RELATÓRIO DE PESQUISA

Asymptotically Linear Elliptic Problems in which the Nonlinearity Crosses at Least Two Eigenvalues

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Asymptotically linear elliptic problems in which the nonlinearity crosses at least two eigenvalues *†

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Abstract

In this paper we establish the existence of multiple solutions for the semilinear elliptic problem

$$\begin{aligned} -\Delta u &= g(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $g(x,0) = 0$ and which is asymptotically linear at infinity.

1 Introduction

Let us consider the problem

$$\begin{aligned} -\Delta u &= g(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a open bounded domain with smooth boundary $\partial\Omega$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that $g(x,0) = 0$. Assume that

$$g_0 := \lim_{t\to 0} \frac{g(x,t)}{t}, \quad \text{uniformly in} \quad \Omega,$$

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$$g_\infty := \lim_{|t|\to\infty} \frac{g(x,t)}{t}, \qquad \text{uniformly in } \Omega.$$

Denote by $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \ldots$ the eigenvalues of $(-\Delta, H_0^1)$, where each λ_j occurs in the sequence as often as its multiplicity. We also denote by φ_j the eigenfunction associated to λ_j.

We will suppose that $\lambda_{k-1} \le g_0 < \lambda_k \le \lambda_m < g_\infty \le \lambda_{m+1}$, and we will look for the existence of two nontrivial solutions of (1). There are many works that prove this kind of results, we can cite [2, 3, 4, 7, 8, 9, 10, 11]. If $g(x,t) = g(t)$ and $g'(t) > g(t)/t$, we can conclude from [3, Theorems 2.2 and 2.3] that if $k = 2$ and $m > 2$, then problem (1) has at least two nontrivial solutions (see [3, Corollary 2.4]). In [3] the authors also consider the resonant case $g_\infty = \lambda_{m+1}$, in this case they assume a Landesman-Lazer condition in order to get a compactness condition. The main aim in this paper is to improve these results in two ways: first we substitute the condition $g'(t) > g(t)/t$, by the weaker one $g'(x,t) \ge \lambda_1$ (see Theorem 2.1); second, we show the same result holds for any $k \ge 2$ and $m > k$ (see Theorem 2.2).

We also obtain a result, Theorem 2.3, related to [11, Theorem 2.1], in this theorem the authors assume that $\lambda_{k-1} < g_\infty < \lambda_k \le \lambda_m < g_0 < \lambda_{m+1}$ and $g'(x,t) \ge \alpha > \lambda_{k-1}$. In Theorem 2.3 we assume that $k = 2$ without the assumption on the $g'(x,t)$. We remark that the Theorem 2.3 in the case $m \ge 3$ follows from Theorems 2.1 and 2.3 in [3].

The classical solutions of the problem (1) correspond to critical points of the functional F defined on $H_0^1 = H_0^1(\Omega)$, by

$$F(u) = \frac{1}{2}\int_\Omega |\nabla \dot{u}|^2 dx - \int_\Omega G(x,u)dx, \quad u \in H_0^1, \tag{2}$$

where $G(x,t) = \int_0^t g(x,s)ds$. Under the above assumptions F is a functional of class C^2.

If U is a neighborhood of an isolated critical point u_0 with $F(u_0) = c$, then by the excision we have for the critical groups of u_0

$$C_p(\Phi, u_0) = H_p(\Phi^c \cap U, \Phi^c \cap U \setminus \{u_0\}).$$

2 Main theorems and proofs

Theorem 2.1 *Suppose that $g'(x,t) \ge \lambda_1$ for all $x \in \Omega$ and $t \in \mathbb{R}$. Assume that $\lambda_1 \le g_0 < \lambda_2$, and that there exists $m \ge 3$ such that $\lambda_m < g_\infty < \lambda_{m+1}$. Then problem (1) has at least two nontrivial solutions.*

Proof: From $\lambda_m < g_\infty < \lambda_{m+1}$ the functional F satisfies the (PS) condition and has the geometry of Saddle Point Theorem. Then there exists u_1, a critical point of F, such that, see [12],

$$C_m(F, u_1) \neq 0. \tag{3}$$

The proof of theorem follows of the next lemma and from the assumption that $m > 2$.

Lemma 2.1 *There exists a critical point u_2 of F such that*

$$C_p(F, u_2) = \delta_{p2}\mathbb{Z}.$$

Proof: The existence of u_2 satisfying

$$C_2(F, u_2) \neq 0, \tag{4}$$

follows as in the proof of Theorem 3.6 in [3] (see also the proof of Theorem 1.3 in [7]). Let be $m(u_2)$, the Morse index of u_2, it follows by Shifting Theorem ([6, Theorem 5.4, Chapter 1]) that

$$C_p(F, u_2) = C_{p-m(u_2)}(\widetilde{F}, 0). \tag{5}$$

So, by (4), we have $m(u_2) \leq 2$. Since $g'(x,t) \geq \lambda_1$ we have $m(u_2) \geq 1$ (the eigenfunction φ_1 satisfies $F''(u_2)(\varphi_1, \varphi_1) < 0$). Then the eigenvalue problem

$$\begin{array}{ll} -\Delta v = \mu g'(x, u_2)v & \text{in} \ \ \Omega \\ v = 0 & \text{on} \ \ \partial\Omega, \end{array} \tag{6}$$

has the first eigenvalue $\mu_1 < 1$. Let be $\phi > 0$ the eigenfunction associated with μ_1. Then by $\mu_1 < 1$ we have $F''(u_2)(\phi, \phi) < 0$, since φ_1 and ϕ are linearly independent, we obtain that $m(u_2) = 2$. Now, by (4) and (5), follows that $C_0(\widetilde{F}, 0) \neq 0$. Therefore 0 is a minimizer of $\widetilde{F}$ then $C_p(\widetilde{F}, 0) = \delta_{p0}\mathbb{Z}$, and we get

$$C_p(F, u_2) = \delta_{p2}\mathbb{Z},$$

and the lemma is proved.

Finally, we show that u_1 and u_2 are nontrivial. Indeed, by $g'(x, 0) < \lambda_2$ follows that $F''(0)(\varphi_j, \varphi_j) < 0$ for all $j \geq 2$. Then $m(0) + n(0) \leq 1$, where $n(0) = \dim\ker F''(0)$, and by a corollary of Shifting Theorem [6, Carollary 5.1, Chapter 1], we have $C_p(F, 0) = 0$ for all $p \geq 1$. Therefore u_1 and u_2 are nontrivial critical points of F. □

Theorem 2.2 *Assume that $g'(x,t) \geq g(x,t)/t$ $\forall\, x \in \Omega$ and $t \in \mathbb{R}$. Suppose that there exist $k \geq 2$, $m \geq k+1$ such that $\lambda_{k-1} \leq g_0 < \lambda_k$ and $\lambda_m < g_\infty < \lambda_{m+1}$. Then problem* (1) *has at least two nontrivial solutions.*

Proof: As in the proof of previous theorem we have a critical point u_1 of F such that

$$C_m(F,u_1) \neq 0. \tag{7}$$

And by the proof of Theorem 1.1 in [7], we have a solution u_2 that satisfies

$$C_k(F,u_2) \neq 0. \tag{8}$$

By $\lambda_{k-1} \leq g'(x,0)$ we have $m(0)+n(0) \leq k-1$, and as in the previous theorem, this implies that u_1 and u_2 are nontrivial.

The theorem follows from the next claim.

Claim: $C_p(F,u_1) = \delta_{pk}G$.

By (7) and the Shifting Theorem we have that $m(u_1) \leq k$. We will show that $m(u_1) = k$. Indeed, by $g(x,t)/t \geq \lambda_{k-1}$ we have that $\mu_i(g(x,u_1)/u_1) < \mu_i(\lambda_{k-1}) \leq 1$ for all $i \leq k-1$. Now, we have that

$$-\Delta u_1 = \frac{g(x,u_1)}{u_1} u_1,$$

this implies that $\mu_k(g(x,u_1)/u_1) \leq 1$. Then $\mu_k(g'(x,u_1)) < 1$, it follows from $g'(x,t) \geq g(x,t)/t$. This implies that $m(u_1) \geq k$, then $m(u_1) = k$. Again, the Shifting Theorem and (7) imply the *Claim.*

□

Theorem 2.3 *Assume that $\lambda_1 < g_\infty < \lambda_2$, and there exists $m \geq 2$ such that $\lambda_m < g_0 < \lambda_{m+1}$. Then problem* (1) *has at least two nontrivial solutions.*

Proof: Let $u_1 \neq 0$ be such that $C_1(F,u_1) \neq 0$. First we prove the claim:

Claim: $C_p(F,u_1) = \delta_{p1}\mathbb{Z}$.

Actually, we have that $m(u_1) \leq 1$. If $m(u_1) = 1$ the claim is proved. If $m(u_1) = 0$, then we have that the first eigenvalue μ_1 of the problem

$$\begin{aligned} -\Delta v &= \mu g'(x,u_1)v \quad &\text{in } \Omega \\ v &= 0 \quad &\text{on } \partial\Omega, \end{aligned} \tag{9}$$

satisfies $\mu_1 = 1$ and is simple. It follows that $n(u_1) = 1$, and so the claim follows by Shifting Theorem.

We also have that $C_p(F,0) = \delta_{pm}\mathbb{Z}$. If u_1 and 0 are the unique critical points of F, then the Morse inequality reads as

$$(-1) = (-1) + (-1)^m.$$

This is a contradiction. □

Remark 2.1 We observe that the resonant case g_0, $g_\infty \in \sigma(-\nabla)$ can be treated as [3] and [11]. We leave for the reader to consider this case

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