Isolating blocks for Morse flows

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Abstract

We present a constructive general procedure to build Morse flows on n-dimensional isolating blocks respecting given dynamical and homological boundary data recorded in abstract Lyapunov semi-graphs. Moreover, we prove a decomposition theorem for handles which, together with a special class of gluings, insures that this construction not only preserves the given ranks of the homology Conley indices, but it also is optimal in the sense that no other Morse flow can preserve this index with fewer singularities.

Introduction

An n-dimensional elementary isolating block can be constructed by considering an $(n-1)$-dimensional surface N^- and an attached handle $h_k = \mathbf{D}^k \times \mathbf{D}^{n-k}$, which contains an index k non-degenerate singularity, attached to the collar $N^- \times [0,1]$. Different attachments may produce non-homeomorphic isolating neighbourhoods with non-homeomorphic boundaries. Any such neighbourhood can be schematically represented by a Lyapunov semi-graph with outgoing edge(s) corresponding to the connected components of $N^- \times [0,1]$, a vertex corresponding to the singularity, while the incoming edge(s) would correspond to the connected components of $N^+ \times [0,1]$, N^+ being the new boundary created by the attachment of h_k to $N^- \times \{1\}$.

An abstract Lyapunov semi-graph has its incoming and outgoing edges labelled only with the Betti numbers of a co-dimension one closed manifold, and its vertices labelled with ranks of the homology Conley indices[1]. In this article we start with the abstract data contained in an abstract Lyapunov semi-graph, and realize them in all their generality. We prove, by giving explicit constructions, that

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[1]Given a manifold M with boundary $\partial M = N^+ \sqcup N^-$, and the flow on M entering through N^+ and exiting throughout N^-, the homology Conley index is the homology of the pair $H_j(M, N^-)$. For details, see [Co].

Main Theorem 1. *Let e^+ and e^- be positive integers. Let $\{(B_j^+ - B_j^-)\}_{j=0}^{n-1}$ be a collection of n integers such that $(B_j^+ - B_j^-) = (B_{n-1-j}^+ - B_{n-1-j}^-)$ for all $j = 0 \ldots n-1$. If $n = 2i+1$, let $(B_i^+ - B_i^- = 0)$ mod 2. Let $\{h_j\}_{j=1}^{n-1}$ be integers satisfying Poincaré-Hopf inequalities[2]. Then there exists an n-dimensional connected manifold M with boundary $\partial M = N^+ \cup N^-$, $N^+ \cap N^- = \emptyset$, and a flow with non-degenerate singularities on M such that*

1. *e^+ is the number of connected components of N^+, the entry boundary of the flow, and e^- is the number of connected components of N^-, the exit boundary of the flow;*
2. *if β_j^+ (respectively β_j^-) denotes the j-th Betti number of N^+ (respectively N^-), then*
$$\beta_j^+ - \beta_j^- = (B_j^+ - B_j^-), \textit{ for all } j = 0 \ldots (n-1);$$
3. *for all $j = 1 \ldots (n-1)$, the rank of the homology Conley index is preserved, that is,*
$$\textit{rank}\,(H_j(M, N^-)) = h_j.$$

The realization we exhibit are "slices" cut off connected sums of generalized tori of dimension n $\sharp_j\ (\sharp^{s_j}\ S^j \times S^{n-j})$ and have boundaries which are themselves connected sums of generalized tori of dimension $(n-1)$. This choice is motivated by the fact that it is the simplest one in terms of the description of the needed gluings, but it is arbitrary. In fact, the result is based on a decomposition theorem (Theorem 2.1) whose interpretation can be adapted to the class of manifolds we wish to work with.

We wish to emphasize the topological meaning of this decomposition theorem: it classifies handles into those which affect the topology of the boundary and those which might increase the topological complexity of the block. It is important to underline that in our construction each handle contributes essentially to the topology of the block, which is reflected in the Conley homology index. This constraint is made explicit in item 3 of our main theorem.

This is one of the main differences between these results and those in [CrMRez], where the focus is on the global realization of an abstract Lyapunov graph of Morse type on a closed manifold without the concern of preserving the Conley homology index. Hence, not only the techniques but also the questions which are addressed differ completely in nature.

Also item 2 deserves some comments. In our realizations we preserve the given differences $\{(B_j^+ - B_j^-)\}_{j=0}^{n-1}$. In general an arbitrary choice of Betti numbers of the edges, respecting the differences, doesn't always correspond to a realization. There are two reasons for this phenomenon. The first one is intrinsic to manifolds of dimension $(n = 0) \mod 4$. For instance the graph in dimension $n = 20$ with one vertex labelled with $h_{10} = 1$, having only one incoming edge and one outgoing edge, both labelled with zero Betti numbers, excepted for $\beta_0 = \beta_{n-1} = 1$, cannot be realized. However, with an appropriate choice of Betti numbers satisfying the same difference, this semi-graph is realized. We'll discuss both situations in Subsection 3.4.

The second reason is due to our choice of gluings preserving the ranks of the homology Conley indices. Nevertheless, if we were to weaken our theorem by not being concerned in preserving the ranks of the homology Conley indices, it is trivial to see from our proof that one could realize arbitrary Betti numbers, satisfying the differences, up to the natural topological restrictions above. In any case, it should be noted that even fixing the Betti numbers of the boundary doesn't mean fixing the boundary.

The manifolds we chose to work with are torsion-free and hence computing homology with $\mathbf{Z}$ or $\mathbf{Z}_2$ coefficients makes no difference. A finer analysis is needed to distinguish manifolds associated with the same Lyapunov semi-graph whose homologies have the same ranks but different torsions.

Last, our construction allows us to answer a question asked in [BeMRez2] concerning a geometric interpretation of some homological data.

[2]This assumption is natural, it guarantees the compatibility of the h_j's with the previous data. Such inequalities can be found in [BeMRez1].

The paper is organized as follows. In the first section we give background material. In the second we prove our decomposition theorem, which motivates the choice of the gluings we use, which we describe in the third section. In the fourth section we prove our main theorem by exhibiting the construction of our isolating blocks. We conclude with final remarks.

1 Background

In this section we give some basic results we shall need in the sequel. The first subsection contains fundamental results in handle theory. The second deals with preliminaries in Lyapunov graph theory.

1.1 Classical handle theory

By completeness, we briefly expose here some known features in Topology and take the occasion to introduce our notation and viewpoint. More details on the subject can be easily found in the classical literature or for instance in [L].

By definition, an n-dimensional *handle* of index k is a product of disks centered at the origin $\mathbf{D}^k \times \mathbf{D}^{n-k}$. The *core* is $\mathbf{D}^k \times 0$ and the *cocore* is $0 \times \mathbf{D}^{n-k}$. We define a flow on the handle by considering a non-degenerate index k singularity at the origin and identifying the core and cocore with its unstable and stable manifold respectively. The *attaching region* of the handle is $\mathbf{S}^{k-1} \times \mathbf{D}^{n-k}$ and corresponds to the part of the boundary of the handle through which the flow exits. Similarly, the *belt region* $\mathbf{D}^k \times \mathbf{S}^{n-k-1}$ corresponds to the part of the boundary of the handle through which the flow enters.

1.1.1 Cancellation lemma

In the following proposition, known as the Cancellation Lemma, it is shown that two handles of consecutive indices can be glued to a ball in such a way that they can be cancelled.

Proposition 1.1. *Let $\mathbf{D}^k$ denote the k-dimensional disk and $D^k_{+\ (resp.\ -)}$ denote the upper (resp. lower) hemisphere of the k-dimensional sphere $S^k = D^k_+ \cup D^k_-$. Then, for all integers $n \geq 2$ the following items hold:*

1. *for all $q = 0 \ldots n$, $\mathbf{D}^n$ admits the q-decomposition below*

$$\mathbf{D}^n = (S^{q-1} \times \mathbf{D}^{n+1-q}) \cup_{S^{q-1} \times D^{n-q}_+} (\mathbf{D}^q \times D^{n-q}_+)$$

2. *in particular, for all $q = 1 \ldots n$ a q-decomposition of D^n can be obtained by gluing to $\mathbf{D}^n$ a $(q-1)$-handle and a q-handle, both of dimension n, in a canonical way.*

Proof: By induction on n, the dimension. For $n = 2$ the proposition is trivial. Assuming the proposition is true for $n-1$, we show it is true for n.

- The 0-decomposition of $\mathbf{D}^n$ is straightforward:

$$\mathbf{D}^n = (S^{-1} \times \mathbf{D}^{n+1}) \cup_{S^{-1} \times D^n_+} (\mathbf{D}^0 \times D^n_+) = \emptyset \cup_\emptyset (\mathbf{D}^0 \times D^n_+)$$

- We show the q-decomposition of $\mathbf{D}^n$ for all $q = 1 \ldots n$. By induction, the q-decomposition

$$\begin{aligned}\mathbf{D}^{n-1} &= (S^{q-1} \times \mathbf{D}^{n-q}) \cup_{S^{q-1} \times D^{n-q-1}_+} (\mathbf{D}^q \times D^{n-q-1}_+) \\ &= (\mathbf{D}^{n-1} \cup_{S^{q-1} \times \mathbf{D}^{n-1-q}} (\mathbf{D}^q \times \mathbf{D}^{n-1-q})) \cup_{S^{q-1} \times D^{n-q-1}_+} (\mathbf{D}^q \times D^{n-q-1}_+)\end{aligned}$$

is obtained by consecutively gluing in a given way a $(q-1)$-handle h_{q-1} and a q-handle h_q, both of dimension $n-1$, to $\mathbf{D}^{n-1}$. Then, by construction,

$$\begin{aligned}\mathbf{D}^n = \mathbf{D}^{n-1} \times D \quad &= (S^{q-1} \times \mathbf{D}^{n-q} \times D) \cup_{S^{q-1} \times D_+^{n-q-1} \times D} (\mathbf{D}^q \times D_+^{n-q-1} \times D) = \\ &= (S^{q-1} \times \mathbf{D}^{n-q+1}) \cup_{S^{q-1} \times D_+^{n-q}} (\mathbf{D}^q \times D_+^{n-q}) \\ &= (\mathbf{D}^n \cup_{S^{q-1} \times \mathbf{D}^{n-q}} (\mathbf{D}^q \times \mathbf{D}^{n-q})) \cup_{S^{q-1} \times D_+^{n-q}} (\mathbf{D}^q \times D_+^{n-q})\end{aligned}$$

is a q-decomposition of $\mathbf{D}^n$ obtained by consecutively gluing to $\mathbf{D}^n$ a $(q-1)$-handle $\tilde{h}_{q-1} = h_{q-1} \times \mathbf{D}$ and a q-handle $\tilde{h}_q = h_q \times \mathbf{D}$, both handles being of dimension n.

$\triangle$

1.1.2 Dual handles

In the following proposition the classical notion of dual handle is introduced.

Proposition 1.2. *Given an $(n-1)$-dimensional manifold N, let M be the n-manifold obtained by attaching a q-handle h_q to $N \times [0,1]$, the collar of N. The boundary of M is the disjoint union of two components $N_0 = N \times \{0\}$ and N_1. Then it is possible to obtain M by gluing to the collar of N_1 a $(n-q)$-handle h_{n-q}.*

The $(n-q)$-handle h_{n-q} in the proposition will be called the *dual handle* of the q-handle h_q. The following example illustrates the concept. Let $n = 3$, $N = S^2$ and $q = 1$. Then $N_0 = S^2$, $N_1 = S^1 \times S^1$ and

$$M = (\mathbf{D}^2 \times S^1) \setminus \mathbf{D}^3$$

Of course, one can start with $S^1 \times S^1$ and glue a 2-handle to the collar of $S^1 \times S^1$ in order to obtain the same manifold M, as shown in Figure 1 below.

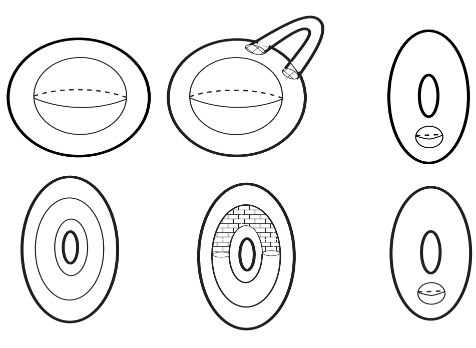

Figure 1: Example of dual handles in dimension 3

Proof: The general idea of the proof is to find the dual $(n-q)$-handle roughly speaking inside $h_q \subset M$. More precisely, let $\tilde{N} = N \setminus (S^{q-1} \times \mathbf{D}^{n-q})$. Hence, by construction

$$N \times [0,1] = (\tilde{N} \times [0,1]) \cup_{S^{q-1} \times S^{n-q-1} \times [0,1]} (S^{q-1} \times \mathbf{D}^{n-q} \times [0,1])$$

After gluing the q-handle to the collar of N in order to obtain the manifold M, we have

$$M = \overbrace{(\tilde{N} \times [0,1]) \cup_{S^{q-1} \times \mathbf{D}^{n-q} \times [0,1]} \mathrlap{\underbrace{\phantom{(S^{q-1} \times \mathbf{D}^{n-q} \times [0,1]) \cup_{S^{q-1} \times \mathbf{D}^{n-q} \times \{1\}} (\mathbf{D}^q \times \mathbf{D}^{n-q})}}_{T}}(S^{q-1} \times \mathbf{D}^{n-q} \times [0,1])}^{N \times [0,1]} \cup_{S^{q-1} \times \mathbf{D}^{n-q} \times \{1\}} \overbrace{(\mathbf{D}^q \times \mathbf{D}^{n-q})}^{h_q}$$

which we shall briefly denote by $M = (\tilde{N} \times [0,1]) \cup_f T$, f being the associated identification map. Let us now study T. Since $(S^{q-1} \times [0,1]) \cup_{S^{q-1} \times \{1\}} \mathbf{D}^q$ is obviously homeomorphic to $\mathbf{D}^q$, we have $T = \mathbf{D}^q \times \mathbf{D}^{n-q}$. Moreover, let $\tilde{\mathbf{D}}^q = [\frac{1}{4}, \frac{3}{4}]^q \subset [0,1]^q = \mathbf{D}^q$. Then the set $\tilde{\mathbf{D}}^q \times \mathbf{D}^{n-q}$ is the $(n-q)$-handle inside T we were looking for (see Figure 2).

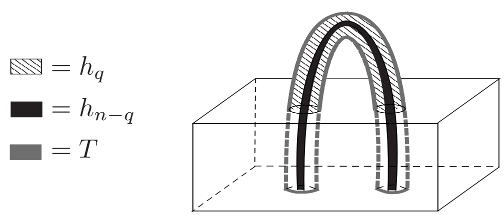

Figure 2: Mutual position of the dual handles

This way, since $(S^{n-q-1} \times [0,1]) \cup_{S^{n-q-1} \times \{1\}} \mathbf{D}^{n-q}$ is obviously homeomorphic to $\mathbf{D}^{n-q}$, we can decompose T as

$$T = (\mathbf{D}^q \times S^{n-q-1} \times [0,1]) \cup_{\mathbf{D}^q \times S^{n-q-1} \times \{1\}} \overbrace{(\mathbf{D}^q \times \mathbf{D}^{n-q})}^{h_{n-q}}$$

Moreover, by construction,

$$N_1 = \tilde{N} \cup_{\mathbf{S}^{q-1} \times S^{n-q-1}} (\mathbf{D}^q \times S^{n-q-1})$$

thus we are done since the last decompositions of T and N_1 imply that

$$M = \overbrace{(\tilde{N} \times [0,1]) \cup_f \mathrlap{\underbrace{\phantom{(\mathbf{D}^q \times S^{n-q-1} \times [0,1]) \cup_{\mathbf{D}^q \times S^{n-q-1} \times \{1\}} (\mathbf{D}^q \times \mathbf{D}^{n-q})}}_{T}}(\mathbf{D}^q \times S^{n-q-1} \times [0,1])}^{N_1 \times [0,1]} \cup_{\mathbf{D}^q \times S^{n-q-1} \times \{1\}} \overbrace{(\mathbf{D}^q \times \mathbf{D}^{n-q})}^{h_{n-q}}$$

that is, M can be obtained by attaching an $(n-q)$-handle to the collar of N_1. $\triangle$

1.1.3 Handle decomposition of some projective spaces

In this subsection we summarize known results on some projective spaces, namely,

- the complex projective spaces $\mathbf{CP}^{2k}$ (of dimension $n = 4k$);

– the Hamiltonian projective spaces $\mathbf{HP}^{2k}$ (of dimension $n = 8k$) obtained by replacing the commutative field $\mathbf{C}$ by the non-commutative field of the quaternions[3], denoted by $\mathbf{H}$;

– the Cayley projective space $\mathbf{OP}^{2k}$ (of dimension $n = 16k$) obtained by replacing the commutative field $\mathbf{C}$ by the non-associative algebra of the octonions[4], denoted by $\mathbf{O}$.

As for the complex projective spaces $\mathbf{CP}^{2k}$, it is shown for instance in [Hu]–III.4 and in [GrHa]–II.19, that the homology of such spaces is non-zero only for even indices. Moreover, the construction of such spaces is shown in terms of cellular attachment via the Morse function defined on $\mathbf{CP}^{2k}$

$$f : [z_0 : z_1 : \ldots : z_{2k}] \to \sum_{j=0}^{2k} c_j \mid z_j \mid^2 \qquad c_j\text{'s all different.}$$

The cellular decomposition can be translated in terms of handle decomposition ([Sm1] and [Sm2]) in such a way that the index of the cell corresponds exactly to the index of the associated handle. In particular we have that $\mathbf{CP}^2$ can be decomposed into the three handles: h_0, h_2 and h_4. The boundary to which h_2 is attached is S^3. The boundary after the attachment is again S^3, and the attachment of the handle generates the second homology group $H_2(\mathbf{CP}^2) = \mathbf{Z}$. More generally, we have that $\mathbf{CP}^{2k}$ can be decomposed into the $(2k + 1)$ handles: $(h_0, h_2, h_4, h_6, \ldots, h_{4k})$.

Analogous results are true for the Hamiltonian projective spaces $\mathbf{HP}^{2k}$. The homology of such spaces is non-zero only for (all) indices which are multiple of 4. In particular we have that $\mathbf{HP}^2$ can be decomposed into the three handles: h_0, h_4 and h_8. The boundary to which h_4 is attached is S^7. The boundary after the attachment is again S^7, and the attachment of the handle generates the fourth homology group $H_4(\mathbf{HP}^2) = \mathbf{Z}$. More generally, we have that $\mathbf{HP}^{2k}$ can be decomposed into the $(2k + 1)$ handles: $(h_0, h_4, h_8, \ldots, h_{8k})$.

Finally, the homology of the Cayley projective spaces $\mathbf{OP}^{2k}$ is non-zero only for indices which are multiple of 8. In particular we have that $\mathbf{OP}^2$ can be decomposed into the three handles: h_0, h_8 and h_{16}. The boundary to which h_8 is attached is S^{15}. The boundary after the attachment is again S^{15}, and the attachment of the handle generates the eighth homology group $H_8(\mathbf{OP}^2) = \mathbf{Z}$. More generally, we have that $\mathbf{OP}^{2k}$ can be decomposed into the $(2k + 1)$ handles: $(h_0, h_8, h_{16}, \ldots, h_{16k})$.

For more details on such projective spaces, see [Ba], and [St]. In [Ba] it is also shown that no other orientable manifold can be seen as a projective space.

1.2 Abstract Lyapunov graphs

The motivation of what follows comes from Frank's idea of *Lyapunov graphs.* In [F] he associates with a continuous flow on a closed manifold and a Lyapunov function on it, the quotient space

$$M/\sim \quad \text{where } x \sim y \Longleftrightarrow x \text{ and } y \text{ are in the same connected component of the level set}$$

and sees such a quotient as a graph L according to the rule that a point of $M/\sim$ is a vertex if and only if it is the equivalence class of a chain recurrent component. Hence all the other points

[3] The quaternions are a 4-dimensional non-commutative field with basis $1, i, j, k$ and their multiplication is given by the following rules:
$i^2 = j^2 = k^2 = -1$;
index cycling identities: $ij = k$, $jk = i$, $ki = j$;
the elements of the basis anticommute : $ij = -ji$, $ik = -ki$, $jk = -kj$.

[4] The octonions are an 8-dimensional non-associative algebra with basis $1, e_1, \ldots, e_7$ and their multiplication is given by the following rules:
$e_1, \ldots, e_7$ are square roots of -1;
$e_1e_2 = e_4$;
e_i and e_j anticommute when $i \neq j$: $e_ie_j = -e_je_i$;
index cycling identities: $e_ie_j = e_k \Rightarrow e_{i+1}e_{j+1} = e_{k+1}$;
index doubling: $e_ie_j = e_k \implies e_{2i}e_{2j} = e_{2k}$.

are edge points of L. Moreover, L can be oriented according to the orientation of the flow. Note that, in order for L to have a finite number of vertices, we have to consider only flows admitting a finite number of chain recurrent components.
One can do the same process when the underlying manifold has some boundary. In this case, we obtain a Lyapunov semi-graph.
The second general idea, in [Rez] and [CrRez], is to enrich a Lyapunov graph with labels concerning the topology of the initial manifold or the original dynamics. Following [Rez] and [CrRez], here we choose to label the graph with some homological information: its vertices are labelled with the ranks of the *homology Conley indices*, and its edges are labelled with the Betti numbers of any of the level sets associated with the given edge.

Of course, one can take the opposite point of view by defining a Lyapunov graph in an abstract way and this is what we start from.

Definition 1.3. *An abstract Lyapunov graph (semi-graph)*[5] *is an oriented graph (semi-graph) with no oriented cycles such that each vertex v is labelled with a list of non-negative integers $\{h_0(v) = k_0, \dots, h_n(v) = k_n\}$. Also, the labels on each edge $\{\beta_0 = 1, \beta_1, \dots, \beta_{n-2}, \beta_{n-1} = 1\}$ must be a collection of non-negative integers satisfying the Poincaré duality (i.e. $\beta_j = \beta_{n-j-1}$ for all j's) and if $n = 2i$ then β_i must be even.*

In this abstract setting, a natural question is whether, given an abstract Lyapunov semi-graph, there exist a manifold, a continuous flow on it and a Lyapunov function, such that the associated Lyapunov semi-graph is the given one. Shortly we speak of the *realization* of an abstract Lyapunov semi-graph.

1.2.1 Continuation results

In this section we want to introduce a special class of abstract Lyapunov semi-graphs, that is, those which can be related to Morse flows on manifolds. In this particular case the singularities of the flow are points, hence the corresponding vertices are labelled with $\{h_j(v) = 1\}$, where j is the dimension of the unstable manifold of this isolated singularity. Passing through a vertex along the opposite orientation of the graph corresponds to attaching a handle of index given by the label of the vertex. As for the boundary, attaching a handle of index j ($j = 1 \dots n-1$) can have one of the following effects:

1. the j-th Betti number of the boundary is increased by 1 (or by 2, if $n = 2j + 1$), and the handle will be said of type j-d (d standing for disconnecting);

2. the $(j-1)$-th Betti number of the boundary is decreased by 1 (or by 2, if $n = 2j - 1$), and the handle will be said of type $(j-1)$-c (c standing for connecting);

3. if $n = 4k$ and $j = 2k$ all the Betti numbers are kept unchanged, and the handle will be said of type β-i (i standing for invariant).

Therefore, we have the following natural abstract definition:

Definition 1.4. *An abstract Lyapunov graph (semi-graph) of Morse type is an abstract Lyapunov graph (semi-graph) that satisfies the following:*

1. *every vertex is labelled with $h_j = 1$ for some $j = 0, \dots, n$.*

2. *the number of incoming edges, e^+, and the number of outgoing edges, e^-, of a vertex must satisfy:*

[5] Given a finite set V we define a *directed semi-graph* $G' = (V', E')$ as a pair of sets $V' = V \cup \{\infty\}$, $E' \subset V' \times V'$. As usual, we call the elements of V' vertices and since we regard the elements of E' as ordered pairs, these are called *directed edges*. Furthermore the edges of the form (∞, v) and (v, ∞) are called *semi-edges* (or dangling edges as in [Rez]). Note that whenever G' does not contain semi-edges, G' is a graph in the usual sense. The graphical representation of the graph will have the semi-edges cut short.

(a) *if* $h_j = 1$ *and* $j \notin \{0, 1, n-1, n\}$ *then* $e^+ = 1$ *and* $e^- = 1$;

(b) *if* $h_1 = 1$ *then* $e^+ = 1$ *and* $0 < e^- \leq 2$; *if* $h_{n-1} = 1$ *then* $e^- = 1$ *and* $0 < e^+ \leq 2$;

(c) *if* $h_0 = 1$ *then* $e^- = 0$ *and* $e^+ = 1$; *if* $h_n = 1$ *then* $e^+ = 0$ *and* $e^- = 1$.

3. *every vertex labelled with* $h_\ell = 1$ *must be of type* ℓ*-d or* $(\ell - 1)$*-c. Furthermore if* $n = 2i$, $(n = 0) \mod 4$ *and* $h_i = 1$, *then* v *may be labelled with* β*-i.*

It was proved in [BeMRez1] that if an abstract Lyapunov semi-graph satisfies the Poincaré-Hopf inequalities, then it can be continued to a Lyapunov semi-graph of Morse type. This means that any vertex of the initial abstract Lyapunov semi-graph L can be replaced by a Lyapunov semi-graph of Morse type L_M, satisfying the same Betti numbers on the e^+ and e^- incoming and outgoing (dangling) edges, and such that the k-th ranks of the Conley homology indices in L are equal to the number of singularities of index k in L_M. Moreover, an algorithm finding all possible continuations of a given graph is described. Observe that this algebraic approach of continuation has a dynamical counterpart in [Rei].

1.2.2 Minimal number of singularities

Another approach toward the realizability of an abstract Lyapunov semi-graph consists in fixing a priori only the homological boundary data and ask what is the minimal number of singularities needed in order to make the homological gaps between the two boundaries vanish. In other words, we want some information about what are the manifolds, if any, with least homology whose boundary satisfies the given data. More precisely,

Definition 1.5. *Given positive integers* e^+ *and* e^-, *and* n *integers* $\{(B_j^+ - B_j^-)\}_{j=0}^{n-1}$ *such that* $(B_j^+ - B_j^-) = (B_{n-1-j}^+ - B_{n-1-j}^-)$ *for all* $j = 0 \ldots n-1$, *and* $(B_i^+ - B_i^- = 0) \mod 2$ *if* $n = 2i+1$, *we say that an* n*-dimensional manifold* M *with boundary* $\partial M = N^+ \cup N^-$ *such that* $N^+ \cap N^- = \emptyset$ *satisfies the given (homological) boundary conditions if* e^+ *is the number of connected components of* N^+, e^- *is the number of connected components of* N^- *and* $(B_j^+ - B_j^-)$ *is the difference of the* j*-th Betti numbers of the boundary components, that is,* $(B_j^+ - B_j^-) = \mathrm{rank}(H_j(N^+)) - \mathrm{rank}(H_j(N^-))$.

In [BeRezVa] it is proved that the loose information about the boundary suffices to determine the abstract minimal number of singularities that must be present in any realization, as well as their indices and types (connecting and disconnecting).

Theorem 1.6. *Let* e^+ *and* e^- *be positive integers. Let* $\{(B_j^+ - B_j^-)\}_{j=0}^{n-1}$ *be integers such that* $(B_j^+ - B_j^-) = (B_{n-1-j}^+ - B_{n-1-j}^-)$ *for all* $j = 0 \ldots n-1$. *If* $n = 2i+1$, *let* $(B_i^+ - B_i^- = 0) \mod 2$. *Then any flow on any* n*-dimensional manifold* M *satisfying the given homological boundary conditions must have at least* $h_{\min}$ *singularities, where*

$$h_{\min} = \begin{cases} e^+ + e^- - 2 + \sum_{j=1}^{i-1} \mid B_j^+ - B_j^- \mid + \left| \dfrac{B_i^+ - B_i^-}{2} \right| & \text{if } n = 2i+1; \\ e^+ + e^- - 2 + \sum_{j=1}^{i} \mid B_j^+ - B_j^- \mid & \text{if } n = 2i. \end{cases}$$

Moreover, such $h_{\min}$ *singularities are of the following indices and types. Let* h_j^d *denote the number of singularities of index* j *and type* j*-d, and let* h_j^c *denote the number of singularities of index* j *and type* $(j-1)$*-c.*

- *We have* $h_1^c = (e^- - 1)$ *and* $h_{n-1}^d = (e^+ - 1)$.
- *For* $j = 1 \ldots \lfloor \frac{n}{2} \rfloor - (n \mod 2)$, *let* k_j *be any integer in* $0 \ldots \mid B_j^+ - B_j^- \mid$:
  if $B_j^+ \geq B_j^-$ *then we have* $h_j^d = k_j$ *and* $h_{n-j-1}^d = (\mid B_j^+ - B_j^- \mid - k_j)$, *else we have* $h_{j+1}^c = k_j$ *and* $h_{n-j}^c = (\mid B_j^+ - B_j^- \mid - k_j)$.

* If $n = 2i+1$, then either $B_i^+ \geq B_i^-$ and we have $h_i^d = \frac{|B_i^+ - B_i^-|}{2}$, or $B_i^+ < B_i^-$ and we have $h_{i+1}^c = \frac{|B_i^+ - B_i^-|}{2}$.

Of course, one knows the indices of the singularities realizing $h_{\min}$ from the above theorem just by forgetting about their types (connecting and disconnecting). Note that the converse is also true: if we apply the algorithm of continuation to any of the admissible lists of h_j's realizing $h_{\min}$, then the associated continuation is unique, that is, the list of the indices uniquely determines the corresponding types.

In this paper we shall construct a Morse flow on an n-dimensional manifold M with boundary satisfying the given homological boundary data and with exactly $h_{\min}$ singularities (Proposition 4.1).

2 Combinatorics of the general vertex label

The fundamental result which allows us to guarantee that we shall be able to build models for our realizations within a given class of manifolds lies on the combinatorial nature of our reference settings.

Theorem 2.1. *Let v be a vertex of a Lyapunov semi-graph. Let $\underline{h} \in \mathbf{R}^{n-1}$ denote its label, with the convention that the j-th coordinate corresponds to the value of h_j. Then $\underline{h}$ is compatible with the boundary conditions if and only if it can be decomposed as*

$$\underline{h} = \underline{h}_{\min} + \underline{h}_{\text{consecutive}} + \underline{h}_{\text{dual}} + \underline{h}_{\text{invariant}}$$

where

$\underline{h}_{\min}$ *is one of the labels associated with $h_{\min}$ and the boundary conditions (theorem 1.6);*

$\underline{h}_{\text{consecutive}}$ *is a vector corresponding to a collection of couples (h_j, h_{j+1}) with adjacent indices (necessarily of types (j-d, j-c));*

$\underline{h}_{\text{dual}}$ *is a vector corresponding to a collection of couples (h_j, h_{n-j}) with dual indices (either of types ($(j-1)$-c, $(n-j)$-d) or of types (j-d, $(n-j-1)$-c);*

$\underline{h}_{\text{invariant}}$ *is a vector which may be non-zero only in dimension $n = 4k$, corresponding to a collection of middle dimension h_{2k}'s of type β-i.*

Proof: Since $\underline{h}$ is compatible with the boundary conditions (i.e, satisfies the Poincaré-Hopf inequalities), the associated graph can be continued to an abstract graph of Morse type. Among the vertices of any continuation, there are $h_{\min}$ of them which are labelled in such a way that their total effect on the Betti numbers is to make the difference of the boundary Betti numbers vanish. The vector associated with these $h_{\min}$ vertices is the vector $\underline{h}_{\min}$ of the decomposition. Next, all the other vertices together must have no total effect on the difference of the Betti numbers that is, either they are β-invariant (and this is possible only for the middle index $j = 2k$ in dimension $n = 4k$), or each variation of one of the Betti numbers caused by a vertex of a given index and type must be cancelled by a vertex of the appropriate index and type. This last situation corresponds to the possibility of pairing up vertices either according to the rule (h_j^d, h_{j+1}^c), which will be taken into account in $\underline{h}_{\text{consecutive}}$, or according to the rules (h_j^c, h_{n-j}^d) and (h_j^d, h_{n-j}^c), which will be taken into account in $\underline{h}_{\text{dual}}$. △

The decomposition of the above proposition is in general not unique. Consider the following example in dimension $n = 5$ (see Figure 3 below, where it is understood that for every edge we have $\beta_0 = \beta_4 = 1$ and $\beta_3 = \beta_1$). In this case $e^+ = 2$, $e^- = 3$, $(B_1^+ - B_1^-) = (1+2) - (0+0+1) = 2$, $(B_2^+ - B_2^-) = (0+10) - (2+2+4) = 2$ and $\underline{h} = (3, 2, 3, 2)$.

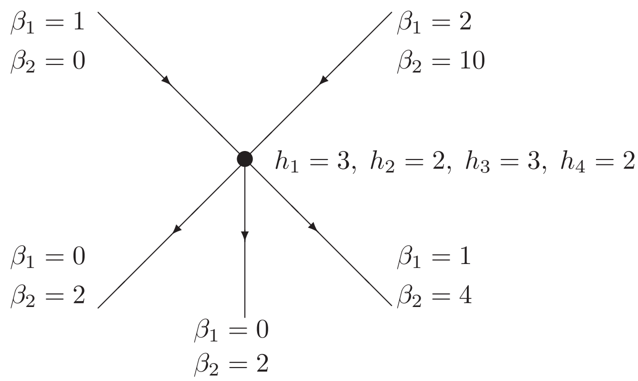

Figure 3: A vertex

For these data we have $h_{\min} = 6$ and three vectors realizing it:

$$\begin{cases} \underline{h}^{(0)}_{\min} = (2,1,2,1) & \text{corresponding to } \{h^c_1 = 2, h^d_2 = 1, h^d_3 = 2, h^d_4 = 1\}; \\ \underline{h}^{(1)}_{\min} = (3,1,1,1) & \text{corresponding to } \{h^c_1 = 2, h^d_1 = 1, h^d_2 = 1, h^d_3 = 1, h^d_4 = 1\}; \\ \underline{h}^{(2)}_{\min} = (4,1,0,1) & \text{corresponding to } \{h^c_1 = 2, h^d_1 = 2, h^d_2 = 1, h^d_4 = 1\}. \end{cases}$$

The semi-graph of Morse type shown in Figure 4 represents one of the possible continuations of the vertex we are studying. By considering only this special continuation it is easy to verify that the label $\underline{h}$ admits at least the three following decompositions:

$$\begin{aligned} \underline{h} = (3,2,3,2) &= \underbrace{\overbrace{(2,1,2,1)}^{(0-\mathrm{c},\,0-\mathrm{c},\,2-\mathrm{d},\,3-\mathrm{d},\,3-\mathrm{d},\,4-\mathrm{d})}}_{\underline{h}^{(0)}_{\min}} + \underbrace{\overbrace{(1,0,0,1)}^{(1-\mathrm{d},\,3-\mathrm{c})} + \overbrace{(0,1,1,0)}^{(2-\mathrm{d},\,2-\mathrm{c})}}_{\underline{h}_{\mathrm{dual}}} \\ &= \underbrace{\overbrace{(2,1,2,1)}^{(0-\mathrm{c},\,0-\mathrm{c},\,2-\mathrm{d},\,3-\mathrm{d},\,3-\mathrm{d},\,4-\mathrm{d})}}_{\underline{h}^{(0)}_{\min}} + \underbrace{\overbrace{(0,1,1,0)}^{(2-\mathrm{d},\,2-\mathrm{c})}}_{\underline{h}_{\mathrm{consecutive}}} + \underbrace{\overbrace{(1,0,0,1)}^{(1-\mathrm{d},\,3-\mathrm{c})}}_{\underline{h}_{\mathrm{dual}}} \\ &= \underbrace{\overbrace{(3,1,1,1)}^{(0-\mathrm{c},\,0-\mathrm{c},\,1-\mathrm{d},\,2-\mathrm{d},\,3-\mathrm{d},\,4-\mathrm{d})}}_{\underline{h}^{(1)}_{\min}} + \underbrace{\overbrace{(0,1,1,0)}^{(2-\mathrm{d},\,2-\mathrm{c})} + \overbrace{(0,0,1,1)}^{(3-\mathrm{d},\,3-\mathrm{c})}}_{\underline{h}_{\mathrm{consecutive}}} \end{aligned}$$

3 Choice of standard gluings

In this section we define the gluings which we allow for the construction of our models and we emphasize the changes produced by these gluings on the boundary and inside the manifold. Note that these choices of gluings are consistent and general because of the decomposition theorem of the previous section.

We keep the same notation: in particular an n-dimensional handle of index q will always be denoted by h_q. Moreover, in what follows, the n-dimensional manifold obtained after step i will be denoted by M_i and its modified boundary by N_i. Since we shall describe the changes produced by the gluings we chose, notice that $\natural$ will denote the connected sum along the boundary, while $\sharp$ will denote the connected sum.

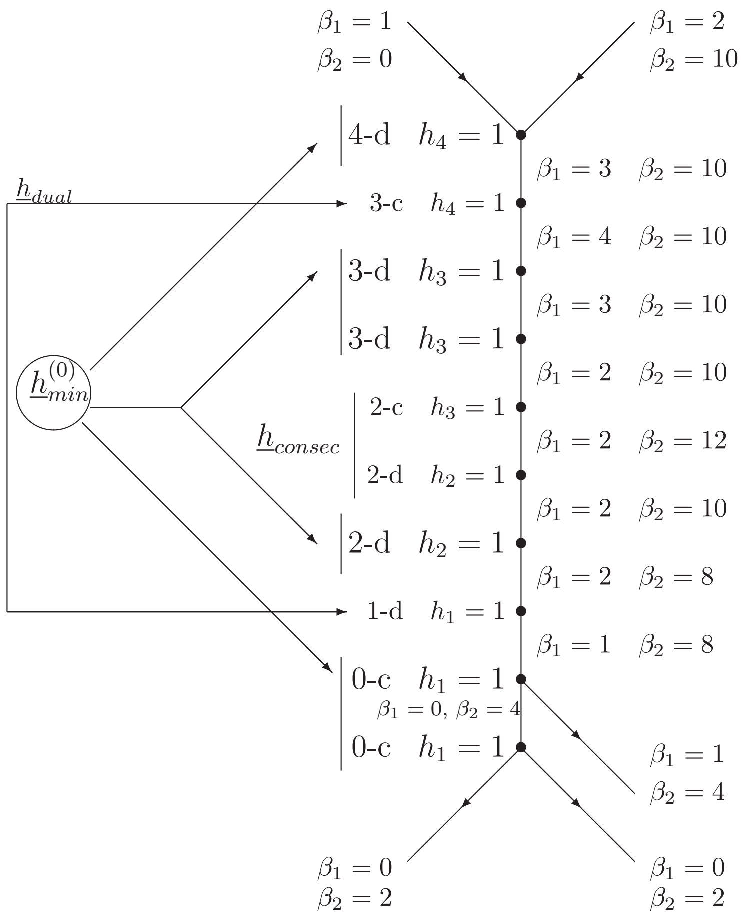

Figure 4: A possible continuation of the vertex of Figure 3

3.1 Trivial gluing

Let us start from M_0 and N_0. A *trivial gluing* is a way of attaching a handle of index q in order to create a q-handlebody. This gluing corresponds to the first step of the 2-step construction detailed in Proposition 1.1. Let B_0 be an n-dimensional ball. After gluing the q-handle to the upper hemisphere of its boundary we obtain the q-handlebody H_q defined by

$$H_q = \underbrace{(\mathbf{D}^q \times \mathbf{D}^{n-q})}_{B_0} \cup_{S^{q-1}\times\mathbf{D}^{n-q}} \underbrace{(\mathbf{D}^q \times \mathbf{D}^{n-q})}_{h_q} = S^q \times \mathbf{D}^{n-q}$$

From the point of view of M_0 we have:

$$\begin{aligned} M_1 &\stackrel{\text{def}}{=} M_0 \cup_{S^{q-1}\times\mathbf{D}^{n-q}} h_q \\ &= M_0 \natural H_q \end{aligned}$$

(to see the connected sum, take an $(n-1)$-dimensional disk in N_0, identify it to the lower hemisphere of B_0, then remove the interior of B_0 and identify the two hemispheres of its boundary). From the point of view of N_0 we have:

$$\begin{aligned} N_1 &\stackrel{\text{def}}{=} N_0 \setminus \underbrace{(S^{q-1} \times \mathbf{D}^{n-q})}_{\text{attaching region of } h_q} \cup_{S^{q-1}\times S^{n-q-1}} \underbrace{(\mathbf{D}^q \times S^{n-q-1})}_{\text{belt region of } h_q} \\ &= N_0 \sharp \partial H_q = N_0 \sharp (S^q \times S^{n-q-1}) \end{aligned}$$

Note that the effect of the trivial gluing on the Betti numbers of the boundary is that only the q-th Betti number β_q and its dual β_{n-q-1} have changed by being increased by 1. For this reason, the trivial gluing of h_q is of type q-d.

3.2 Null gluing

Let us start from M_0 and N_0. A *null gluing* concerns two handles of consecutive indices, say, q and $q+1$. It is an application of the 2-steps construction detailed in Proposition 1.1:

1. Trivial gluing (Subsection 3.1) of the q-handle h_q on a trivial disc $\mathbf{D}^{n-1}$ of the boundary N_0. We get:
$$\begin{cases} M_1 = & M_0 \natural H_q \\ N_1 = & N_0 \sharp \partial H_q = N_0 \sharp (S^q \times S^{n-q-1}) \end{cases}$$

2. According to Proposition 1.1, we glue h_{q+1} to H_q in order to obtain an n-dimensional disk. Note that during this process the lower hemisphere of B_0 of the previous step (same notation as in Subsection 3.1) is never modified. Hence, in the same way as before, from the point of view of M_2 we have:
$$\begin{aligned} M_2 \overset{\text{def}}{=} & \; M_1 \cup^{(\text{null})}_{S^q \times \mathbf{D}^{n-q-1}} h_{q+1} = \\ = & \; M_0 \natural \underbrace{(H_q \cup^{(\text{null})}_{S^q \times \mathbf{D}^{n-q-1}} h_{q+1})}_{\mathbf{D}^n} = \\ = & \; M_0 \natural \mathbf{D}^n = M_0 \end{aligned}$$
while from the point of view of N_2 we have:
$$N_2 = N_0 \sharp \partial \mathbf{D}^n = N_0$$

Note that the effect of the null gluing on the Betti numbers of the boundary is globally null. After the first step, only the q-th Betti number β_q and its dual β_{n-q-1} have changed by being increased by 1 (trivial gluing of h_q of type q-d). After the second step the gluing of h_{q+1} decreases by 1 the same Betti numbers β_q and β_{n-q-1} (gluing of h_{q+1} of type q-c).

3.3 Dual gluing

A *dual gluing* can only be performed by using two handles of complementary indices q and $(n-q)$.

1. The first step consists in gluing a q-handle h_q to M_0 via a trivial gluing (Subsection 3.1). We hence create a q-handlebody H_q and the global result of the gluing is that
$$\begin{cases} M_1 = & M_0 \natural H_q \\ N_1 = & N_0 \sharp \partial H_q = N_0 \sharp (S^q \times S^{n-q-1}) \end{cases}$$

2. We want now to attach the $(n-q)$-handle h_{n-q} by identifying its attaching region $S^{n-q-1} \times \mathbf{D}^q$ to the belt region of h_q:
$$\partial h_q \cap N_1 = \partial h_q \setminus (S^{q-1} \times \mathbf{D}^{n-q}) = \mathbf{D}^q \times S^{n-q-1} = S^{n-q-1} \times \mathbf{D}^q$$
The resulting manifold is
$$M_2 = M_0 \sharp S^q \times S^{n-q}$$
and its boundary is
$$N_2 = N_0$$
In order to prove these claims, consider the q-handlebody H_q obtained in step 1 :
$$H_q = B_0 \cup_{S^{q-1} \times \mathbf{D}^{n-q}} h_q = S^q \times \mathbf{D}^{n-q}$$

Take a copy of it, called $\tilde{H}_q$, which can be obtained by successively gluing to ∂H_q an $(n-q)$-handle h_{n-q} and an n-handle B_n (just take the definition of dual handles in Subsection 1.1.2). Then

$$H_q \cup_{\partial H_q} \tilde{H}_q = (\underbrace{S^q \times \mathbf{D}^{n-q}}_{H_q}) \cup_{S^q \times S^{n-q-1}} (\underbrace{S^q \times \mathbf{D}^{n-q}}_{\tilde{H}_q}) = S^q \times S^{n-q}$$

This means that

$$H_q \cup_{S^{n-q-1} \times \mathbf{D}^q} h_{n-q} = (S^q \times S^{n-q}) \setminus B_n$$

where B_n is an n-dimensional ball.

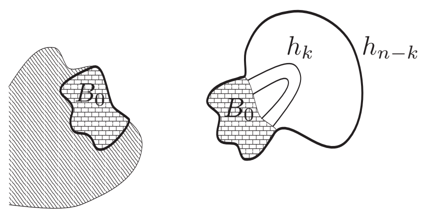

Figure 5: Dual gluing

Summarizing our steps (see Fig. 5), we started from M_0 and took away an n-dimensional ball B_0 such that upper hemisphere of ∂B_0 is a $\mathbf{D}^{n-1}$ in ∂M_0. Therefore we can consider M_0 as the first member of the connected sum describing M_2, and B_0 the ball used to perform the connected sum (even if B_0 is not in the interior of M_0, by taking collars one can show that in our context the final result would be the same).
On the other hand, by attaching h_q and h_{n-q} to B_0 we obtain a manifold homeomorphic to $(S^q \times S^{n-q}) \setminus B_n$ (therefore, $(S^q \times S^{n-q})$ is the second member of the connected sum, and B_n the ball used to perform the connected sum). Note that in this process the lower hemisphere of ∂B_0 has never been modified. By replacing B_0 at its place in M_0, we have proved the first claim:

$$M_2 = M_0 \sharp S^q \times S^{n-q}$$

As for the second step, it suffices to see that after the gluing of h_q and h_{n-q}, we have

$$N_2 = N_0 \sharp \partial \mathbf{D}^{n-1} = N_0$$

Note that the effect of the dual gluing on the Betti numbers of the boundary is globally null. After the first step, only the q-th Betti numbers β_q and its dual β_{n-q-1} have changed by being increased by 1 (trivial gluing of h_q of type q-d). After the second step the gluing of h_{n-q} decreases by 1 the same Betti numbers β_q and β_{n-q-1} (gluing of h_{n-q} of type $(n-q-1)$-c).

3.4 Invariant gluing

When the ambient dimension n is of the form $n = 4k$ and the index of the singularity is the middle dimension $2k$, then there is the possibility of gluing the corresponding handle in an invariant way, that is, in such a way that the Betti numbers of the boundary after such a gluing are the same as those of the boundary before the gluing. For this reason, all gluings of a single handle h_{2k} like these are of type β-i.

We have examples of invariant gluings in the construction of the projective spaces $\mathbf{CP}^{2k}$, $\mathbf{HP}^{2k}$ and $\mathbf{OP}^{2k}$ described is subsection 1.1.3: in all these cases the middle dimensional handle is necessarily of type β-i. We need to divide our study in different subcases according to the ambient dimension.

Let us start from M_0 and N_0. If $n = 4$, take a 4-dimensional ball B_0 out of M_0 such that upper hemisphere of ∂B_0 is a $\mathbf{D}^3$ in $\partial M_0 = N_0$. Glue to B_0 a 2-handle as in the handle decomposition of $\mathbf{CP}^2$ (by identifying B_0 to h_0), thus obtaining the complement of a ball in the complex plane, that is, $\mathbf{CP}^2 \setminus B_n$. By identifying the boundary of this manifold to that of B_0 in M_0, we have made the following connected sum:

$$M_1 = M_0 \sharp \mathbf{CP}^2$$

whose boundary is $N_1 = N_0$.

In the same way, if $n = 8$, by considering the middle gluing in the handle decomposition of $\mathbf{HP}^2$, we have:

$$\begin{cases} M_1 = & M_0 \sharp \mathbf{HP}^2 \\ N_1 = & N_0 \end{cases}$$

The same argument applies for $n = 16$ when considering the handle decomposition of $\mathbf{OP}^2$. In this case we have:

$$\begin{cases} M_1 = & M_0 \sharp \mathbf{OP}^2 \\ N_1 = & N_0 \end{cases}$$

In the remaining cases ($n = 12$ or $n = 4k$ with $k \geq 5$) no invariant gluing can be expressed in such a direct form. It is proved in [EK] that in dimension $n = 4k$ with $k = 3$ or $k \geq 5$ there exists no orientable manifold having a handle decomposition of the form (h_0, h_{2k}, h_n). Nevertheless, invariant gluings are possible also in these dimensions, up to adding some hypotheses on the boundary to which the invariant gluing is performed. For instance, in dimension $n = 4k$, consider the handle decomposition $(h_0, \ldots, h_{2k-2}, h_{2k}, h_{2k+2}, \ldots, h_n)$ of $\mathbf{CP}^{2k}$ and let W be the boundary obtained after the gluing of the first k handles. Let V be the manifold obtained by gluing the handle h_{2k} as in the construction of $\mathbf{CP}^{2k}$. Because of the symmetry of the construction, the new boundary is again W. To prove this claim, consider the Morse function defined on $\mathbf{CP}^{2k}$

$$f : [z_0 : z_1 : \ldots : z_{2k}] \to \sum_{j=0}^{2k} (-2k + 2j) \ \mid z_j \mid^2$$

whose critical values are $\{-2k, -2k+2, -2k+4, \ldots, 2k-4, 2k-2, 2k\}$. The attaching of the handle h_{2k} corresponds to passing through the critical value 0, that is, W can be thought of as $f^{-1}(-1)$ and the new boundary after the attachment can be thought of as $f^{-1}(1)$. These two level sets are homeomorphic via the map

$$[z_0 : z_1 : \ldots : z_{2k}] \to [z_{2k} : z_{2k-1} : \ldots : z_0]$$

Now, if we start from any boundary of the form $N_0 = N \sharp W$, N being any $(n-1)$-dimensional manifold, we can glue the handle h_{2k} to the "W part", as in the construction of $\mathbf{CP}^{2k}$. Hence the result of the gluing is

$$\begin{cases} M_1 = & M_0 \natural V \\ N_1 = & N_0 \end{cases}$$

4 Isolating blocks

In this section we prove our main theorem by realizing step by step the isolating block. Each group of handles appearing in the decomposition theorem (Theorem 2.1) will be treated separately in a specific subsection.

4.1 Realizations of $\underline{h}_{\min}$

Proposition 4.1. *Let v be a vertex of a Lyapunov semi-graph labelled by a vector $\underline{h}_{\min}$ compatible with the boundary data and satisfying $h_{\min}$. Then such a vertex can be realized explicitly as a manifold with boundary contained in the n-dimensional sphere S^n.*

Proof: Let Y denote the block realizing $\underline{h}_{\min}$. In the first step we build the exit boundary $\partial^- Y$, while in the second step, starting from a collar of $\partial^- Y$, we build the block Y itself. In order to define $\partial^- Y$, we do the following operations. We perform a continuation of the given vertex v. From theorem 1.6 we know the explicit general form of the continuation in terms of the boundary data. By using the same notation, we start building an auxiliary manifold X with empty exit boundary and nonempty entry boundary $\partial^+ X$. Such $\partial^+ X$ will be trivially identified to the exit boundary $\partial^- Y$ we are looking for. Here is the construction of X by attaching the following handles in such a way that all their attaching regions are disjoint:

* Take e^- attracting balls (singularities h_0) and do the following operations on the appropriate ball (use the boundary Betti numbers to choose);
* For $j = 1 \ldots \lfloor \frac{n}{2} \rfloor - (n \mod 2)$, let k_j be the integer in $0 \ldots \mid B_j^+ - B_j^- \mid$ associated with the given label of v:
  if $B_j^+ \geq B_j^-$ then attach in a trivial way B_j^- handles h_j of type j-d,
  else if $B_j^+ < B_j^-$ attach in a trivial way k_j handles h_j of type j-d and $(B_j^- - k_j)$ handles h_{n-j-1} of type $(n-j-1)$-d.
* If $n = 2i+1$, attach in a trivial way $\frac{B_i^-}{2}$ handles h_i and type i-d.

By construction $\partial^+ X$ is an $(n-1)$-dimensional manifold which can be described in terms of connected sums of $(n-1)$-dimensional generalized tori and whose Betti numbers are the ones we need to define $\partial^- Y$ by $\partial^- Y := \partial^+ X$. More precisely, for each incoming edge of v labelled by the Betti numbers

$$(\hat\beta_0 = 1, \hat\beta_1, \ldots, \hat\beta_{n-2} = \hat\beta_1, \hat\beta_{n-1} = \hat\beta_0 = 1),$$

the associated component will be the connected sums of generalized tori:

$$\begin{cases} \sharp_{j=1}^{i} \sharp^{\hat\beta_j}\ S^j \times S^{n-j-1} & \text{if } n = 2i; \\ (\sharp_{j=1}^{i-1} \sharp^{\hat\beta_j}\ S^j \times S^{n-j-1}) \sharp\ (\sharp^{\frac{\hat\beta_i}{2}}\ S^i \times S^{n-i-1}) & \text{if } n = 2i+1. \end{cases}$$

We shall refer to the handles used in the construction of the auxiliary manifold X as *ghost handles* because they will not be seen in Y, the realization of $\underline{h}_{\min}$.

The second step consists in gluing the handles of $\underline{h}_{\min}$ to a collar of $\partial^- Y$. First we glue the $e^- - 1$ handles of index 1 and type 0-c in order to connect all the components of $\partial^- Y$. Then, for each one of the remaining handles, if it is of type j-d and $j < n-1$, we glue it in a trivial way, thus performing the connected sum with a j-handlebody, else, if it is of (index $(j+1)$ and) type j-c, by construction it can be paired up with a ghost handle h_j of type j-d and will be glued in such a way that the two of them are glued in a null way (Subsection 3.2). We are left with $e^+ - 1$ handles of index $(n-1)$ and type $(n-1)$-d, which we'll use to disconnect – according to the desired components and associated Betti numbers – the connected sum of generalized tori

$$\begin{cases} \sharp_{j=1}^{i} \sharp^{B_j^+}\ S^j \times S^{n-j-1} & \text{if } n = 2i; \\ (\sharp_{j=1}^{i-1} \sharp^{B_j^+}\ S^j \times S^{n-j-1}) \sharp\ (\sharp^{\frac{B_j^+}{2}}\ S^i \times S^{n-i-1}) & \text{if } n = 2i+1; \end{cases}$$

obtained in the last step. After gluing all the $h_{\min}$ handles, we have determined a manifold Y whose entry and exit boundaries have the needed Betti numbers. Note that in particular the manifold $X \cup_{\partial^- Y} Y$ can be completed to a sphere by gluing handles which can be paired in a null way with those which have not been paired yet and we are done. △

It is worth pointing out that by construction a realization of $\underline{h}_{\min}$ respecting given Betti numbers (and not only their differences) is always possible.

Here we have made the choice of realizing $\underline{h}_{\min}$ by a "slice" cut off from the sphere, other analogous constructions are of course possible (for instance as "slices" cut off from connected sums of generalized tori).

An important observation is that, since the attaching regions can all be chosen disjoint, the rank of the Conley index is preserved at this stage (see [Co]), that is,

$$\text{rank } (H_j(Y, \partial^- Y)) = \text{number of handles of index } j \text{ appearing in } \underline{h}_{\min}.$$

4.2 Realization of $\underline{h}_{\text{consecutive}}$

We recall that, by Theorem 2.1, handles in this class are couples (h_j, h_{j+1}) with adjacent indices and of types j-d and j-c respectively. Of course one could realize them by considering them as null pairs and glue them in a null way (Subsection 3.2), but in our setting this solution cannot be accepted because handles like these can be removed from the isolating block without altering its topology, hence a realization of this type would not match item 3 of our main theorem.

To solve the problem, we exhibit another way of gluing two consecutive handles which contributes to the Conley index but which requires conditions on the Betti numbers of the exit boundary.

The underlying idea is the following (to fix ideas let us consider dimension $n = 3$ and $j = 1$). Suppose the initial boundary N_0 is a torus $S^1 \times S^1$, the boundary of the 1-handlebody $H_1 = S^1 \times \mathbf{D}^2$, and take a collar of N_0. In order to glue the consecutive handles (h_1, h_2) in a non-null way, use the handle of index 2 to "fill the hole" as in the second step of the null gluing, thus obtaining a new boundary $N_1 = S^2$, then glue the handle of index 2 in a trivial way, thus obtaining a new boundary $N_2 = S^1 \times S^1$ (see Figure 6).

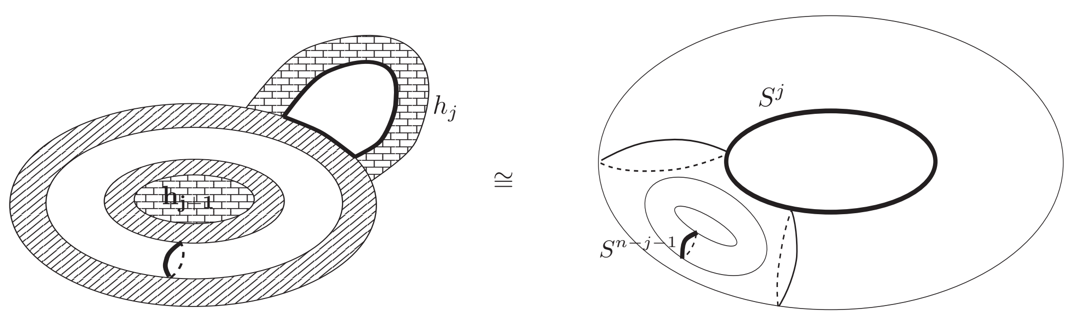

Figure 6: Isolating block of a consecutive pair of handles

Therefore, by definition, the homotopy Conley index of the block M is the homotopy type of the wedge of spheres $S^1 \bigvee S^2$ (see Figure 8). The homology Conley index is: $H_1(M, N_0) = \mathbf{Z}$, $H_2(M, N_0) = \mathbf{Z}$. The associated ranks are $h_1 = h_2 = 1$ as we needed.

The above construction can be easily adapted to the general dimension.

Proposition 4.2. *Let* e^+ *and* e^- *be positive integers. Let* $\{(B_j^+ - B_j^-)\}_{j=0}^{n-1}$ *be a collection of* n *integers such that* $(B_j^+ - B_j^-) = (B_{n-1-j}^+ - B_{n-1-j}^-)$ *for all* $j = 0 \ldots n-1$. *If* $n = 2i+1$, *let*

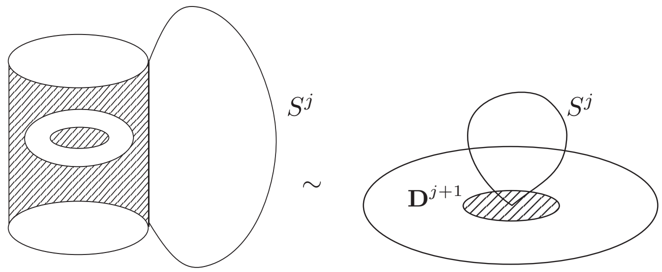

Figure 7: Homotopy type of the block

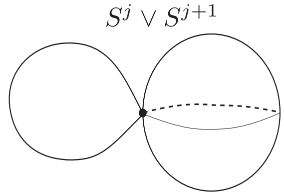

Figure 8: Conley index of the block

$(B_i^+ - B_i^- = 0)$ mod 2. Moreover, let $\underline{h}$ be of the form

$$\underline{h} = \underline{h}_{\min} + \underline{h}_{\text{consecutive}}$$

Then such data are realizable in the sense of the Main Theorem.

Proof: Let $\partial^- Y$ be the exit boundary of the block realizing $\underline{h}_{\min}$ as in Proposition 4.1. For all $j = 1 \dots n-2$, let c_j be the number of consecutive couples of the form (h_j, h_{j+1}) appearing in the class $\underline{h}_{\text{consecutive}}$. Define N_0 as the connected sum:

$$N_0 = \partial^- Y \,\sharp\, (\sharp^{c_j} S^j \times S^{n-j-1})$$

and think of it as the boundary of the auxiliary manifold $X \,\sharp\, (\sharp^{c_j} H_j)$ where X is the auxiliary manifold appearing in Proposition 4.1 and H_j denotes the j-handlebody $S^j \times \mathbf{D}^{n-j}$ obtained by gluing a *ghost handle* of index j and type j-d in a trivial way to a ball (Subsection 3.1).

Take a collar of N_0 and glue the handles of the class $\underline{h}_{\min}$ to the "$\partial^- Y$ part" as detailed in the proof of Proposition 4.1. Now, for all handles (h_j, h_{j+1}) of the class $\underline{h}_{\text{consecutive}}$, pair up the one of index $(j+1)$ and type j-c with a ghost handle $\hat{h}_j$ of one of the H_j's and glue it in such a way that the total gluing of $(\hat{h}_j, h_{j+1})$ is null (Subsection 3.2). Glue the remaining handle h_j in a trivial way. The result of this double gluing applied $\sum_j c_j$ times is that the new boundary N_1 is homeomorphic to N_0 (implying no effect on the Betti numbers of the boundary).

On the other hand, after each of the consecutive gluings (h_j, h_{j+1}), the ranks of the homology Conley indices have been increased by 1 at dimensions j and $(j+1)$, which can again be proved by considering disjoint attaching regions (see [Co]) or by studying long exact sequences as shown below.

Consider $N = S^j \times S^{n-j-1}$ as the boundary of the j-handlebody $H_j^{(0)} = S^j_{(0)} \times \mathbf{D}^{n-j}_{(0)}$. Let M be the result of gluing two handles, of indices j and $j+1$, on a collar of N in the way described above. This means that we obtain

$$M = (S^j \times \mathbf{D}^{n-j}) \ \setminus \ H_j^{(0)} = (S^j \times \mathbf{D}^{n-j}) \ \setminus \ (S^j_0 \times D^{n-j}_0)$$

where $H_j^{(0)} \subset \mathbf{D}^n \subset M$ as in Figure 6.

If $n \neq 2j+1$, the only non-zero homology groups of M are (besides $H_0(M) = H_n(M) = \mathbf{Z}$ and $H_{n-1}(M) = [\partial M] = \mathbf{Z}$):

$$\left.\begin{array}{lll} H_j(M) & = [S^j] & = \mathbf{Z} \\ H_{n-j-1}(M) & = [S^{n-j-1}] & = \mathbf{Z} \end{array}\right\} \text{(see Figure 6)}$$

while if $n = 2j+1$, since $j = n-j-1$, the only difference is that

$$H_j(M) = H_{n-j-1}(M) = [S^j, S^{n-j-1}] = \mathbf{Z} \oplus \mathbf{Z}.$$

Consider now the following long exact sequence of the pair (M, N):

$$\begin{array}{cccccccc} \dots & \xrightarrow{\partial_{k+2}} & H_{k+1}(N) & \xrightarrow{\iota_*^{k+1}} & H_{k+1}(M) & \xrightarrow{\sigma_*^{k+1}} & H_{k+1}(M,N) & \xrightarrow{\partial_{k+1}} \\ & \xrightarrow{\partial_{k+1}} & H_k(N) & \xrightarrow{\iota_*^{k}} & H_k(M) & \xrightarrow{\sigma_*^{k}} & H_k(M,N) & \xrightarrow{\partial_{k}} \\ & \xrightarrow{\partial_{k}} & H_{k-1}(N) & \xrightarrow{\iota_*^{k-1}} & H_{k-1}(M) & \xrightarrow{\sigma_*^{k-1}} & H_{k-1}(M,N) & \xrightarrow{\partial_{k-1}} & \dots \\ \dots & \xrightarrow{\partial_{2}} & H_1(N) & \xrightarrow{\iota_*^{1}} & H_1(M) & \xrightarrow{\sigma_*^{1}} & H_1(M,N) & \xrightarrow{\bar{\partial}_1} & \{0\} \end{array}$$

We shall make our computations in details in the case $n \neq 2j+1$, assuming without loss of generality that $j < n-j-1$. The remaining cases can be treated similarly.

First we show that $H_j(M,N) = \mathbf{Z}$ by showing that we have the following short exact sequence:

$$\{0\} \to H_j(M) \xrightarrow{\sigma_*^j} H_j(M,N) \to \{0\}$$

In fact, since $H_j(N)$ is generated by $[S^j]$, its image under ι_*^j is zero because of the choice of the gluing of h_{j+1}, making it homologous to zero in M. Also, $H_{j-1}(N) = \{0\}$ (if $j=1$ there is nothing to prove). Therefore σ_*^j is an isomorphism and we are done.

Next we show that $H_{j+1}(M,N) = \mathbf{Z}$ by showing the following short exact sequence:

$$\{0\} \to H_{j+1}(M,N) \xrightarrow{\partial_{j+1}} H_j(N) \to \{0\}$$

The right null arrow has just been justified above. As for the left one, either $H_{j+1}(M)$ is trivial and there is nothing to add, or $j+1 = n-j-1$ and $H_{j+1}(M)$ and $H_{j+1}(N)$ are generated by $[S^{n-j-1}]$, hence, i_*^{j+1} is an isomorphism and in this case the image of $[S^{n-j-1}]$ under σ_*^{j+1} is zero and we obtain the exact sequence above.

$\triangle$

4.3 Realization of $\underline{h}_{\mathrm{dual}}$

The realization of the class $\underline{h}_{\mathrm{dual}}$ is straightforward and no extra condition on the Betti numbers of the boundary is needed. Just consider for (h_q, h_{n-q}) in $\underline{h}_{\mathrm{dual}}$ the dual gluing of Subsection 3.3. It has already been explained there that the total effect on the Betti numbers of the boundary is globally null. Moreover, each dual gluing (h_q, h_{n-q}) applied to a block M_0 corresponds to taking the connected sum of it with a generalized torus $S^q \times S^{n-q}$. Therefore, the effect of each dual gluing (h_q, h_{n-q}) on the Conley index of the isolating block is non-trivial only at ranks q and $(n-q)$, that is,

$$\left\{\begin{array}{lll} \text{rank } H_q(M_2, N_0) & = \text{rank } H_q(M_0 \sharp S^q \times S^{n-q}, N_0) & = \text{rank } H_q(M_0, N_0) + 1 \\ \text{rank } H_{n-q}(M_2, N_0) & = \text{rank } H_{n-q}(M_0 \sharp S^q \times S^{n-q}, N_0) & = \text{rank } H_{n-q}(M_0, N_0) + 1 \end{array}\right.$$

If the ambient dimension n is not a multiple of 4, then the construction of the isolating block is done, otherwise another class of handles must be treated.

4.4 Realization $\underline{h}_{\text{invariant}}$

For $n = 4$, $n = 8$ and $n = 16$, the realization of the class $\underline{h}_{\text{invariant}}$ is also straightforward and no extra condition on the Betti numbers of the boundary is needed. In these dimensions, let M_0 be an isolating block realizing $\underline{h}_{\min} + \underline{h}_{\text{consecutive}} + \underline{h}_{\text{dual}}$. Let b_i denote the number of middle-dimensional invariant handles appearing in the decomposition of $\underline{h}$. Then, by Subsection 3.4, the realization M_1 of our main theorem is achieved by applying b_i invariant gluings and has the form:

$$M_1 = M_0 \sharp (\sharp^{b_i} \mathbf{PP})$$

where $\mathbf{PP}$ is the complex projective plane $\mathbf{CP}^2$ if $n = 4$, the Hamiltonian projective plane $\mathbf{HP}^2$ if $n = 8$, or the Cayley projective plane $\mathbf{OP}^2$ if $n = 16$. In all these cases we have

$$\text{rank } H_{\frac{n}{2}}(M_1, N_0) = \text{rank } H_{\frac{n}{2}}(M_0 \sharp (\sharp^{b_i} PP), N_0) = \text{rank } H_{\frac{n}{2}}(M_0, N_0) + b_i$$

In general, if $n = 4k$, $k \neq 1, 2, 4$, we need to construct boundaries ad hoc for the realization. Let M_0 be an isolating block realizing $\underline{h}_{\min} + \underline{h}_{\text{consecutive}} + \underline{h}_{\text{dual}}$, and let N^- denote the exit boundary of such a realization. If b_i denotes the number of middle-dimensional invariant handles appearing in the decomposition of $\underline{h}$, consider for instance the boundary $\partial^+ W$ of the manifold obtained in the construction of $\mathbf{CP}^{2k}$ after gluing the handles $h_0, \ldots, h_{2k-2}$, and define

$$N_0 = N^- \sharp (\sharp^{b_i} \partial^+ W)$$

Consider the collar of N_0, glue to the "N^- part" of it all the non-invariant handles as in the realization of M_0. Then glue each of the b_i handles h_{2k} of type β-i to one of the $\partial^+ W$ of the collar and do this by using an invariant gluing (as for the construction of $\mathbf{CP}^{2k}$). At the end, the realization M_1 is achieved after these last operations and has the form:

$$M_1 = M_0 \natural \natural^{b_i} V$$

where V denotes the manifold obtained by gluing h_{2k} on $\partial^+ W$ in an invariant way.

In particular, the Betti numbers of the boundary are kept unchanged, while the homology Conley index of the isolating block has been modified at position $2k$ by the invariant gluings :

$$\text{rank } H_{2k}(M_1, N_0) = \text{rank } H_{2k}(M_0 \natural \natural^{b_i} V, N_0) = \text{rank } H_{2k}(M_0, N_0) + b_i$$

Our main theorem is now completely proved.

5 Final remarks

The explicit realization of isolating blocks described in this paper allows us to answer a question asked in [BeMRez2]. Given a closed manifold and a continuous flow it is known that the Morse inequalities are satisfied. So, in order to have a necessary condition for the realization of abstract Lyapunov graphs it was important to consider when the Morse inequalities are satisfied. In [BeMRez2] it was shown that given abstract data $(h_0, h_1, \ldots, h_n)$ satisfying the Poincaré-Hopf inequalities we can find a Betti number vector $(\gamma_0, \ldots, \gamma_n)$ that satisfies the Morse inequalities, and vice-versa. The collection in the positive orthant can create all the possible Betti number vectors that satisfy the Morse inequalities for the initial data. In this same paper it was shown that it is possible to have negative γ's and the authors asked about an interpretation of these negative numbers. For instance for $n = 2i + 1$ the definition of γ is:

$$\gamma_0(h^{cd}) = \gamma_{2i+1}(h^{cd}) = 1$$
$$\gamma_j(h^{cd}) = \begin{cases} h_j^d - h_{j+1}^c, & \text{if } 1 \leq j < i \\ h_i^d, & \text{if } j = i \\ h_{i+1}^c, & \text{if } j = i + 1 \\ -h_{j-1}^d + h_j^c, & \text{if } i + 2 \leq j \leq 2i \end{cases}$$

Take $n = 7$ and the Lyapunov graph of Figure 9.

Such a graph can be realized as $S^2 \times S^5$ by attaching to h_0 the dual handles h_5 of type 5-d

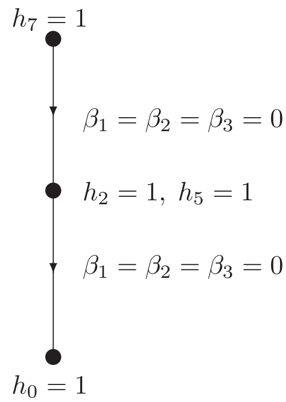

Figure 9: A Lyapunov graph

(hence $h_5^d = 1$) and h_2 of type 1-c (hence $h_2^c = 1$) with the dual gluing and by closing the manifold with h_7. In this case the only non-zero γ_j's are $\gamma_0 = \gamma_7 = 1$ and $\gamma_1 = \gamma_6 = -1$. Of course one can realize the same manifold with handles with the same index by attaching to h_0 the dual handles h_2 of type 2-d (hence $h_2^d = 1$) and h_5 of type 4-c (hence $h_5^c = 1$) with the dual gluing and by closing the manifold with h_7. In this case all the γ's are positive and the only non-zero γ_j's are $\gamma_0 = \gamma_7 = 1$ and $\gamma_2 = \gamma_5 = 1$. Of course these examples as closed manifolds can be transformed into examples of manifolds with boundary by taking h_0 and h_7 away from the construction.

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