Stability of periodic travelling wave solutions for the Korteweg - de Vries equation *

Lynnyngs Kelly Arruda †
IMECC - UNICAMP,
Caixa Postal 6065.
13081-970 Campinas-SP, Brazil
E-mail address: lynnyngs@ime.unicamp.br

Abstract

This paper is concerned with nonlinear stability properties of periodic travelling waves solutions of the classical Korteweg - de Vries equation,

$$u_t + uu_x + u_{xxx} = 0, \quad x, t \in \mathbb{R}.$$

It is shown the existence of a nontrivial smooth curve of periodic travelling wave solutions depending on the classical Jacobian elliptic functions. We find positive cnoidal wave solutions. Then we prove, by using the framework established in [15] by Grillakis, Shatah and Strauss, the nonlinear stability of the cnoidal wave solutions in the space $H^1_{per}([0, L])$.

1 Introduction

The Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

where $u = u(x, t)$ is a real-valued scalar variable, t the time and x a scalar spatial variable, provides a simple and useful model for describing the long-time evolution of wave phenomena in which the steepening effect of the nonlinear term uu_x is counterbalanced by the dispersion term u_{xxx}. It was

* *Key words and phrases: Korteweg - de Vries equation, cnoidal waves, nonlinear stability.*

† The author was supported by CNPq/Brazil

originally derived by D. J. Korteweg and G. de Vries in 1895 [21] to describe the unidirectional propagation of small but finite amplitude waves in a nonlinear dispersive medium. This equation has many other direct physical applications to solids, liquids, gases and plasma: magnetohydrodynamic waves in a cold plasma (Gardner & Morikawa [14]), longitudinal waves propagating in a one-dimensional lattice of equal masses coupled by nonlinear springs, the Fermi & Pasta & Ulam problem [13] (Kruskal & Zabusky [22]), ion-acoustic waves in a cold plasma (Taniuti and Washini [28]), rotating flow in tube (Leibovich [23]) and longitudinal dispersive waves in elastic rods (Nariboli [27]). We refer the reader to Jeffrey & Kakutani [19] and Miura [26] for more applications.

The first interesting property of the KdV equation is the existence of steady progressing wave solutions. These permanent waves are fundamental quantities of the general solution to the KdV-equation and they are known as solitary waves or solitons (the latter name, which suggests an analogy with particles, is appropriate since the solitary waves retain their form even after joint interactions) and cnoidal waves (Korteweg and de Vries's generalization of the sinusoidal wave). Our main focus is the study of periodic travelling waves of (1), which are solutions of the form $u(x,t) = \phi_c(x-ct)$, where $\phi_c : \mathbb{R} \to \mathbb{R}$ and $c > 0$ represents the velocity relative to the velocity of infinitesimal long waves. So, ϕ_c must satisfy the differential equation

$$\phi_c'' + \frac{1}{2}\phi_c^2 - c\phi_c = A_{\phi_c}, \tag{2}$$

where A_{ϕ_c} is an integration constant, which will be considered equal to zero here (one may always perform the change of unknown $\widetilde{\phi}_c = \phi_c + \sqrt{c^2+2A_{\phi_c}} - c$). An explicit form for ϕ_c is well known (KdV [21], Benjamin [5]) and is given in function of the family of parameters $\beta_1, \beta_2, \beta_3$:

$$\phi_c(\xi) = \beta_2 + (\beta_3 - \beta_2)cn^2\Big[\sqrt{\frac{\beta_3-\beta_1}{12}}\,\xi; k\Big], \tag{3}$$

where cn is the *Jacobian elliptic function* (see Appendix or Byrd and Friedman [9]) and k is the *modulus* defined by

$$k^2 = \frac{\beta_3-\beta_2}{\beta_3-\beta_1},$$

with $\beta_1+\beta_2+\beta_3 = 3c$, $\beta_1 < \beta_2 < \beta_3$ and $\beta_3 > 0$. Since $cn(u+2K) = -cn(u)$, where $K(k)$ is the *complete elliptic integral of the first kind* determined by the modulus k, we have that ϕ_c has fundamental period (wavelength) T_{ϕ_c} equal to

$$T_{\phi_c} \equiv \frac{4\sqrt{3}}{\sqrt{\beta_3 - \beta_1}}\, K(k). \tag{4}$$

Moreover, an important feature of the fundamental period T_{ϕ_c} is that it depends on the speed c and satisfies the inequality

$$T_{\phi_c} > \frac{2\pi}{\sqrt{c}}. \tag{5}$$

(See Section 2).

We note that equation (3) contains a huge of periodic travelling wave solutions for the KdV equation, which are obtained basically by varying the modulus k. Moreover, formula (3) contains, at least formally, a basic solution of the KdV equation which is obtained by approximation with periodic solutions. In fact, if we let β_1 and β_2 tend to zero through positive values, we get $k^2 \to 1^-$ and $\beta_3 \to 3c^-$. The elliptic functions and their periods also become simpler in this limit, in fact, $cn(u; 1^-) \sim \operatorname{sech} u$ and $K(k) \to +\infty$ as $k \to 1^-$. The cnoidal wave looses its periodicity in this limit and we obtain a wave form with a single hump and with “infinit period, of the form

$$\phi_c(\xi; 0, 0, 3c) = 3c\, sech^2\Big(\frac{\sqrt{c}}{2}\xi\Big), \tag{6}$$

which is exactly the classical solitary wave solution of the KdV equation with speed c.

The unique studies regarding periodic travelling wave solutions associated to (1) in literature are, at our knowledge, those by Benjamin in [4] and by Angulo & Bona in [3] (see also Angulo & Alvarez [2]). In the last one, the authors considered the constant of integration A_ϕ in (2) different from zero and they presented a theory of existence and nonlinear stability of cnoidal wave solutions for the KdV equation, which are defined in an *a priori* fundamental interval $[0, L]$ and have mean zero on it.

Our main result here will be Theorem 3.1, where we show that the orbit generated by the periodic travelling wave solution (3), namely the set

$$\Omega_{\phi_c} = \{\phi_c(\cdot + s);\quad s \in \mathbb{R}\},$$

is $H^1_{per}([0, L])$-stable with respect to the flow of the KdV equation.

In order to prove this result we recall that the equation (1) has two basic constants of motion, namely

$$E(u) = \frac{1}{2}\int_0^L [u_x^2 - \frac{1}{3}u^3]dx \quad \text{and} \quad F(u) = \frac{1}{2}\int_0^L u^2 dx. \tag{7}$$

This means that if u is a smooth solution of (1), then $E(u(\cdot,t)) = E(u(\cdot,0))$ and $F(u(\cdot,t)) = F(u(\cdot,0))$ for all t. We also recall that the nonlinear stability criterion given by Grillakis *et al.* is based on the convexity property of the classical function $d(c) = E(\phi_c) + cF(\phi_c)$, and that the travelling wave solution (3) belongs to the range of the linearized operator around it, which we denote by $\mathcal{L}_{cn}$.

We remark that our theory of stability for ϕ_c is established with respect to perturbations of the same wavelenght L in $H^1_{per}([0,L])$.

The plan of this paper is as follows. In section 2, we prove the existence of a smooth curve of cnoidal waves with a fixed period L. Section 3 is devoted to establish the nonlinear stability of periodic travelling wave solutions based on the Grillakis *et al.* theory. The paper concludes with a review about Jacobian elliptic functions.

2 Existence of a smooth curve of cnoidal waves with a fixed period L

In this section we establish the existence of a family of even periodic solutions $\phi = \phi_c$ for the equation

$$\phi'' - c\phi + \frac{\phi^2}{2} = 0, \tag{8}$$

such that the mapping $c \to \phi_c$ is C^1.

First, we observe that multiplying (8) by ϕ', a second integration is possible yielding the first-order equation

$$\begin{aligned}(\phi'(\xi))^2 &= \frac{1}{3}[-\phi^3(\xi) + 3c\phi^2(\xi) + 6B_\phi] \equiv \frac{1}{3}p_\phi(\phi(\xi)) \\ &= \frac{1}{3}(\phi-\beta_1)(\phi-\beta_2)(\beta_3-\phi), \end{aligned} \tag{9}$$

where B_{ϕ_c} is an integration constant and $\beta_1, \beta_2, \beta_3$ are the zeros of the polynomial $p_\phi(t) = -t^3 + 3ct^2 + 6B_\phi$, so they satisfy the relations

$$\begin{cases} 3c = \beta_1 + \beta_2 + \beta_3 \\ 0 = \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 \\ B_\phi = \frac{1}{6}\,\beta_1\beta_2\beta_3. \end{cases} \tag{10}$$

Moreover, we assume that $\beta_1 < \beta_2 < \beta_3$ and $\beta_3 > 0$, and we obtain from (9) that $\beta_2 \leqq \phi \leqq \beta_3$. By defining $\varphi = \phi/\beta_3$, (9) becomes

$$(\varphi')^2 = \frac{\beta_3}{3}(\varphi-\eta_1)(\varphi-\eta_2)(1-\varphi)$$

where $\eta_i = \beta_i/\beta_3$, $i = 0, 1$. We also impose the crest of the wave to be at $\xi = 0$, that is $\varphi(0) = 1$. Now we define a further variable ψ via the relation

$$\varphi = 1 + (\eta_2 - 1)sin^2\psi,$$

and so we get that

$$(\psi')^2 = \frac{\beta_3}{12}(1-\eta_1)\Big[1 - \Big(\frac{1-\eta_2}{1-\eta_1}\Big)sin^2\psi\Big],$$

and $\psi(0) = 0$. In order to write this in a standard form we define

$$k^2 = \frac{1-\eta_2}{1-\eta_1}, \qquad l = \frac{\beta_3}{12}(1-\eta_1).$$

It follows that $0 \leqq k^2 \leqq 1$ and $l > 0$ and we obtain

$$\int_0^\psi \frac{dt}{\sqrt{1-k^2 sin^2 t}} = \sqrt{l}\,\xi.$$

Therefore, from the definition of the Jacobian elliptic function $y = sn\ (u;k)$ (see (30)), we can write the last equality as

$$sin\ \psi = sn\ (\sqrt{l}\,\xi;k),$$

and hence

$$\varphi = 1 + (\eta_2 - 1)sn^2\ (\sqrt{l}\,\xi;k).$$

Using the relation $sn^2 + cn^2 = 1$, we arrive finally to the conventional form

$$\phi(\xi) = \phi(\xi,\beta_1,\beta_2,\beta_3) = \beta_2 + (\beta_3-\beta_2)cn^2\Big[\sqrt{\frac{\beta_3-\beta_1}{12}}\,\xi;k\Big], \tag{11}$$

where

$$k^2 = \frac{\beta_3-\beta_2}{\beta_3-\beta_1}, \quad -\beta_1 = \beta_2+\beta_3-3c = \frac{\beta_2\beta_3}{\beta_2+\beta_3}, \quad \beta_1 < \beta_2 < \beta_3. \tag{12}$$

From (12), we have that β_2, β_3 belong to the rotated ellipse Σ given by

$$\beta_2^2 + \beta_3^2 + \beta_2\beta_3 - 3c(\beta_2+\beta_3) = 0, \tag{13}$$

and since $\beta_2 < \beta_3$, it follows that $0 < \beta_2 < 2c < \beta_3 < 3c$.

Next, since cn^2 has fundamental period $2K(k)$, then ϕ has fundamental period T_ϕ equal to

$$T_\phi \equiv \frac{4\sqrt{3}}{\sqrt{\beta_3-\beta_1}}\,K(k). \tag{14}$$

Now, we prove that $T_\phi > \frac{2\pi}{\sqrt{c}}$. Initially we express T_ϕ as a function of β_3 and c. In fact, for every $\beta_3 \in (2c, 3c)$ there is a unique $\beta_2 \in (0, 2c)$ such $(\beta_2, \beta_3) \in \Sigma$ where

$$2\beta_2 = 3c - \beta_3 + \sqrt{9c^2 + 6c\beta_3 - 3\beta_3^2}. \tag{15}$$

So, by defining $\beta_1 \equiv 3c - \beta_2 - \beta_3$, we obtain for

$$k^2(\beta_3, c) = \frac{3\beta_3 - 3c - \sqrt{9c^2 + 6c\beta_3 - 3\beta_3^2}}{3\beta_3 - 3c + \sqrt{9c^2 + 6c\beta_3 - 3\beta_3^2}} \tag{16}$$

that

$$T_\phi(\beta_3, c) = \frac{4\sqrt{6}}{\sqrt{3\beta_3 - 3c + \sqrt{9c^2 + 6c\beta_3 - 3\beta_3^2}}} K(k(\beta_3), c). \tag{17}$$

Then by fixing $c > 0$, we have that $T_\phi(\beta_3) \to \infty$ as $\beta_3 \to 3c$ and $T_\phi(\beta_3) \to \frac{2\pi}{\sqrt{c}}$ as $\beta_3 \to 2c$. So, since the mapping $\beta_3 \in (2c, 3c) \to T_\phi(\beta_3)$ is strictly increasing (see proof of Theorem 2.1), it follows that $T_\phi > \frac{2\pi}{\sqrt{c}}$.

Now we obtain a cnoidal wave solution with period L. For $c_0 > \frac{4\pi^2}{L^2}$ there is a unique $\beta_{3,0} \in (2c, 3c)$ such that $T_\phi(\beta_{3,0}) = L$. So, for c_0 and $\beta_{3,0}$ such that $(\beta_{2,0}, \beta_{3,0}) \in \Sigma(c_0)$, we have that the cnoidal wave $\phi(\cdot) = \phi(\cdot; \beta_{1,0}, \beta_{2,0}, \beta_{3,0})$ with $\beta_{1,0} = 3c_0 - \beta_{2,0} - \beta_{3,0}$, has fundamental period L and satisfies (8) with $c = c_0$.

We note that by the analysis above we can consider the cnoidal wave $\phi(\cdot; \beta_1, \beta_2, \beta_3)$ in (11) as a function depending only on c and β_3. So we will denote it by $\phi_c(\cdot; \beta_3)$ or ϕ_c.

Next we show the existence of a smooth curve of cnoidal wave solutions for equation (8), in other words, we show that at least locally the choice of $\beta_{3,0}$ above depends smoothly of c_0.

Proposition 2.1 *Let $L > 0$ arbitrary but fixed. Consider $c_0 > \frac{4\pi^2}{L^2}$ and $\beta_{3,0} = \beta_3(c_0) \in (2c_0, 3c_0)$ such that $T_{\phi_{c_0}} = L$. Then,*

(1) there exists an interval $I(c_0)$ around c_0, an interval $J(\beta_{3,0})$ around $\beta_{3,0}(c_0)$, and a unique smooth function $\Lambda : I(c_0) \to J(\beta_{3,0})$ such that $\Lambda(c_0) = \beta_{3,0}$ and

$$\frac{4\sqrt{6}}{\sqrt{3\beta_3 - 3c + \sqrt{9c^2 + 6c\beta_3 - 3\beta_3^2}}} K(k) = L, \tag{18}$$

where $c \in I(c_0)$, $\beta_3 = \Lambda(c)$, and $k^2 \equiv k^2(c) \in (0, 1)$ is defined in (16).

(2) *The cnoidal wave solution given by (11),* $\phi_c(\cdot;\beta_1,\beta_2,\beta_3)$*, determined by* $\beta_1 \equiv \beta_1(c)$*,* $\beta_2 \equiv \beta_2(c)$ *and* $\beta_3 \equiv \beta_3(c)$*, has fundamental period* L *and satisfies the equation (8). Moreover, the mapping*

$$c \in I(c_0) \to \phi_c \in H^1_{per}([0,L]) \tag{19}$$

is a smooth function.

(3) $I(c_0)$ *can be chosen as* $(\frac{4\pi^2}{L^2}, +\infty)$.

Proof: The idea of the proof is to apply implicit function theorem. We consider the open set $\Omega = \{(\beta,c); c > \frac{4\pi^2}{L^2}, \beta \in (2c,3c)\} \subseteq \mathbb{R}^2$ and define $\Psi : \Omega \to \mathbb{R}$ by

$$\Psi(\beta,c) = \frac{4\sqrt{6}}{\sqrt{3\beta - 3c + \sqrt{9c^2 + 6c\beta - 3\beta^2}}} K(k(\beta,c)), \tag{20}$$

where $k(\beta,c)$ is defined in (16), with $\beta_3 = \beta$. By hypotheses, $\Psi(\beta_{3,0},c_0) = L$.

Now we calculate $\partial_\beta \Psi(\beta,c)$. By denoting $a \equiv a(\beta) = 3\beta - 3c$ and $b \equiv b(\beta) = 9c^2 + 6c\beta - 3\beta^2$, we have that

$$\frac{\partial\Psi}{\partial\beta} = \frac{\sqrt{a+\sqrt{b}}\left[4\sqrt{6}\frac{dK}{dk}\frac{dk}{d\beta}\right] - 4\sqrt{6}K\frac{1}{2}\left[a+\sqrt{b}\right]^{-\frac{1}{2}}\left[3 + \frac{1}{2}b^{-\frac{1}{2}}(6c - 6\beta)\right]}{a+\sqrt{b}}.$$

Now from (16) it follows that

$$\frac{dk^2}{d\beta} = \frac{2a^2 + 6b}{\sqrt{b}(a+\sqrt{b})^2}.$$

Since $\frac{dk^2}{d\beta} = 2k\frac{dk}{d\beta}$, we obtain

$$\frac{dk}{d\beta} = \frac{1}{2k}\frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^2} > 0.$$

Thus, $\frac{\partial\Psi}{\partial\beta} > 0$. In fact,

$$\begin{aligned}
\frac{\partial\Psi}{\partial\beta} &= 4\sqrt{6}\frac{dK}{dk}\frac{1}{2k}\frac{(2a^2+6b)}{\sqrt{b}(a+\sqrt{b})^{\frac{5}{2}}} - 4\sqrt{6}\frac{1}{2}\frac{(3\sqrt{b}-a)}{\sqrt{b}(a+\sqrt{b})^{\frac{3}{2}}}K > 0 \\
&\Leftrightarrow \frac{dK}{dk}\frac{1}{k}\frac{(2a^2+6b)}{(a+\sqrt{b})} > (3\sqrt{b}-a)K \\
&\Leftrightarrow (2a^2+6b)(E - k'^2K) > (3\sqrt{b}-a)(a+\sqrt{b})k^2k'^2K \\
&\Leftrightarrow (2a^2+6b)E > (3\sqrt{b}-a)(a+\sqrt{b})k^2k'^2K + (2a^2+6b)k'^2K \\
&\Leftrightarrow (2a^2+6b)E > (2a\sqrt{b}+3b-a^2)k^2k'^2K + (2a^2+6b)k'^2K \\
&\Leftrightarrow (2a^2+6b)E > (2a\sqrt{b}+3b)k^2k'^2K + a^2k'^2K + a^2k'^4K + 6bk'^2K.
\end{aligned}$$

Now,

$$\begin{aligned}(2a^2+6b)E &= (1+k'^2)a^2E+k^2a^2E+6bE\\ &= k^2(1+k'^2)a^2E+k'^2(1+k'^2)a^2E+k^2a^2E+6bE.\end{aligned}$$

Since $a>\sqrt{b}$ and the fact that $k\to E(k)+K(k)$ is strictly increasing implies that $(1+k'^2)E>2k'^2K$, we have that

$$k^2(1+k'^2)a^2E>2k^2k'^2a^2K>2a\sqrt{b}k^2k'^2K.$$

We have that $6bE=6k^2bE+6k'^2bE$ and $E-k'^2K>0$ implies that $3k^2bE>3bk^2k'^2K$. Also, by using the inequality $(1+k'^2)E>2k'^2K$, we obtain

$$3k^2bE+6k'^2bE=3bE+3k'^2bE=3(1+k'^2)bE>6bk'^2K.$$

Now we have to show that

$$k^2a^2E+k'^2(1+k'^2)a^2E-a^2k'^2K-a^2k'^4K>0:$$

this follows from $k'^2(1+k'^2)a^2E>2k'^4a^2K$ and

$$k^2a^2E+k'^4a^2K-a^2k'^2K=k^2a^2E-k^2k'^2a^2K=k^2a^2(E-k'^2K)>0.$$

Therefore, there exists a unique smooth function Λ, defined in a neighborhood $I(c_0)$ of c_0, such that $\Psi(\Lambda(c),c)=L$ for every $c\in I(c_0)$. So, we obtain (18). Finally, since c_0 was chosen arbitrary in the interval $\mathcal{I}=(\frac{4\pi^2}{L^2},+\infty)$, it follows that Λ can be extended to $\mathcal{I}$. This completes the proof of the Proposition.

□

Corollary 2.1 *Consider the mapping $\Lambda: I(c_0)\to J(\beta_{3,0})$ determined by Theorem 2.1. Then, Λ is a strictly increasing function in $I(c_0)$.*

Proof: By Theorem 2.1 we have that $\Psi(\Lambda(c),c)=L$ for every $c\in I(c_0)$ and so

$$\frac{d}{dc}\Lambda(c)=-\frac{\partial\Psi/\partial c}{\partial\Psi/\partial\beta}. \tag{21}$$

We will show that $\partial\Psi/\partial c<0$. In order to do this, we denote again $a(c)=3\beta-3c$ and $b(c)=9c^2+6c\beta-3\beta^2$, and we note that

$$\begin{aligned}\frac{\partial\Psi}{\partial c} &= \frac{4\sqrt{6}\sqrt{a(c)+\sqrt{b(c)}}\frac{dK}{dk}\frac{dk}{dc}}{a(c)+\sqrt{b(c)}}\\ &+ \frac{-4\sqrt{6}K\frac{1}{2}(a(c)+\sqrt{b(c)})^{-\frac{1}{2}}(-3+\frac{1}{2}b(c)^{-\frac{1}{2}}(18c+6\beta))}{a(c)+\sqrt{b(c)}}\end{aligned}$$

where $k(\beta, c)$ is defined by (16), with $\beta_3 = \beta$. Now,

$$\frac{dk}{dc} = \frac{1}{2k}\frac{(3c-3\beta)b(c)^{-\frac{1}{2}}(18c+6\beta) - 6\sqrt{b(c)}}{[a(c)+\sqrt{b(c)}]^2} < 0$$

and

$$-3 + \frac{1}{2}b(c)^{-\frac{1}{2}}(18c+6\beta) = \frac{-6\sqrt{b(c)}+18c+6\beta}{2\sqrt{b(c)}} > 0$$

because $-6\sqrt{b(c)} + 18c + 6\beta > 0 \Leftrightarrow 3c + \beta > \sqrt{b(c)} \Leftrightarrow 9c^2 + 6c\beta + \beta^2 > 9c^2 + 6c\beta - 3\beta^2 \Leftrightarrow 4\beta^2 > 0$. So it follows that $\frac{d\Psi}{dc} < 0$. And the proof is completed. $\square$

Now, we prove that the modulus function $k(c)$ is strictly increasing.

Lemma 2.1 *Consider* $c \in (\frac{4\pi^2}{L^2}, \infty)$, $\beta_3 = \Lambda(c)$ *and the modulus function*

$$k(c) \equiv k(\Lambda(c)) = \sqrt{\frac{3\beta_3(c) - 3c - \sqrt{9c^2 + 6c\beta_3(c) - 3\beta_3(c)^2}}{3\beta_3(c) - 3c + \sqrt{9c^2 + 6c\beta_3(c) - 3\beta_3(c)^2}}}.$$

Therefore, $\frac{d}{dc}k(c) > 0$.

Proof: Denoting by $a(c) = 3\beta_3(c) - 3c$ and $b(c) = 9c^2 + 6w\beta_3(c) - 3\beta_3(c)^2$, we have that

$$\frac{dk}{dc}(c) = \frac{1}{2k}\left[\frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^2}\right]\beta_3'(c) + \frac{1}{2k}\left[\frac{-ab^{\frac{-1}{2}}(18c+6\beta_3) - 6\sqrt{b}}{(a+\sqrt{b})^2}\right].$$

Using that $\beta_3'(c) = \frac{d}{dc}\Lambda(c)$, by (21), we get that $\frac{dk}{dc}(c) > 0 \Leftrightarrow$

$$\frac{1}{2k}\left\{\frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^2}\right\}\left\{\frac{\frac{dK}{dk}\left[\frac{1}{2k}\frac{(ab^{-1/2}(18c+6\beta_3)+6\sqrt{b})}{(a+\sqrt{b})^{3/2}}\right] + \frac{K(a+\sqrt{b})^{-1/2}(-3+\frac{1}{2}b^{-1/2}(18c+6\beta_3))}{2}}{\frac{dK}{dk}\left[\frac{1}{2k}\frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^{3/2}}\right] + \frac{K(a+\sqrt{b})^{-1/2}(-3+\frac{1}{2}b^{-1/2}(6\beta_3-6c))}{2}}\right\}$$
$$+\frac{1}{2k}\left[\frac{-ab^{\frac{-1}{2}}(18c+6\beta_3)-6\sqrt{b}}{(a+\sqrt{b})^2}\right] > 0.$$

Now, the last inequality is true if, and only if

$$\frac{1}{2k}\left\{\frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^2}\right\}\frac{K}{2}(a+\sqrt{b})^{-1/2}(-3+\frac{1}{2}b^{-1/2}(18c+6\beta_3)) >$$
$$\frac{1}{2k}\left[\frac{ab^{\frac{-1}{2}}(18c+6\beta_3)+6\sqrt{b}}{(a+\sqrt{b})^2}\right]\frac{K}{2}(a+\sqrt{b})^{-1/2}(-3+\frac{1}{2}b^{-1/2}(6\beta_3-6c)),$$

and this happens if, and only if

$$(2a^2+6b)[-3+\frac{1}{2}b^{-1/2}(18c+6\beta_3)] >$$
$$[a(18c+6\beta_3)+6b][-3+\frac{1}{2}b^{-1/2}(6\beta_3-6c)]. \qquad (22)$$

Now, (22) is equivalent to

$$-6a^2+4\times 18c\sqrt{b}+3a(18c+6\beta_3)>0,$$

which is satisfied since

$$3a(3c+\beta_3)-a^2=a(9c+3\beta_3-3\beta_3+3c)=12ac>0.$$

This completes the proof. □

3 Stability theorem

The next theorem is the main result of this paper.

Theorem 3.1 *Let $c \in (\frac{4\pi^2}{L^2}, +\infty)$. Then the orbit Ω_ϕ is $H^1_{per}([0,L])$ -stable with respect to the flow of Korteweg - de Vries equation.*

In order to prove our theorem of nonlinear stability, we only need to show that the function $d(c)$ defined by

$$d(c)=E(\phi_c)+cF(\phi_c)\equiv \frac{1}{2}\int_0^L[(\phi_c')^2-\frac{1}{3}\phi_c^3]dx+c\frac{1}{2}\int_0^L\phi_c^2dx,$$

is convex. Actually, the well-posedness of the KdV equation (1), in the periodic case was established by Colliander, Keel, Staffilani, Takaoka & Tao in [11] (or see [20]), and the spectral analysis of the operator

$$L_{cn}=-\frac{d^2}{dx^2}+c-\phi_c,$$

that we need in order to apply the theory of Grillakis, Shatah and Strauss, was proved in [3] and [2].

3.1 Convexity of the function $d(c)$

Observe that, since $c \in (\frac{4\pi^2}{L^2}, +\infty) \to \phi_c \in H^n_{per}([0, L)$ is a C^1-function, we have that

$$d'(c) = \langle E'(\phi_c) + F'(\phi_c), \frac{d}{dc}\phi_c\rangle_1 + F(\phi_c) = F(\phi_c) = \frac{1}{2}\int_0^L \phi_c^2(x)dx. \quad (23)$$

So, we have the following Lemma.

Lemma 3.1 *If $c \in (\frac{4\pi^2}{L^2}, +\infty)$ then $d(c)$ is a convex function.*

Proof: By (18), we have that $a + \sqrt{b} = \frac{48\times 2K^2}{L^2}$, and by (16), we have that $a - \sqrt{b} = \frac{48\times 2k^2K^2}{L^2}$. So we conclude that

$$a = \frac{48(1+k^2)K^2}{L^2} \quad \text{e} \quad \sqrt{b} = \frac{48(1-k^2)K^2}{L^2}. \quad (24)$$

Thus,

$$\begin{cases} 3\beta_3 - 3c = \frac{48(1+k^2)K^2}{L^2} \\ 9c^2 + 6c\beta_3 - 3\beta_3^2 = \frac{48^2(1-k^2)^2K^4}{L^4}; \end{cases}$$

solving the system above, we get

$$c = \frac{8K^2\sqrt{3k'^4 + (1+k^2)^2}}{L^2} \quad (25)$$

and

$$\beta_3 = \frac{8K^2[2(1+k^2) + \sqrt{3k'^4 + (1+k^2)^2}]}{L^2}. \quad (26)$$

Now, by (8), we have that $\int_0^L \phi_c^2 d\xi = 2c\int_0^L \phi_c d\xi$, from which, by using also (23), we obtain

$$d'(c) = c\int_0^L \phi_c dx. \quad (27)$$

Using that

$$\int_0^L cn^2\Big[\sqrt{\frac{\beta_3 - \beta_1}{12}}\,\xi; k\Big]d\xi \quad = \quad \frac{L}{K}\Big[\frac{E(k) - k'^2K(k)}{k^2}\Big],$$

we get

$$\begin{aligned}
\frac{d}{dc}\Big(\int_0^L \phi_c^2 d\xi\Big) &= 2\int_0^L \phi_c(x)d\xi + 2c\frac{d}{dc}\Big(\int_0^L \phi_c d\xi\Big) \\
&= 2\beta_2 L + 2(\beta_3-\beta_2)\frac{L}{K}\Big[\frac{E(k)-k'^2K(k)}{k^2}\Big] \\
&+ 2cL\frac{d}{dk}\Big\{\beta_2 + (\beta_3-\beta_2)\frac{1}{K}\Big[\frac{E(k)-k'^2K(k)}{k^2}\Big\}\frac{dk}{dc}.
\end{aligned} \tag{28}$$

Now, from the relations (15), (24), (25) and (26), we have that

$$\begin{aligned}
\beta_2 &= \frac{1}{2}\Big[\frac{16K^2\sqrt{3k'^4+(1+k^2)^2}}{L^2} - \frac{16K^2[(1+k^2)}{L^2} + \frac{48(1-k^2)K^2}{L^2}\Big] \\
&= \frac{8K^2}{L^2}\Big[\sqrt{3k'^4+(1+k^2)^2} - (1+k^2) + 3(1-k^2)\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{d}{dk}\beta_2(k) &= \frac{16K}{L^2}\frac{dK}{dk}[\sqrt{3k'^4+(1+k^2)^2} - (1+k^2) + 3k'^2] \\
&+ \frac{8K^2}{L^2}\Big\{\frac{1}{2}[3k'^4+(1+k^2)^2]^{-\frac{1}{2}}[-8k+16k^3] - 8k\Big\}.
\end{aligned}$$

Moreover, as

$$\begin{aligned}
(\beta_3-\beta_2)\frac{1}{K}\Big[\frac{E(k)-k'^2K(k)}{k^2}\Big] &= \frac{48k^2K^2}{L^2}\frac{1}{K}\Big[\frac{E(k)-k'^2K(k)}{k^2}\Big] \\
&= \frac{48K}{L^2}[E(k)-k'^2K(k)],
\end{aligned}$$

we conclude that

$$\frac{d}{dk}\Big\{(\beta_3-\beta_2)\frac{1}{K}\Big[\frac{E(k)-k'^2K(k)}{k^2}\Big]\Big\} = \frac{48}{L^2}kK(k)^2 > 0.$$

It follows that

$$
\begin{aligned}
\frac{d}{dc}\Big(\int_0^L \phi_c^2 d\xi\Big) &= \frac{16K^2}{L}\Big[\sqrt{3k'^4+(1+k^2)^2}-(1+k^2)+3k'^2\Big] \\
&+ \frac{96K}{L}[E(k)-k'^2K(k)] \\
&+ 2c\Big\{\frac{16K}{L}\frac{dK}{dk}[\sqrt{3k'^4+(1+k^2)^2}-(1+k^2)+3k'^2] \\
&+ \frac{8K^2}{L}\Big[\frac{(-4k+8k^3)}{\sqrt{3k'^4+(1+k^2)^2}}-8k\Big]+\frac{48}{L}kK^2\Big\}\frac{dk}{dc}.
\end{aligned} \tag{29}
$$

Denoting by

$$
\begin{aligned}
f(k): &= \frac{16K}{L}\frac{dK}{dk}[\sqrt{3k'^4+(1+k^2)^2}-(1+k^2)+3k'^2] \\
&+\frac{8K^2}{L}\frac{(-4k+8k^3)}{\sqrt{3k'^4+(1+k^2)^2}}-\frac{16}{L}kK^2 \\
&= \frac{16K}{L}\Big\{\frac{dK}{dk}[\sqrt{3k'^4+(1+k^2)^2}-(1+k^2)+3k'^2] \\
&+K\frac{(-2k+4k^3)}{\sqrt{3k'^4+(1+k^2)^2}}-kK\Big\},
\end{aligned}
$$

from $E^2 > k'^2K^2$ we have that $E > k'K$ and thus $\frac{E-k'^2K}{kk'^2} > \frac{k'(1-k')K}{kk'^2} = \frac{(1-k')K}{kk'}$. Now, we show that $f(k)$ is positive. It is enough to show that

$$
(1-k')[\sqrt{3k'^4+(1+k^2)^2}-(1+k^2)+3k'^2] > kk'[\frac{(2k-4k^3)}{\sqrt{3k'^4+(1+k^2)^2}}+k].
$$

Denoting by $D = 3k'^4+(1+k^2)^2 = 4k'^4+4-4k'^2$ and

$$
p(k') = (1-k')[\sqrt{D}-(1+k^2)+3k'^2]\sqrt{D}-kk'(2k-4k^3)-k^2k'\sqrt{D},
$$

where $k=\sqrt{1-k'^2}$, we deduce that $p'(k)>0$ implies $f(k)>0$. Now, we claim that $p(k')>0$. Actually,

$$
\begin{aligned}
p(k') &= (1-k')\{[\sqrt{D}-2+4k'^2]\sqrt{D}-2k'(1+k')+4k(1-k')(1+k')^2 \\
&\quad -(1+k')k'\sqrt{D}\} \\
&= (1-k')(k'-1)\{4k'^3+4k'^2-2k'-4-4k(1+k')^2 \\
&\quad +(3k'+2)\sqrt{D}\} \\
&= -(1-k')^2\{-4k'^2-6k'-4+(3k'+2)\sqrt{D}\}.
\end{aligned}
$$

Now $-4k'^2 - 6k' - 4 + (3k'+2)\sqrt{D} < 0 \Leftrightarrow (3k'+2)\sqrt{D} < 4k'^2 + 6k' + 4 \Leftrightarrow (9k'^2 + 12k' + 4)(4k'^4 + 4 - 4k'^2) < (4k'^2 + 6k' + 4)^2 \Leftrightarrow 36k'^4(k'^2 - 1) + 48k'^3(k'^2 - 1) - 48k'^2 < 0$. It follows that $p(k') > 0 \ \forall k' \in (0,1)$.

Finally, it follows that $\frac{d}{dc}\Big(\int_0^L \phi_c dx \Big) > 0$, and so we conclude by (27) that

$$d''(c) = c\frac{d}{dc}\Big(\int_0^L \phi_c dx \Big) + \int_0^L \phi_c dx > 0.$$

□

4 APPENDIX

In this Appendix we recall some properties of the Jacobian Elliptic Integrals that have been used in this work (see [9]).

First, we define *the normal elliptic integral of the first kind*:

$$\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\ \sin^2\theta}} \equiv F(\varphi, k),$$

where $y = sin\varphi$, and *the normal elliptic integral of the second kind*,

$$\int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}}\ dt = \int_0^\varphi \sqrt{1-k^2\ \sin^2\theta}\ d\theta \equiv E(\varphi, k).$$

In their algebraic forms, these two integrals possess the following properties: the first is finite for all real (or complex) values of y, including infinity; the second has a simple pole of order 1 for $y = \infty$. The number k is called the *modulus*. This number may take any real or imaginary value. Here we will to take $0 < k < 1$. The number k' is called the *complementary modulus* and is related to k by $k' = \sqrt{1-k^2}$. The variable φ is the *argument* of the normal elliptic integrals, and may be of course either real or complex, but it is usually understood that $0 < y \leqq 1$ or $0 < \varphi \leqq \pi/2$.

When $y = 1$, the integrals above are said to be *complete*. In this case, one writes:

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\ \sin^2\theta}} = F(\pi/2, k) \equiv K(k) \equiv K,$$

and

$$\int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\ dt = \int_0^{\pi/2} \sqrt{1-k^2\ \sin^2\theta}\ d\theta = E(\pi/2, k) \equiv E(k) \equiv E.$$

Some special values of K and E are: $K(0) = E(0) = \pi/2$, $E(1) = 1$ and $K(1) = +\infty$. For $k \in (0,1)$, one has $K'(k) > 0$, $K''(k) > 0$, $E'(k) < 0$, $E''(k) < 0$ and $E(k) < K(k)$.

Now, we give some derivatives of the complete elliptical integrals K and E, that we used in this work:

$$\frac{dK}{dk} = \frac{E-k'^2K}{kk'^2}, \quad \frac{dE}{dk} = \frac{E-K}{k}, \quad \frac{d^2E}{dk^2} = -\frac{1}{k}\frac{dK}{dk} = -\frac{E-k'^2K}{k^2k'^2}.$$

We will now define the *Jacobian Elliptic Functions.* Initially, we consider the elliptic integral

$$u(y_1;k) \equiv u = \int_0^{y_1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\ \sin^2\theta}} = F(\varphi,k) \tag{30}$$

which is a strictly increasing function of the real variable y_1, hence we can define its inverse function by $y_1 = \sin\varphi \equiv sn(u;k)$ (or briefly $y_1 = sn\ u$, when it is not necessary to emphasize the modulus). The function $sn\ u$ is a odd elliptic function. Other two basic functions can be defined by

$$\begin{aligned} cn\ (u;k) &= \sqrt{1-y_1^2} = \sqrt{1-sn^2(u;k)} \\ dn\ (u;k) &= \sqrt{1-k^2y_1^2} = \sqrt{1-k^2sn^2(u;k)}, \end{aligned}$$

requiring that $sn(0,k) = 0$, $cn(0,k) = 1$ and $dn(0,k) = 1$. The functions $cn\ u$ and $dn\ u$ are therefore even functions. The functions $sn\ u$, $cn\ u$, and $dn\ u$ are called *Jacobian elliptic functions* and are one-valued functions of the argument u. These functions have a real period, namely $4K, 4K$ and $2K$ respectively. The most important properties of the Jacobian elliptic functions which have been used in this work are summarized by the formulas given below.

1. Fundamental Relations:

$$\begin{aligned} & sn^2u + cn^2u = 1, \\ & k^2sn^2u + dn^2u = 1 \\ & k'^2sn^2u + cn^2u = dn^2u \\ & -1 \leqq sn\ u \leqq 1, \quad -1 \leqq cn\ u \leqq 1, \quad k'^2 \leqq dn\ u \leqq 1. \end{aligned} \tag{31}$$

2. Special Values:

$$\begin{aligned} & sn(-u) = -sn\ u, \quad cn(-u) = cn\ u, \quad dn(-u) = dn\ u, \quad sn\ 0 = 0, \\ & cn\ 0 = 1, \quad sn\ K = 1, \quad cn\ K = 0. \\ & sn(u+4K) = sn\ u, \quad cn(u+4K) = cn\ u, \quad dn(u+2K) = dn\ u \\ & sn(u+2K) = -sn\ u, \quad cn(u+2K) = -cn\ u. \end{aligned}$$

Finally, we have

$$sn\ (u,0) = \sin\ u,\ \ cn\ (u,0) = \cos\ u,$$
$$sn\ (u,1) = \tanh\ u,\ \ cn\ (u,1) = sech\ u.$$

3. Differentiation of the Jacobian Elliptic Functions:

$$\begin{array}{l} \frac{\partial}{\partial u} sn(u) = cn\ u\ dn\ u,\ \ \frac{\partial}{\partial u} cn(u) = -sn\ u\ dn\ u, \\ \frac{\partial}{\partial u} dn(u) = -k^2 sn\ u\ cn\ u. \end{array} \tag{32}$$

References

[1] J. Albert and J. Bona, *Total positivity and the stability of internal waves in stratified fluids of finite depth*, IMA J. Appl. Math., **46** (1991), 1-19.

[2] B. Alvarez and J. Angulo, *Existence and stability of periodic travelling waves solutions of the Benjamin equation*, Communications on Pure and Applied Analysis, **4**, Number 2 (2005), 367- 388.

[3] J. Angulo and J. Bona, *Nonlinear stability of periodic travelling waves solutions to the Korteweg - de Vries equation*, preprint/2003.

[4] T. B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. London A, **338** (1972), 153-183.

[5] T. B. Benjamin, *Lectures on nonlinear wave motion*, Nonlinear Wave Motion, A. C. Newell, ed., AMS, Providence, R. I., **15** (1974), 3-47.

[6] J. L. Bona, *On the stability theory of solitary waves*, Proc. Roy. Soc. London A, **344** (1975), 363-374.

[7] J. L. Bona, P. E. Souganidis and W. A. Strauss, *Stability and instability of solitary waves of Korteweg - de Vries type*, Proc. Royal. Soc. London Ser. A **411** (1987), 395-412.

[8] J. Bourgain, *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations*, Geometric and Functional Anal., **3** (1993) 107–156, 209–262.

[9] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, sec. ed., Springer-Verlag, New York (1971).

[10] T. Cazenave and P.-L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys., **85** (1982), 549-561.

[11] J. Colliander, M. Kell, G. Staffilani, H. Takaoka, and T. Tao, *Sharp global well-posedness for KDV and modified KDV on* $\mathbb{R}$ *and* $\mathbb{T}$

[12] A. Erdélyi, *On Lamé functions*, Philos. Mag. **7**, 31 (1941), 123-130.

[13] E. Fermi, J. Pasta and S. Ulam, *Studies of nonlinear problems, I*, Los Alamos Report LA 1940 (1955).

[14] C. S. Gardner and G.K. Morikawa, *Similarity in the asymptotic behavior of collision-free hydromagnetic waves and water waves*, New York Univ., Courant Inst. Math. Sci. Res. Rep. NYO-9082 (1960)

[15] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry I.*, J. Funct. Anal. **74** (1987), 160-197.

[16] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry II.*, J. Funct. Anal. **94** (1990), 308-348.

[17] E. L. Ince, *The periodic Lamé functions*, Proc. Roy. Soc. Edin. **60** (1940), 47-63.

[18] R. Iorio and V. de Magalhães Iorio, *Fourier Analysis and Partial Differential Equations*, Cambridge stud. in advan. math. **70** (2001)

[19] A. Jeffrey and T. Kakutani, *Weak nonlinear dispersive waves: a discussion centred around the Korteweg-de Vries*, S.I.A.M. Rev., **14** (1972), 582.

[20] E. C. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc, **9** (1996), 573-603.

[21] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and a new type of long stationary waves*, Philos. Mag. **39**, (1895), 422-443.

[22] M. D. Kruskal and N. J. Zabusky, *Progress on the Fermi-Pasta-Ulam nonlinear string problem*, Princeton Plasma Physics Laboratory Annual Rep. MATT-Q-21, Princeton, N.J., (1963) 301-308.

[23] S. Leibovich, *Weakly non-linear waves in rotating fluids* J. Fluid Mech., **42** (1970), 803-822.

[24] O. Lopes, *A linearized instability result for solitary waves* Discrete Contin. Dyn. Syst., **8** (2002), 115-119.

[25] W. Magnus and S. Winkler, *Hill's Equation*, Interscience, Tracts in Pure and Appl. Math. Wiley, NY. **20** (1976).

[26] R. M. Miura, *The Korteweg-de Vries equation: A survey of results*, S.I.A.M. Rev. **18** (1976), 412-459.

[27] G. A. Nariboli, *Nonlinear longitudinal dispersive waves in elastic rods*, Iowa State Univ. Engineering Res. Inst Preprint **442** (1969).

[28] T. Taniuti and H. Washimi, *Propagation of ion-acoustic solitary waves of small amplitude*, Phys. Rev. Lett., **17** (1966), 996-998.

[29] M. I. Weinstein, *Modulational stability of ground states of nonlinear Schrödinger equations* SIAM J. Math. Anal., **16** (1985), 472-491.

[30] M. I. Weinstein, *Liapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math., **39** (1986), 51-68.