Invexity and the Kuhn-Tucker Theorem in the Continuous-time Context

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Abstract

In this work we study the relationship between the concept of invexity and the Kuhn-Tucker optimality conditions to the continuous-time nonlinear programming problem . We use duality results to establish these relations. We also show that the invexity concept is a kind of constraint qualification.

Key words: Continuous-time nonlinear programming, Kuhn-Tucker conditions, invex type functions.

1 Introduction

We regard the continuous-time nonlinear programming problem below.

$$\left.\begin{aligned} &\text{Minimize}\ \ \phi(x) = \int_0^T f(x(t),t)dt, \\ &\text{subject to } g(x(t),t) \leq 0 \text{ a.e. in } [0,T], \\ &\qquad\qquad x \in X. \end{aligned}\right\} \qquad \text{(CNP)}$$

Here X is a nonempty open convex subset of the Banach space $L_\infty^n[0,T]$, $\phi : X \to \mathbb{R}$, $g(x(t),t) = \gamma(x)(t)$, $f(x(t),t) = \xi(x)(t)$, $\gamma : X \to \Lambda_1^m[0,T]$ and

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$\xi : X \to \Lambda_1^1[0,T]$, where $L_\infty^n[0,T]$ denotes the space of all n-dimensional vector valued Lebesgue measurable functions, which are essentially bounded, defined on the compact interval $[0,T] \subset \mathbb{R}$, with norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1 \le j \le n} \operatorname{ess\,sup}\{|x_j(t)|,\ 0 \le t \le T\},$$

where for each $t \in [0,T]$, $x_j(t)$ is the j-th component of $x(t) \in \mathbb{R}^n$ and $\Lambda_1^m[0,T]$ denotes the space of all m-dimensional vector functions which are essentially bounded and Lebesgue measurable, defined on $[0,T]$, with the norm $\|\cdot\|_1$ defined by

$$\|y\|_1 = \max_{1 \le j \le m} \int_0^T |y_j(t)| dt.$$

This class of problems was introduced in 1953 by Bellman [4] in connection with production-inventory "botleneck processes". He considered a type of optimization problems, which is now known as continuous-time linear programming, he formulated its dual and provided duality relations. He also suggested some computational procedure. Since then, a lot of authors have extended his theory to wider classes of continuous-time linear problems (e.g. [2], [3], [8], [9], [12], [15], [16], [20] and [21]). On the other hand, optimality conditions in the spirit of Kuhn-Tucker type for continuous nonlinear problems were first investigated by Hanson and Mond [11]. They considered a class of linear constrained nonlinear programming problems. Assuming a nonlinear integrand in the cost function twice differentiable, they linearized the cost function and applied Levinson's duality theory [12] to obtain the Karush-Kuhn-Tucker optimality conditions. Also applying linearization, Farr and Hanson [7] obtained necessary and sufficient optimality conditions for a more general class of continuous-time nonlinear problems (both cost function and constraints were nonlinear). Assuming some kind of constraint qualifications and using direct methods, further generalizations of the theory of optimality conditions for continuous-time nonlinear problems are to be found in Scott and Jefferson [19], Abraham and Buie [1], Reiland and Hanson [18] and Zalmai [26], [24], [25], [23], [22]. The development of nonsmooth necessary optimality conditions for problem (CNP) was given in [5]. The sufficient conditions for the nonsmooth case was given in [17]. Related results can be found in Craven [6]. However, his arguments are via approximation of smooth functions rather than alternative theorems. None the above works established necessary and sufficient conditions for a Kuhn-Tucker point be a global solution of (CNP). In [14] we introduced the concept of KT-invexity for Problem (CNP) and show that a KT-point is a global minimizer if and only if the Problem (CNP) is KT-invex. In [14] we also introduced the concept of WD-invexity and show that holds weak duality if and only if the Problem (CNP) is WD-invex. We observe that in the case of mathematical programming these results was given by [13].

In this work, inspired by one previous work of Hanson [10], we study the relationship between the concept of invexity and the Kuhn-Tucker optimality conditions to the continuous-time nonlinear programming problem (CNP). We use duality results given in [26] to establish this relations. We show that invexity kernel η and the Lagrange multiplier λ of the Kuhn-Tucker theory are dual variables. We know of the Kuhn-Tucker theory [25] that if the constraints of the problem satisfies a constraint qualification (Slater or Karlin) then the Kuhn-Tucker conditions are necessary for optimality. We show that the invexity concept is also a kind of constraint qualification.

This work is organized as follows. In Section 1, we give the preliminaries and recall the notion of stability for (CNP) (given in [26]), the definition of invexity (given in [17]) and KT-invexity (given in [14]). In Section 2, we establish and prove our main results and we show an example.

2 Preliminaries

In this section we fix some basic concepts and notation adhered to in this paper.

Let $\mathbb{F}$ be the set of all feasible solutions to Problem (CNP) (which we suppose nonempty), i.e.,

$$\mathbb{F} = \{x \in X : g(x(t), t) \leq 0 \text{ a.e. in } [0, T]\}.$$

Let V be an open subset of $\mathbb{R}^n$ containing the set $\{x(t) \in \mathbb{R}^n : x \in X,\ t \in [0, T]\}$. We assume that f and g_i (the i-th component of g), $i = 1, 2, \ldots, m$, are real functions defined on $V \times [0, T]$. The functions $t \mapsto f(x(t), t)$ and $t \mapsto g(x(t), t)$ are assumed to be Lebesgue measurable and integrable for all $x \in X$. In this paper we assume also that the functions f and g are continuously differentiable (in the Fréchet sense) with respect to their first arguments. We denote by $\nabla f(x(t), t)$ and $\nabla g(x(t), t)$ these derivatives, respectively.

Let

$$I = \{1, 2, \ldots, m\}.$$

For any $x \in \mathbb{F}$, we denote by $I(x)$ the index set of all the binding constraints at x:

$$I(x) = \{i \in I : \ g_i(x(t), t) = 0 \text{ a.e. in } [0, T]\}.$$

About vectors, in this paper they are all collum vectors. We use a prime to denote transposition. Besides, $w \leq 0$ means that $w_i \leq 0$ for all i, and $w < 0$ means that $w_i < 0$ for all i.

Now, we introduce the concept of stability for nonlinear continuous-time programming problems (more details can be found in [26]), recall the notions of invexity for functions (given in [17]) in the continuous-time context and of KT-invexity (given in [14]) for Problem (CNP). This concept and notions will be needed in the next section.

We say that the Problem (CNP) is *stable* if $p(0) < \infty$ and there exists a constant $M > 0$ such that

$$p(0) \leq p(y) + M\|y\|_1, \ \forall y \in \Lambda_1^m[0,T],$$

where

$$p(y) = \inf_{x \in X} \left\{ \int_0^T f(x(t),t)dt : g(x(t),t) \leq y(t) \text{ a.e. in } [0,T] \right\}.$$

p is called *perturbation function* associated with Problem (CNP) and $y(t)$ is called *perturbation vector*.

We will denote by Y the feasible set of the perturbed problem, that is,

$$Y = \{y \in \Lambda_1^m[0,T] : g(x(t),t) \leq y(t) \text{ a.e. in } [0,T] \text{ for some } x \in X\}.$$

Proposition 2.1 *Y is a convex set and p is a convex function in Y.*

PROOF. See [26].

Let $\psi : U \times [0,T] \to \mathbb{R}$ be a differentiable function with respect to its first argument, where $U \subset \mathbb{R}^n$ is a nonempty subset. Let $y \in X$. We say that the function $\psi(\cdot,t)$ is *invex* in $y(t)$ (with respect to U) if there exists $\eta : U \times U \to \mathbb{R}^n$ such that the function $t \mapsto \eta(x(t),y(t)) \in L_\infty^n[0,T]$ and

$$\psi(x(t),t) - \psi(y(t),t) \geq \nabla\psi'(y(t),t)\eta(x(t),y(t)) \text{ a.e. in } [0,T],$$

for all $x \in X$.

We say that Problem (CNP) is *KT-invex* in $y(t)$ (with respect to U) if there exists $\eta : U \times U \to \mathbb{R}^n$ such that the function $t \mapsto \eta(x(t),y(t)) \in L_\infty^n[0,T]$ and

$$\phi(x) - \phi(y) \geq \int_0^T \nabla f'(y(t),t)\eta(x(t),y(t))dt$$

$$-\nabla g_i'(y(t),t)\eta(x(t),y(t)) \geq 0 \text{ a.e. in } [0,T], \ i \in I(y),$$

for all $x \in \mathbb{F}$.

We say that the Kuhn-Tucker conditions apply at $x \in X$ if

$$\int_0^T \left[\nabla f'(x(t),t) + \sum_{i \in I} \lambda_i(t) \nabla g_i'(x(t),t) \right] z(t)dt = 0, \ \forall z \in L_\infty^n[0,T],$$
$$\lambda_i(t) g_i(x(t),t) = 0 \text{ a.e. in } [0,T], \ i \in I,$$
$$\lambda_i(t) \geq 0 \text{ a.e. in } [0,T], \ i \in I,$$

for some $\lambda \in L_\infty^n[0,T]$.

3 Main Results

In this section we state our results and give an example.

Theorem 3.1 *Let $y \in \mathbb{F}$ be an optimal solution for Problem (CNP). If the Kuhn-Tucker conditions apply at y and all $\lambda \in L_\infty^n[0,T]$ satisfying them are bounded, then the active constraints at y are invex functions at $y(t)$ (with respect to V) with a common η.*

PROOF. Since the Kuhn-Tucker conditions apply at y, there exists $\lambda \in L_\infty^n[0,T]$ such that

$$\int_0^T \left[\nabla f'(y(t),t) + \sum_{i \in I} \lambda_i(t) \nabla g_i'(y(t),t) \right] z(t)dt = 0, \ \forall z \in L_\infty^n[0,T], \tag{1}$$
$$\lambda_i(t) g_i(y(t),t) = 0 \text{ a.e. in } [0,T], \ i \in I, \tag{2}$$
$$\lambda_i(t) \geq 0 \text{ a.e. in } [0,T], \ i \in I. \tag{3}$$

Consider any fixed $x \in X$. Let $b, \tilde{\lambda} \in L_\infty^{m+1}[0,T]$, $c \in L_\infty^{n+2}[0,T]$ and $A \in L_\infty^{(n+2)\times(m+1)}[0,T]$ defined for each $t \in [0,T]$ by

$$b(t) = \begin{bmatrix} f(x(t),t) - f(y(t),t) \\ g_1(x(t),t) - g_1(y(t),t) \\ \vdots \\ g_m(x(t),t) - g_m(y(t),t) \end{bmatrix}, \quad \tilde{\lambda}(t) = \begin{bmatrix} \lambda_0 \\ \lambda_1(t) \\ \vdots \\ \lambda_m(t) \end{bmatrix},$$

$$A(t) = \begin{bmatrix} \nabla f(y(t),t) & \nabla g_1(y(t),t) & \cdots & \nabla g_m(y(t),t) \\ 0 & g_1(y(t),t) & \cdots & g_m(y(t),t) \\ 1 & 0 & \cdots & 0 \end{bmatrix} \quad \text{e} \quad c(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Consider the linear continuous-time programming problem

$$\left.\begin{aligned} &\text{Minimize } \int_0^t b'(t)\tilde{\lambda}(t)dt \\ &\text{subject to } A(t)\tilde{\lambda}(t) = c(t) \text{ a.e. in } [0,T], \\ &\qquad\qquad \tilde{\lambda}(t) \geq 0 \text{ a.e. in } [0,T]. \end{aligned}\right\} \tag{4}$$

We can suppose that there does not exist $v \in L_\infty^{n+2}[0,T]$, $v(t) \neq 0$ a.e. in $[0,T]$, such that

$$v'(t)A(t) = 0 \text{ a.e. in } [0,T], \tag{5}$$

because otherwise there are redundant constraints.

Since the equality in (1) holds for all $z \in L_\infty^n[0,T]$, we have

$$\nabla f(y(t),t) + \sum_{i \in I} \lambda_i(t) \nabla g_i(y(t),t) = 0 \text{ a.e. in } [0,T]. \tag{6}$$

It follows from (2), (3) and (6) that the problem is feasible. By hypothesis all λ satisfying the Kuhn-Tucker conditions are bounded. Therefore all $\tilde{\lambda}$ feasible to the problem in (4) are bounded, that is, for all $\tilde{\lambda}$ satisfying the constraints in (4) there exists $K > 0$ such that

$$\|\tilde{\lambda}\|_\infty \leq K.$$

Using this, we obtain

$$\begin{aligned} \left| \int_0^T b'(t)\tilde{\lambda}(t)dt \right| &\leq \int_0^T |b'(t)\tilde{\lambda}(t)|dt = \int_0^T \left| \sum_{j=1}^{m+1} b_j(t)\tilde{\lambda}_j(t) \right| dt \\ &\leq \int_0^T \sum_{j=1}^{m+1} |b_j(t)||\tilde{\lambda}_j(t)|dt \leq \int_0^T \sum_{j=1}^{m+1} |b_j(t)|\|\tilde{\lambda}_j\|_\infty dt \end{aligned}$$

$$= \sum_{j=1}^{m+1} \|\tilde{\lambda}_j\|_\infty \int_0^T |b_j(t)| dt \leq \sum_{j=1}^{m+1} \|\tilde{\lambda}\|_\infty \|b\|_1$$
$$= (m+1)\|\tilde{\lambda}\|_\infty \|b\|_1 \leq (m+1)K\|b\|_1.$$

Thus for all fixed $x \in X$ the objective in (4) is bounded, so that the problem has an optimal solution. Let $\tilde{\lambda}_*$ be an optimal solution. By Fritz John Theorem it follows that there exist $u_0 \in \mathbb{R}$ and $u \in L_\infty^{n+2}[0,T]$ such that

$$\int_0^T [u_0 b'(t) + u'(t)A(t)]h(t)dt = 0, \ \forall h \in L_\infty^{m+1}[0,T], \tag{7}$$

and

$$(u_0, u(t)) \neq 0 \text{ a.e. in } [0,T].$$

Suppose that $u_0 = 0$. From (7) comes

$$\int_0^T u'(t)A(t)h(t)dt = 0, \ \forall h \in L_\infty^{m+1}[0,T],$$

where $u(t) \neq 0$ a.e. in $[0,T]$. Then

$$u'(t)A(t) = 0 \text{ with } u(t) \neq 0 \text{ a.e. in } [0,T],$$

which contradicts (5). Therefore $u_0 \neq 0$. Setting $u_* = u/u_0$ we obtain

$$\int_0^T [b'(t) + u_*'(t)A(t)]h(t)dt = 0, \ \forall h \in L_\infty^{m+1}[0,T]. \tag{8}$$

Let $\tilde{\lambda} \in L_\infty^{m+1}[0,T]$. We have clearly that

$$\int_0^T b'(t)\tilde{\lambda}(t)dt = \int_0^T b'(t)\tilde{\lambda}_*(t)dt + \int_0^T b'(t)[\tilde{\lambda}(t) - \tilde{\lambda}_*(t)]dt \tag{9}$$

and

$$A(t)\tilde{\lambda}(t) - c(t) = A(t)[\tilde{\lambda}(t) - \tilde{\lambda}_*(t)] \text{ a.e. in } [0,T]. \tag{10}$$

Multiplying the expression in (10) by $u_*'(t)$ and integrating, we obtain

$$\int_0^T u_*'(t)[A(t)\tilde{\lambda}(t) - c(t)]dt = \int_0^T u_*'(t)A(t)[\tilde{\lambda}(t) - \tilde{\lambda}_*(t)]dt. \tag{11}$$

From (9) and (11) it follows that

$$\int_0^T \{b'(t)\tilde{\lambda}(t) + u'_*(t)[A(t)\tilde{\lambda}(t) - c(t)]\}dt$$

$$= \int_0^T b'(t)\tilde{\lambda}_*(t)dt + \int_0^T [b'(t) + u'_*(t)A(t)][\tilde{\lambda}(t) - \tilde{\lambda}_*(t)]dt.$$

Using (8) we obtain

$$\int_0^T b'(t)\tilde{\lambda}(t)dt = \int_0^T \{b'(t)\tilde{\lambda}_*(t) - u'_*(t)[A(t)\tilde{\lambda}(t) - c(t)]\}dt. \qquad (12)$$

The perturbation function $p : \Lambda_1^{n+2}[0,T] \to \mathbb{R}$ associated with the problem in (4) is given by

$$p(z) = \inf_{\tilde{\lambda} \in L_\infty^{m+1}[0,T]} \left\{ \int_0^T b'(t)\tilde{\lambda}(t)dt : A(t)\tilde{\lambda}(t) - c(t) = z(t) \text{ a.e. in } [0,T] \right\},$$

where $z(t)$ is the perturbation vector. It is clear that

$$p(0) = \inf_{\tilde{\lambda} \in L_\infty^{m+1}[0,T]} \left\{ \int_0^T b'(t)\tilde{\lambda}(t)dt : A(t)\tilde{\lambda}(t) = c(t) \text{ a.e. in } [0,T] \right\}$$

$$= \int_0^T b'(t)\tilde{\lambda}_*(t)dt < \infty.$$

In this case the feasible set of the perturbed problem is given by

$$Z = \{z \in \Lambda_1^{n+2}[0,T] : A(t)\tilde{\lambda}(t) - c(t) = z(t) \text{ a.e. in } [0,T], \text{ for some } \tilde{\lambda} \in L_\infty^{m+1}[0,T]\}.$$

From (12) we have

$$\int_0^T b'(t)\tilde{\lambda}(t)dt = \int_0^T [b'(t)\tilde{\lambda}_*(t) - u'_*(t)z(t)]dt,$$

for all $\tilde{\lambda} \in L_\infty^{m+1}[0,T]$ such that $A(t)\tilde{\lambda}(t) - c(t) = z(t)$ a.e. in $[0,T]$. Thus

$$\inf_{\tilde{\lambda} \in L_\infty^{m+1}[0,T]} \left\{ \int_0^T b'(t)\tilde{\lambda}(t)dt : A(t)\tilde{\lambda}(t) - c(t) = z(t) \text{ a.e. in } [0,T] \right\}$$

$$= \int_0^T [b'(t)\tilde{\lambda}_*(t) - u_*'(t)z(t)]dt,\ \forall z \in Z.$$

Thence

$$p(z) = p(0) + \int_0^T [-u_*'(t)]z(t)dt,\ \forall z \in Z.$$

Therefore $-u_* \in \partial p(0)$ [3]. From Proposition 2.1 we have that Z is convex and p is convex in Z. Then follows from Lema 3.2 (page 433) in [26] that there exists a constant $M > 0$ such that

$$p(0) \le p(z) + M\|y\|_1,\ \forall z \in Z.$$

Thus (4) is stable. So in accordance with the Strong Duality Theorem in [26] the associated dual problem to the primal problem in (4) has an optimal solution. The dual problem is given by

$$\left.\begin{array}{l} \text{Maximize } \displaystyle\int_0^T c'(t)\tilde{\eta}(x(t),y(t))dt \\ \text{subject to } A'(t)\tilde{\eta}(x(t),y(t)) \le b(t) \text{ a.e. in } [0,T]. \end{array}\right\} \quad (13)$$

Rewriting, we have

$$\text{Maximize } \int_0^T \eta_{n+2}(x(t),y(t))dt$$

$$\text{subject to } \begin{bmatrix} \nabla f'(y(t),t) & 0 & 1 \\ \nabla g_1'(y(t),t) & g_1(y(t),t) & 0 \\ \vdots & \vdots & \vdots \\ \nabla g_m'(y(t),t) & g_m(y(t),t) & 0 \end{bmatrix} \begin{bmatrix} \eta_1(x(t),y(t)) \\ \vdots \\ \eta_n(x(t),y(t)) \\ \eta_{n+1}(x(t),y(t)) \\ \eta_{n+2}(x(t),y(t)) \end{bmatrix}$$

$$\le \begin{bmatrix} f(x(t),t) - f(y(t),t) \\ g_1(x(t),t) - g_1(y(t),t) \\ \vdots \\ g_m(x(t),t) - g_m(y(t),t) \end{bmatrix} \text{ a.e. in } [0,T].$$

Therefore the problem below has an optimal solution.

[3] $\partial p(0)$ denotes the sub-differential of p at 0.

$$
\begin{aligned}
&\text{Maximize} \int_0^T \eta_{n+2}(x(t), y(t))dt \\
&\text{subject to} \, \nabla f'(y(t), t)\eta(x(t), y(t)) + \eta_{n+2}(x(t), y(t)) \\
&\qquad\qquad \leq f(x(t), t) - f(y(t), t) \text{ a.e. in } [0, T] \qquad (14) \\
&\qquad \nabla g_1'(y(t), t)\eta(x(t), y(t)) + g_1(y(t), t)\eta_{n+1}(x(t), y(t)) \\
&\qquad\qquad \leq g_1(x(t), t) - g_1(y(t), t) \text{ a.e. in } [0, T] \\
&\qquad \vdots \\
&\qquad \nabla g_m'(y(t), t)\eta(x(t), y(t)) + g_m(y(t), t)\eta_{n+1}(x(t), y(t)) \\
&\qquad\qquad \leq g_m(x(t), t) - g_m(y(t), t) \text{ a.e. in } [0, T],
\end{aligned}
$$

where $\eta'(x(t), y(t)) = \begin{bmatrix} \eta_1(x(t), y(t)) & \eta_2(x(t), y(t)) & \cdots & \eta_n(x(t), y(t)) \end{bmatrix}$. For $i \in I(y)$ we have $g_i(y(t), t) = 0$ a.e. in $[0, T]$. So we show that there exists η such that

$$
g_i(x(t), t) - g_i(y(t), t) \geq \nabla g_i'(y(t), t)\eta(x(t), y(t)) \text{ a.e. in } [0, T], \ i \in I(y), \qquad (15)
$$

for all $x \in X$, that is, the active constraints at y are invex functions at $y(t)$ (with respect to V) with the same η.

Theorem 3.2 *Let $y \in \mathbb{F}$ be an optimal solution for Problem (CNP). Suppose that there exists $x \in X$ such that $g_{i_0}(x(t), t) < 0$ a.e. in $[0, T]$ for some $i_0 \in I(y)$. If the Kuhn-Tucker conditions apply at y and all $\lambda \in L_\infty^m[0, T]$ satisfying them are bounded, then the Problem (CNP) is KT-invex at $y(t)$ (with respect to V) with a nontrivial η.*

PROOF. Consider any fixed $x \in \mathbb{F}$. Define $\tilde{\eta} \in L_\infty^{m+1}[0, T], \ c \in L_\infty^{n+2}[0, T]$ and $A \in L_\infty^{(n+2)\times(m+1)}[0, T]$ as in the proof of Theorem 3.1 and $b \in L_\infty^{m+1}[0, T]$, for all $t \in [0, T]$, by

$$
b(t) = \begin{bmatrix} f(x(t), t) - f(y(t), t) \\ -g_1(y(t), t) \\ \vdots \\ -g_m(y(t), t) \end{bmatrix}.
$$

In a similar way as in the proof of Theorem 3.1, we have that

$$
\nabla g_i'(y(t), t)\eta(x(t), y(t)) \leq -g_i(y(t), t) = 0 \text{ a.e. in } [0, T], \ i \in I(y), \qquad (16)
$$

for all $x \in \mathbb{F}$. Besides, the Strong Duality Theorem in [26] says that the optimal values of primal problem in (4) and dual problem in (13) are equal, that is,

$$\begin{aligned}\int_0^T \eta_{n+2}(x(t),y(t))dt &= \int_0^T \left\{[f(x(t),t)-f(y(t),t)] \right.\\ &\quad \left. -\sum_{i\in I}\lambda_i(t)g_i(y(t),t)\right\}dt \\ &= \int_0^T [f(x(t),t)-f(y(t),t)]dt,\end{aligned}$$

where we use the slackness condition (2) in the second equality. Thus from (14) we obtain

$$\begin{aligned}\int_0^T \nabla f'(y(t),t)\eta(x(t),y(t))dt + \int_0^T [f(x(t),t)-f(y(t),t)]dt \\ \leq \int_0^T [f(x(t),t)-f(y(t),t)]dt.\end{aligned}$$

Thence, since y is an optimal solution, we have

$$\begin{aligned}&\int_0^T \nabla f'(y(t),t)\eta(x(t),y(t))dt \\ &\quad \leq \int_0^T \nabla f'(y(t),t)\eta(x(t),y(t))dt + \int_0^T [f(x(t),t)-f(y(t),t)]dt \\ &\quad \leq \int_0^T [f(x(t),t)-f(y(t),t)]dt.\end{aligned}$$

So

$$\phi(x)-\phi(y) \geq \int_0^T \nabla f'(y(t),t)\eta(x(t),y(t))dt. \tag{17}$$

for all $x \in \mathbb{F}$. From (16) and (17) it follows that the Problem (CNP) is KT-invex at $y(t)$ (with respect to V).

Suppose that $\eta(x(t), y(t)) = 0$ a.e. in $[0, T]$ is the only solution of (16). From (15) we have that

$$0 > g_{i_0}(x(t), t) \geq \nabla g'_{i_0}(y(t), t)\eta(x(t), y(t)) \text{ a.e. in } [0, T]$$

has a solution $\eta(x(t), y(t))$. It clear that $\eta(x(t), y(t)) \neq 0$ a.e. in $[0, T]$ and that $\eta(x(t), y(t))$ is also solution of (16). This contradicts what we suppose initially. Therefore $\eta(x(t), y(t)) = 0$ a.e. in $[0, T]$ is not the only solution of (16).

The example below shows that the hypothesis that there exists $x \in X$ such that $g_{i_0}(x(t), t) < 0$ a.e. in $[0, T]$ for some $i_0 \in I(y)$ in general cannot be weakened to guarantee the existence of a nontrivial η.

Example 3.3 *Consider the nonlinear continuous-time programming problem below:*

$$\begin{aligned}
&\textit{Minimize} \int_0^1 \exp(-x_1(t))dt \\
&\textit{subject to} +4[x_1(t)]^2 + [x_2(t)]^2 - 1 \leq 0 \ \textit{a.e. in } [0, T], \\
&\qquad -4[x_1(t)]^2 - [x_2(t)]^2 + 1 \leq 0 \ \textit{a.e. in } [0, T], \\
&\qquad +x_1(t) - x_2(t) \leq 0 \ \textit{a.e. in } [0, T], \\
&\qquad -x_1(t) + x_2(t) \leq 0 \ \textit{a.e. in } [0, T].
\end{aligned}$$

Clearly this problem does not satisfies the hypothesis of Theorem 3.2. It easy to see that $\bar{x}_1(t) = \bar{x}_2(t) = \frac{\sqrt{5}}{5}$ *a.e. in* $[0, T]$ *is the optimal solution of the problem. If the problem is KT-invex at* $(\bar{x}_1, \bar{x}_2)$ *then the system below has a solution:*

$$\begin{aligned}
&\int_0^1 \exp(-x_1(t))dt - \int_0^1 \exp\left(-\frac{\sqrt{5}}{5}\right) dt \geq \int_0^1 -\exp\left(-\frac{\sqrt{5}}{5}\right) \eta_1(t)dt, \\
&0 \geq +8\frac{\sqrt{5}}{5}\eta_1(t) + 2\frac{\sqrt{5}}{5}\eta_2(t) \ \textit{a.e. in } [0, T], \\
&0 \geq -8\frac{\sqrt{5}}{5}\eta_1(t) - 2\frac{\sqrt{5}}{5}\eta_2(t) \ \textit{a.e. in } [0, T], \\
&0 \geq +\eta_1(t) - \eta_2(t) \ \textit{a.e. in } [0, T], \\
&0 \geq -\eta_1(t) + \eta_2(t) \ \textit{a.e. in } [0, T].
\end{aligned}$$

The only solution of this problem is the trivial one

$$\eta_1(t) = \eta_2(t) = 0 \ \textit{a.e. in } [0, T].$$

In the next theorem we have a converse of Theorem 3.1. Furthermore, it shows that invexity is a constraint qualification.

Theorem 3.4 *Suppose that the problem (CNP) has an optimal solution $y \in \mathbb{F}$. If the active constraints at y are invex functions at $y(t)$ (with respect to V) with the same η, then the Kuhn-Tucker conditions apply at y.*

PROOF. Let $x \in \mathbb{F}$. By hypothesis there exists $\eta : V \times V \to \mathbb{R}^n$ such that the function $t \mapsto \eta(x(t), y(t)) \in L^n_\infty[0, T]$ and

$$\nabla g_i'(y(t), t)\eta(x(t), y(t)) \leq g_i(x(t), t) - g_i(y(t), t) \leq 0 \text{ a.e. in } [0, T],$$

for $i \in I(y)$. For $i \notin I(y)$, we have $g_i(y(t), t) < 0$ a.e. in $[0, T]$, so that we can choose a escalar $\eta_{n+1}(x(t), y(t))$ large enough such that

$$\nabla g_i'(y(t), t)\eta(x(t), y(t)) - g_i(x(t), t) + g_i(y(t), t)\eta_{n+1}(x(t), y(t)) < 0 \text{ a.e. in } [0, T],$$

for $i \notin I(y)$. Since $g_i(y(t), t) < 0$ a.e. in $[0, T]$, we have

$$\begin{aligned}\nabla g_i'(y(t), t)\eta(x(t), y(t)) - g_i(x(t), t) + g_i(y(t), t) + g_i(y(t), t)\eta_{n+1}(x(t), y(t))\\ < \nabla g_i'(y(t), t)\eta(x(t), y(t)) - g_i(x(t), t) + g_i(y(t), t)\eta_{n+1}(x(t), y(t)) < 0\end{aligned}$$

a.e. in $[0, T]$. Thus there exists $\eta_{n+1}(x(t), y(t))$ such that

$$\nabla g_i'(y(t), t)\eta(x(t), y(t)) + g_i(y(t), t)\eta_{n+1}(x(t), y(t)) < g_i(x(t), t) - g_i(y(t), t),$$

a.e. in $[0, T]$, for $i \notin I(y)$. Take $\eta_{n+2}(x(t), y(t))$ such that

$$\nabla f'(y(t), t))\eta(x(t), y(t)) + \eta_{n+2}(x(t), y(t)) \leq f(x(t), t) - f(y(t), t) \text{ a.e. in } [0, T].$$

So the linear continuous-time programming problem below has an optimal solution:

$$\text{Maximize} \int_0^T \eta_{n+2}(x(t), y(t))dt$$

$$\text{subject to} \begin{bmatrix} \nabla f'(y(t), t)\eta(x(t), y(t)) + \eta_{n+2}(x(t), y(t)) \\ \nabla g_1'(y(t), t)\eta(x(t), y(t)) + g_1(y(t), t)\eta_{n+1}(x(t), y(t)) \\ \vdots \\ \nabla g_m'(y(t), t)\eta(x(t), y(t)) + g_m(y(t), t)\eta_{n+1}(x(t), y(t)) \end{bmatrix}$$

$$\leq \begin{bmatrix} f(x(t),t) - f(y(t),t) \\ g_1(x(t),t) - g_1(y(t),t) \\ \vdots \\ g_m(x(t),t) - g_m(y(t),t) \end{bmatrix} \text{ a.e. in } [0,T].$$

Using the notation of the proof of Theorem 3.1, it follows that the problem below has an optimal solution:

$$\text{Maximize} \int_0^T c'(t)\tilde{\eta}(x(t),y(t))dt$$
$$\text{subject to } A'(t)\tilde{\eta}(x(t),y(t)) \leq b(t) \text{ a.e. in } [0,T].$$

In an analogous way as in the proof of Theorem 3.1 (just remembering that in this case when we apply the Fritz John Theorem the slackness condition holds), we can show that the problem above is stable. Therefore, by the Strong Duality Theorem in [26], the associated dual problem has an optimal solution. The dual problem is given by

$$\text{Minimize} \int_0^T b'(t)\tilde{\lambda}(t)dt$$
$$\text{subject to } A(t)\tilde{\lambda}(t) = c(t) \text{ a.e. in } [0,T],$$
$$\tilde{\lambda}(t) \geq 0 \text{ a.e. in } [0,T].$$

Rewriting the constraints of this problem we have

$$\begin{bmatrix} \nabla f(y(t),t) & \nabla g_1(y(t),t) & \cdots & \nabla g_m(y(t),t) \\ 0 & g_1(y(t),t) & \cdots & g_m(y(t),t) \\ 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \lambda_0(t) \\ \lambda_1(t) \\ \vdots \\ \lambda_m(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \text{ a.e. in } [0,T],$$

and

$$\lambda_i(t) \geq 0 \text{ a.e. in } [0,T],\ i = 0,1,\ldots,m.$$

Then there exists $\lambda = (\lambda_1,\ldots,\lambda_m) \in L^m_\infty[0,T]$ such that

$$\int_0^T \left[\nabla f'(y(t),t) + \sum_{i\in I} \lambda_i(t)\nabla g_i'(y(t),t)\right] z(t)dt = 0,\ \forall z \in L^n_\infty[0,T],$$
$$\lambda_i(t)g_i(y(t),t) = 0 \text{ a.e. in } [0,T],$$

$$\lambda_i(t) \geq 0 \text{ a.e. in } [0,T],\ i \in I.$$

Thus the Kuhn-Tucker conditions apply at y.

Watching the proofs of Theorems 3.1 and 3.4, we can notice that η and λ are dual variables.

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