and (see (Román-Flores and Bassanezi, 2001))

$$\left[\hat{f}(u)\right]^\alpha = f\left([u]^\alpha\right)\ ,\quad \forall \alpha \in [0, 1],\ \forall u \in \mathbb{R}^n. \qquad (1)$$

Consequently, if u is a fuzzy number with support $[u]^0 = [a, b]$ then we have that

$$\left[\hat{f}(u)\right]^0 = \left[\inf_{x \in [a,b]} f(x), \sup_{x \in [a,b]} f(x)\right].$$

3 DECOMPOSITION AND MULTILINEARIZATION

In this section we present a proposal for obtain a approximation of $\hat{f}(u)$, using decomposition of fuzzy sets and multilinearization of f.

Let u be a fuzzy number with support $[a,b]$ and suppose that f is differentiable at $[a,b]$. We decompose u in n compact fuzzy sets, i.e., we divide $[a,b]$ in n subintervals $[x_i, x_{i+1}]$ and we consider the fuzzy sets u_i defined by $u_i(x) = u(x)$ for all $x \in [x_i, x_{i+1}]$ and $u_i(x) = 0$ another case. Consequently we have

$$u(x) = \bigvee_{i=1,\dots,n} u_i(x).$$

For $i = 1, 2, ..., n$ we take

$$x_i^* = \frac{x_{i+1} + x_i}{2}$$

and we consider the tangent straight line to f in the point x_i^*, defined in the interval $[x_i, x_{i+1}]$, and we denote it by

$$f_i(x) = f^{'}(x_i^*)(x - x_i^*) + f(x_i^*).$$

Let us observe that f_i is only defined in the interval $[x_i, x_{i+1}]$, outside of this interval $f_i = 0$.

Now, for each $i = 1, 2, ..., n$ we obtain $\hat{f}_i(u_i)$, that it is defined by

$$\hat{f}_i(u_i)(y) = u_i\left(\frac{y - f(x_i^*)}{f^{'}(x_i^*)} + x_i^*\right);$$

if $f^{'}(x_i^*) \neq 0$ and $\hat{f}_i(u_i)(y) = u(x_i^*)$ if $f^{'}(x_i^*) = 0$. In general we have that

$$\hat{f}_i(u)(y) = \hat{f}_i(u_i)(y).$$

Finally, we define $H : \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ by

$$H = \bigvee_{i=1,2,\dots,n} \hat{f}_i\ .$$

This way, $H(u)$ is an approximation of $\hat{f}(u)$.

If we take $g_n : \mathbb{R} \to \mathbb{R}$ defined by

$$g_n(x) = \begin{cases} f_1(x) & if \quad x \in [x_1, x_2] \\ f_2(x) & if \quad x \in [x_2, x_3] \\ \vdots & \vdots \quad\quad \vdots \\ f_n(x) & if \quad x \in [x_n, x_{n+1}] \end{cases} \tag{2}$$

Then we have the following

Proposition 1 Given f, f_i, g_n as above, then

$$\hat{g_n}(u) = H(u) = \bigvee_{i=1,2,\dots,n} \hat{f}_i(u).$$

Proof Let $\alpha \in [0,1]$. Then,

$$[\hat{g_n}(u)]^\alpha = g_n([u]^\alpha) = f_1([u]^\alpha) \cup f_2([u]^\alpha) \cup ... \cup f_n([u]^\alpha)$$
$$= \bigvee_{i=1,2,\dots,n} \hat{f}_i([u]^\alpha) = \left[\bigvee_{i=1,2,\dots,n} \hat{f}_i(u)\right]^\alpha .$$

Therefore

$$\hat{g_n}(u) = \bigvee_{i=1,2,\dots,n} \hat{f}_i(u).\square$$

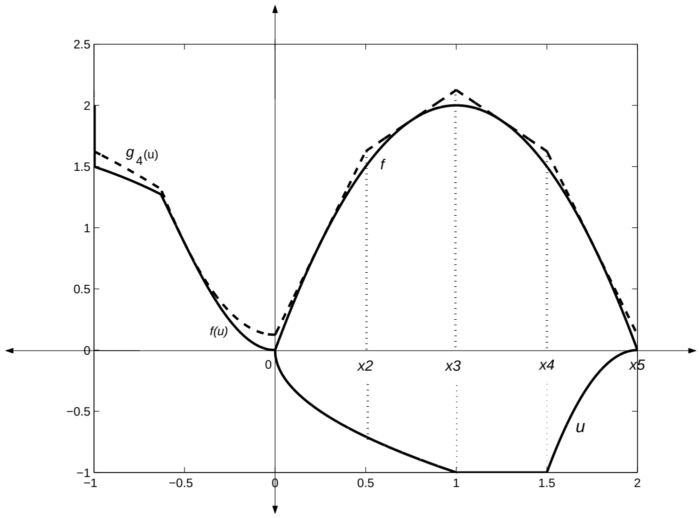

Figure 2: Process of decomposition and multilinearization for obtain $\hat{f}(u)$

Example 1 We consider the fuzzy number u defined by

$$u(x) = \begin{cases} \sqrt{x} & if \quad x \in [0,1] \\ 1 & if \quad x \in [0,1.5] \\ 4(x-2)^2 & if \quad x \in [1.5,2]. \end{cases}$$

We have that the uncertainty interval is $[u]^0 = [0,2]$.

Now, we descompose u in 4 compact fuzzy sets u_1, u_2, u_3, u_4. For this we consider a partion regular of the interval $[0,2]$ in 4 subintervals with same degree of uncertainties, these subintervals are:

$$[0, 0.5]\ ,\ [0.5, 1]\ ,\ [1, 1.5]\ ,\ [1.5, 2].$$

For each $i = 1, 2, 3, 4$ we consider $u_i(x) = u(x)$ for all $x \in [x_i, x_{i+1}]$ and $u_i(x) = 0$ another case, for example,

$$u_2(x) = \begin{cases} \sqrt{x} & if \quad x \in [0.5, 1] \\ 0 & if \quad x \neq [0.5, 1]. \end{cases}$$

Next, let us take the function f defined by $f(x) = -2x^2 + 4x$. Then, we take the tangent straight line f_i to the curve f in the point $x_i^* = \frac{x_i + x_{i+1}}{2}$, the half point of the interval

$[x_i, x_{i+1}]$. For example, for $i = 2$ we have that $x_2^* = 0.75$ and

$$f_2(x) = x + 9/8.$$

Now, for each $i = 1, 2, 3, 4$, we obtain $\hat{f}_i(u_i)$. For example

$$\hat{f}_2(u_2)(y) = u_2(y - 9/8) = \sqrt{y - 9/8}, \ \text{ with } \ y \in f_2([0.5, 1]).$$

Finally, we consider $\hat{g_n}(u) = \hat{f}_1(u_1) \vee \hat{f}_2(u_2) \vee \hat{f}_3(u_3) \vee \hat{f}_4(u_4)$, which is an approximation of $\hat{f}(u)$. See Figure 2.

Theorem 1 *Let u be a fuzzy number with $[u]^0 = [a,b]$. If f, g are two function continuous in $[a,b]$ such that $\|f - g\|_\infty \leq M$. Then*

$$D\left(\hat{f}(u), \hat{g}(u)\right) \leq M.$$

Proof From (1) we have that

$$D\left(\hat{f}(u),\hat{g}(u)\right)=\sup_{\alpha\in[0,1]} H\left(f\left([u]^{\alpha}\right),g\left([u]^{\alpha}\right)\right).$$

Now, given $z\in f\left([u]^{\alpha}\right)$ there exists $y\in[u]^{\alpha}$ such that

$$f(y)=z \quad \text{and} \quad g(y)\in g\left([u]^{\alpha}\right).$$

Therefore

$$d\left(z,g\left([u]^{\alpha}\right)\right)=\inf_{c\in g([u]^{\alpha})} d(z,c)\le|f(y)-g(y)|\le M.$$

Consequently,

$$d\left(f\left([u]^{\alpha}\right),g\left([u]^{\alpha}\right)\right)=\sup_{z\in f([u]^{\alpha})} d\left(z,g\left([u]^{\alpha}\right)\right)\le M.$$

Similarly, we obtain that

$$d\left(g\left([u]^{\alpha}\right),f\left([u]^{\alpha}\right)\right)\le M,$$

Thus,

$$D\left(\hat{f}(u),\hat{g}(u)\right)\le M.\square$$

Note that g_n defined in (2) converge uniformly to f when $n\to\infty$. Therefore, we have the following immediate consequence of the Theorem 1.

Corollary 1 *Given f and g_n as above (2), then*

$$\hat{g_n}(u)\stackrel{D}{\to}\hat{f}(u) \quad \text{as} \quad n\to\infty.$$

4 PRESERVING DIFFERENTIABILITY

As was said in the introduction, differentiable fuzzy sets are an important tool for the implementation of fuzzy expert systems and its applications. Then, would be interesting to know if $\hat{f}$ preserves differentiability, i.e., the following question arises:
If u is a differentiable fuzzy sets, $\hat{f}(u)$ is a differentiable fuzzy sets?.
In general, the answer is negative as can be seen in the following example. But imposing some conditions on f, an affirmative answer is obtained as we will see in the following Theorem.

Example 2 We consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^3$. Is well know that f is differentiable and bijective in $\mathbb{R}$. We take the fuzzy numbers u defined by

$$u(x)=\begin{cases}1-x^2, & if \quad x\in[-1,1]\\ 0, & if \quad x\notin[-1,1]\end{cases}$$

It fuzzy set represent "the sets of real numbers around of zero". We can see that u is a differentiable fuzzy sets.

Now, $\hat{f}(u)$ is defined by

$$\hat{f}(u)(x)=\begin{cases}1-\sqrt[3]{x^2}, & if \quad x\in[-1,1]\\ 0, & if \quad x\notin[-1,1]\end{cases}$$

that is not a differentiable fuzzy set, because the derive at $0\in[\hat{f}(u)]^0$ doesn't exist.

Note that, in this case $[u]^0=[-1,1]$ and $f^{'}(0)=0$, i.e., f has a critical point in the support of u.

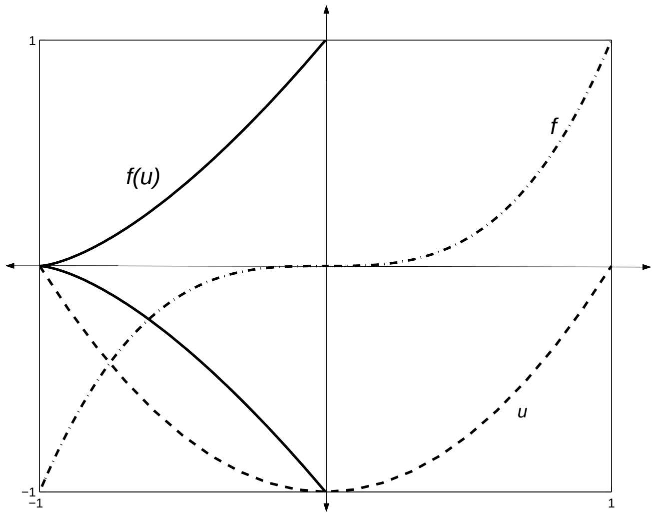

Figure 3: The differentiable fuzzy set u and the fuzzy set $\hat{f}(u)$ not differentiable

Theorem 2 *Let u be a differentiable fuzzy sets with interval of confidence $[u]^0=[a,b]$. Let f be a differentiable function in $[a,b]$ with $f^{'}(x)\neq 0$ for all $x\in[a,b]$. Then, $\hat{f}(u)$ is a differentiable fuzzy sets in $f([a,b])$ and*

$$\hat{f}(u)^{'}(y)=\frac{u^{'}}{f^{'}}(x) \quad \textit{with} \quad y=f(x).$$

Proof If $f^{'}(x)\neq 0$ for all $x\in[a,b]$, then f is monotone in $[a,b]$. Therefore, f is bijective and there exist f^{-1}, the inverse of f, in $[a,b]$. Also, f^{-1} is differentiable and

$$\left(f^{-1}(y)\right)^{'}=\frac{1}{f^{'}(x)} \quad \text{with } f(x)=y.$$

Now, as f is bijectiva, we have that

$$\hat{f}(u)(y)=u\left(f^{-1}(y)\right)=u\circ f^{-1}(y).$$

As u is differentiable in $[a,b]$ and f^{-1} is differentiable in $f([a,b])$, we have that $u\circ f^{-1}$ is differentiable in $f([a,b])$ and

$$\hat{f}(u)^{'}(y)=\left(u\circ f^{-1}\right)^{'}(y)=u^{'}(f^{-1}(y))\cdot\left(f^{-1}\right)^{'}(y)=\frac{u^{'}}{f^{'}}(x),$$

with $y=f(x)$. $\square$

5 CONCLUSIONS

In this paper we presents a proposal for decomposition of a fuzzy number u in n compact fuzzy sets $u_1,u_2,...,u_n$ such a way that the uncertainty interval of each u_i has smaller incertainty than the original u and

$$u=\vee u_i.$$

Taking advantage of the differentiability of f obtains a multilinearization of f and, taking into account the decomposition de u, we presents the fuzzy set $\hat{g_n}(u)$ which is an approximation of $\hat{f}(u)$ with the property that $\hat{g_n}(u)\to\hat{f}(u)$ as $n\to\infty$, Corollary 1, i.e.,

$$D\left(\hat{g_n}(u),\hat{f}(u)\right)\to 0 \quad \text{as} \quad n\to\infty,$$

where D is the metric in the space $\mathcal{F}(\mathbb{R})$.

In the last section of this work, we present conditions so that $\hat{f}$ preserves differentiability, because this is not always true as can be seen in the Example 2. The condition imposed on f, is that f doesn't have not critical points in the uncertainty interval (support) of u.

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