

Spacetime Deformations and Electromagnetism in Material Media

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Abstract

This paper is intended to investigate the relation between electrodynamics in anisotropic material media and its analogous formulation in an spacetime, with non-null Riemann curvature tensor. After discussing the electromagnetism via chiral differential forms, we point out the optical activity of a given material medium, closely related to topological spin, and the Faraday rotation, associated to topological torsion. Both quantities are defined in terms of the magnetic potential and the electric and magnetic fields and excitations. We revisit some properties of material media and the associated Green dyadics. Some related features of ferrite are also investigated. It is well-known that the constitutive tensor is essentially equivalent to the Riemann curvature tensor. In order to investigate the propagation of electromagnetic waves in material media, we prove that it is analogous to consider the electromagnetic wave propagation in the vacuum, but this time in a curved spacetime, which is obtained by a deformation of the Lorentzian metric of Minkowski spacetime. Spacetime deformations leave invariant the form of Maxwell equations. Also, there exists a close relation between Maxwell equations in curved spacetime and in an anisotropic material medium, indicating that electromagnetism and spacetime properties are deeply related. For instance, the equations of holomorphy in Minkowski spacetime are essentially Maxwell equations in vacuum. Besides, the geometrical aspects of wave propagation can be described by an effective geometry which represents a modification of the Lorentzian metric of Minkowski spacetime, i.e., a kind of spacetime deformation.

Key words: electromagnetism, constitutive tensor, Green dyadic, optical activity, Faraday rotation, spacetime deformations.

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Introduction

The metric-free formulation of electromagnetism is an old concept [1, 6, 9, 10, 22, 28, 32, 33, 34, 35, 36, 38, 40], and it is more natural, correct, precise and geometrically sensible if differential forms, intrinsically endowed with chirality [16, 28], are used [13, 38]. The excitations¹ $\mathbf{D}(x)$, $\mathbf{H}(x)$ (respectively the electric displacement and the magnetic field) and the fields $\mathbf{E}(x)$, $\mathbf{B}(x)$ (respectively the electric field and the magnetic induction) are naturally described in a spacetime destituted of a metric, which is introduced only when the constitutive relations are to be defined. The metric-free (and affine-free) formulation of electrodynamics brings a geometric character and a clear physical interpretation. But if we want to relate the Faraday excitation 2-form field $G(x)$ and the electromagnetic field strength 2-form field $F(x)$, we need to consider a constitutive tensor χ , that gives the relation $G(x) = \chi F(x)$. Equivalently, spacetime must be endowed with a metric, since the constitutive law depends on the spacetime metric [9].

The constitutive tensor (CT) is more than a relation between $F(x)$ and $G(x)$, describing physical intrinsic properties of matter or spacetime. It is essentially the Hodge star operator [15] that, besides the well-known duality between k -forms and $(n - k)$ -forms in a n -dimensional vector space (endowed with a metric), changes the parity of differential forms, but with an additional information about the medium structure [4]. In this sense, CT describes the properties of spacetime (magnetic) permeability and (electric) permittivity, since a general medium can be arbitrarily anisotropic. Under this viewpoint, a CT immediately brings a light-cone conformal structure to spacetime [15]. Formally, the link between CT and spacetime structure is expected, since CT and the Riemann curvature tensor have the same mathematical properties. A CT can reveal precious informations about spacetime, for example, the CT scalar curvature is identically null in any medium possessing central symmetry [38].

In this paper the CT χ that describes any linear (in particular, crystalline) media, in the general case presenting optical activity, is expressed by a conformal transformation of the vacuum CT χ_0 . The metric associated to the medium is derived from the CT (up to a conformal factor) and vice-versa [20, 22, 23, 25]. In this sense, the Lorentzian metric of Minkowski spacetime, associated with χ_0 , is deformed into a general metric of a riemannian spacetime, related to χ .

We prove that in order to completely describe the CT of any linear medium presenting natural optical activity, we only need the matrix γ , that describes the optical

¹We follow the nomenclature given in [22].

activity of such a medium, and χ_0 . In the particular investigation of crystalline media, we describe the constitutive tensor associated to the 32 crystal classes presenting natural optical activity uniquely from χ_0 , i.e., from the spacetime metric, since χ_0 can be written as second order metric tensor combination [20, ?]. This paper is organized as follows: in Sec. 1 we review the geometric description of fields and excitations in electromagnetic theory, well-established originally in the papers by Maxwell [21], Schönberg [10], Hehl [22], Kiehn [32, 33, 34, 35, 36], Post [38] and Jancewicz [27, 28], among others. The Maxwell equations carry information about the nature of the different fields and excitations, and the theory arises with an essential geometric character, if the fields and excitations are correctly interpreted, as in [22, 27, 28, 38], as even and odd differential form fields. In Sec. 2 we revisit the main features of electromagnetism in anisotropic material media. The Green dyadic is obtained and we treat the particular case of a electrically anisotropic material media. Plane waves are investigated, illustrating the present approach. Also, optically active media are treated. For instance, ferrite is explicitly studied. In Sec. 3 Maxwell equations in curved spacetime and in an anisotropic material medium are proved to be equivalent. It sheds some new light on the differential geometric aspects of electromagnetic wave propagation, that can be described by a geometry which represents a modification of the metric in Minkowski spacetime. We prove that all the information contained in the constitutive tensor associated to any linear media (arbitrarily describing optical activity), is precisely given by the *vacuum* constitutive tensor and by the matrix that describes the optical activity associated with the medium. Such matrix is essentially viewed as a perturbation of the original constitutive tensor that does not describe optically active media.

1 Electromagnetism in the exterior algebra

Heretofore $\Omega^k(M)$ denotes the space of k -form fields defined on a manifold M . Given the map $\mathbf{E} : M \rightarrow \mathbb{R}^3$, the electric field $\mathbf{E}(x)$, $x \in M$, is an even 1-form field ($\mathbf{E}(x) \in \Omega^1(\mathbb{R}^3)$), since $\mathbf{E}(x)$ is a linear map from the infinitesimal vector $d\mathbf{r}(x)$ to the infinitesimal scalar potential $dV(x)$, given by $dV(x) = -\mathbf{E}(x) \cdot d\mathbf{r}(x)$. The physical dimension of $\mathbf{E}(x)$ in the SI, $[\mathbf{E}(x)] = \text{Vm}^{-1}$, agrees with this interpretation. Analogously the magnetic induction $\mathbf{B}(x)$ is an even 2-form field ($\mathbf{B}(x) \in \Omega^2(\mathbb{R}^3)$), since $\mathbf{B}(x)$ is a linear map from the infinitesimal bivector $d\mathcal{S}(x)$ to the infinitesimal scalar $d\phi(x)$. Explicitly we have $d\phi(x) = -\mathbf{B}(x) \cdot d\mathcal{S}(x)$, where $\phi(x)$ is the magnetic flux. The physical dimension $\mathbf{B}(x)$ in the SI, $[\mathbf{B}(x)] = \text{Wbm}^{-2} = \text{T}$ (= Tesla), again agrees with such an interpretation [21, 28].

From now on we call an even (odd) differential form field the one that doesn't (does) change sign under parity transformations² [16, 28]. Even form fields are elements of $\Omega_+(M)$, hereon simply denoted by $\Omega(M)$, while odd form fields are elements of $\Omega_-(M)$.

²A parity transformation is defined in \mathbb{R}^n as the inversion of an odd number of basis vectors of \mathbb{R}^n .

Such forms are called *chiral* differential forms.

1.1 Homogeneous Maxwell equations and potentials

The differential operator $d : \Omega_{\pm}^p(M) \rightarrow \Omega_{\pm}^{p+1}(M)$ does not change the differential forms parity. From now on we denote $\partial_{\alpha} = \partial/\partial\alpha$. We also adopt natural units, in particular the speed of light $c = 1$, in what follows.

The first homogeneous Maxwell equation is an expression relating even 2-form fields:

$$d\mathbf{E}(x) + \partial_t\mathbf{B}(x) = 0. \quad (1)$$

The absence of magnetic monopoles can be described by the equation

$$d\mathbf{B}(x) = 0. \quad (2)$$

Eqs.(1) and (2) are the homogeneous Maxwell equations. From eq.(2), using the Poincaré lemma³, there exists an even 1-form field $\mathbf{A}(x)$ (the magnetic potential) satisfying the relation

$$\mathbf{B}(x) = d\mathbf{A}(x). \quad (3)$$

Substituting in eq.(1), one obtains the expression $d\mathbf{E}(x) + \partial_t d\mathbf{A}(x) = 0$, or $d(\mathbf{E}(x) + \partial_t\mathbf{A}(x)) = 0$. Using the Poincaré lemma, there exists a scalar field potential $\Phi(x) \in \Omega^0(\mathbb{R}^3)$ such that $\mathbf{E}(x) + \partial_t\mathbf{A}(x) = -d\Phi(x)$, implying that

$$\mathbf{E}(x) = -\partial_t\mathbf{A}(x) - d\Phi(x). \quad (4)$$

1.2 Non-homogeneous Maxwell equations

The electric current density $\mathbf{j}(x)$ is an element of $\Omega_{-}^2(\mathbb{R}^3)$, an odd 2-form field, which changes sign under parity transformations. It is clear that $d\mathbf{j} \in \Omega_{-}^3(\mathbb{R}^3)$, and then the continuity equation (local form of electric charge conservation) can be written as

$$d\mathbf{j}(x) + \partial_t\rho(x) = 0, \quad (5)$$

where $\rho(x)$, the electric charge density, is an odd 3-form field. Obviously $d\rho(x) = 0$, and the Poincaré lemma again asserts that there exists $\mathbf{D}(x) \in \Omega_{-}^2(\mathbb{R}^3)$ such that

$$\rho(x) = d\mathbf{D}(x). \quad (6)$$

The 1-form field $\mathbf{D}(x)$ is called electric displacement. The unity of $\mathbf{D}(x)$ in the SI is $[\mathbf{D}(x)] = \text{Cm}^{-2}$ ($\text{C} = \text{Coulomb}$). Substituting in eq.(5), we have $d\mathbf{j}(x) + \partial_t(d\mathbf{D}(x)) = 0$,

³The Poincaré lemma asserts that if an open set $\mathcal{U} \subset \mathbb{R}^n$ is *star-shaped*, every closed form is exact in \mathcal{U} . ($\omega \in \Omega^k(M)$ is closed if $d\omega = 0$, and exact if there exists a form $\eta \in \Omega^{k-1}$ such that $\omega = d\eta$. An open set $\mathcal{U} \subset \mathbb{R}^n$ is star-shaped with respect to the origin if, for all $\mathbf{x} \in \mathcal{U}$, the line from the origin to \mathbf{x} is in \mathcal{U} .) The quotient between the group of the closed forms and the group of the exact forms defines the *de Rham cohomology group* [29], which is regarded as an obstruction to the global existence of closed forms.

and $d(\mathbf{j}(x) + \partial_t \mathbf{D}(x)) = 0$. There exists an odd 1-form field $\mathbf{H}(x) \in \Omega_-^1(\mathbb{R}^3)$ such that

$$\mathbf{j}(x) + \partial_t \mathbf{D}(x) = d\mathbf{H}(x). \quad (7)$$

It describes the Ampère-Oersted law. The SI unit of $\mathbf{H}(x)$ is $[\mathbf{H}(x)] = \text{Am}^{-1}$ (A = Ampère). The odd form fields $\mathbf{D}(x)$ and $\mathbf{H}(x)$ are potentials with sources $\rho(x)$ and $\mathbf{j}(x)$, respectively.

The Poynting vector $\mathbf{S}(x)$ describes the electromagnetic strenght energy flux density. It is possible to write

$$\mathbf{S}(x) = \mathbf{E}(x) \wedge \mathbf{H}^*(x), \quad (8)$$

where heretofore $*$ denotes the complex conjugate of a form field. From the algebraic viewpoint, the product above is the unique possibility, since quantities representing flux densities are described by odd differential 3-form fields [16, 19, 23, 28, 39], elements of $\Omega_-^3(\mathbb{R}^3)$. The electric ($\mathbf{w}_e(x)$) and magnetic ($\mathbf{w}_m(x)$) energy densities are elements of $\Omega_-^3(\mathbb{R}^3)$, expressed by

$$\mathbf{w}_e(x) = \frac{1}{2} \mathbf{E}(x) \wedge \mathbf{D}^*(x) = \frac{1}{2} \mathbf{D}(x) \wedge \mathbf{E}^*(x), \quad \mathbf{w}_m = \frac{1}{2} \mathbf{B}(x) \wedge \mathbf{H}^*(x) = \frac{1}{2} \mathbf{H}(x) \wedge \mathbf{B}^*(x). \quad (9)$$

The electromagnetic field energy density is written as [7, 28]

$$\mathbf{w} = \mathbf{w}_e + \mathbf{w}_m = \frac{1}{2} (\mathbf{E}(x) \wedge \mathbf{D}^*(x) + \mathbf{B}(x) \wedge \mathbf{H}^*(x)). \quad (10)$$

1.3 The Hodge star operator

It is well-known [41] that the vector spaces $\Omega^k(\mathbb{R}^3)$ and $\Omega^{3-k}(\mathbb{R}^3)$ have the same dimension, since $\dim \Omega^k(\mathbb{R}^3) = \binom{3}{k} = \binom{3}{3-k} = \dim \Omega^{3-k}(\mathbb{R}^3)$. The same result is valid to any n -dimensional space (see Appendix). Meanwhile, it does not exist any canonical isomorphism between $\Omega^k(\mathbb{R}^3)$ and $\Omega^{3-k}(\mathbb{R}^3)$. The isomorphism given by the *Hodge star operator* $\star : \Omega^k(V) \rightarrow \Omega^{n-k}(V)$ [29, 41] always satisfies $\star\star = \pm id$. The contraction is a generalization of the interior product, and it can be written in terms of the Hodge star operator and the exterior product, as

$$\psi \lrcorner \phi = \star((\star\phi) \wedge \psi), \quad (11)$$

where $\psi, \phi \in \Omega(\mathbb{R}^3)$. For more details see, e.g., [22, 41, 42].

Denoting $g = g_{ij} dx^i \otimes dx^j$ a metric over V and $g^{-1} : V^* \times V^* \rightarrow \mathbb{R}$, with components g^{ij} the metric over the dual space V^* , the action of the star Hodge operator \star in a p -form is explicitly expressed as

$$\star dx^{i_1} \wedge \dots \wedge dx^{i_p} = g^{i_1 j_1} \dots g^{i_p j_p} \varepsilon_{j_1 \dots j_n} \frac{\sqrt{|\hat{g}|}}{(n-p)!} dx^{j_{p+1}} \wedge \dots \wedge dx^{j_n} \quad (12)$$

where ε denotes the Levi-Civita tensor, $\hat{g} := \det(g)$ and $n = \dim V$. We say that a hermitian matrix $A \in \text{End}(V)$ is positive defined if for any non-null vector v in a complex n -dimensional, we have

$$v^* A v > 0. \quad (13)$$

Now, the electric star operator \star_ε and the magnetic star operator \star_h are defined respectively by eq.(12) with $g^{ij} = \frac{\varepsilon_{ji}^*}{\det(\varepsilon_{ij})}$ and $g^{ij} = \frac{\mu_{ji}^*}{\det(\mu_{ij})}$, respectively. We can prove that \star_ε and \star_h satisfy $\star_\varepsilon = \star\varepsilon = \varepsilon\star$ and $\star_h = \star\mu = \mu\star$, and if in eq.(12) we substitute $g_{ij} = \delta_{ij}$, then $\star_\varepsilon\star = \star\star_\varepsilon = \varepsilon$ and $\star_h\star = \star\star_h = \mu$. [7]. By eq.(12), since $E \in \Omega^1(V)$ and $B \in \Omega^2(V)$, then

$$\star_\varepsilon(E_1dx + E_2dy + E_3dz) = (\varepsilon E)_1dy \wedge dz + (\varepsilon E)_2dz \wedge dx + (\varepsilon E)_3dx \wedge dy, \quad (14)$$

where $(\varepsilon E)_i = \sum_{j=1}^3 \varepsilon_{ij} E_j$, and

$$\star_h(B_1dy \wedge dz + B_2dz \wedge dx + B_3dx \wedge dy) = \sum_{i=1}^3 (\mu^{-1}B)_i dx^i, \quad (15)$$

where $(\mu^{-1}B)_i = \sum_{j=1}^3 (\mu^{ji})^* B_j$.

1.4 The Poynting theorem

If we take the \mathbb{C} -conjugation of eqs.(1) and (7) and respectively multiply by $\mathbf{E}^*(x)$ and $\mathbf{H}^*(x)$ we obtain

$$\begin{aligned} \mathbf{E}(x) \wedge d\mathbf{H}^*(x) - \mathbf{H}^*(x) \wedge d\mathbf{E}(x) &= \left[\mathbf{E}(x) \wedge (\partial_t \star \varepsilon \mathbf{E}^*(x)) + \star \mu^{-1} \mathbf{B}(x) \wedge (\partial_t \mathbf{B}^*(x)) \right] \\ &= \star \left[\mathbf{E}(x) \lrcorner (\partial_t \varepsilon \mathbf{E}^*(x)) + (\mu^{-1} \mathbf{B}(x)) \lrcorner (\partial_t \mathbf{B}^*(x)) \right] \end{aligned} \quad (16)$$

Eq.(16) can be written in a coordinate system as

$$\frac{1}{2} \partial_t \left(\mathbf{E}(x) \lrcorner (\varepsilon \mathbf{E}^*(x)) + \mathbf{B}(x) \lrcorner (\mu^{-1} \mathbf{B}^*(x)) \right) dx \wedge dy \wedge dz.$$

and from the expression

$$-d\mathbf{S}(x) = \mathbf{H}^*(x) \wedge d\mathbf{E}(x) - \mathbf{E}(x) \wedge d\mathbf{H}^*(x), \quad (17)$$

it follows that

$$d\mathbf{S}(x) = \partial_t \mathbf{w}(x), \quad (18)$$

the so-called *Poynting theorem* [2, 7, 13, 14].

1.5 Electromagnetic Intensity and Excitation

The electromagnetic field strength $F(x) \in \Omega^2(\mathbb{R}^{1,3})$ is an even 2-form in $\mathbb{R}^{1,3}$, also called the *Faraday 2-form field* [19]. If an arbitrary, but fixed, time vector is chosen in $\mathbb{R}^{1,3}$, we can split spacetime in space plus time. Then it is possible to use $\mathbf{E}(x)$ and $\mathbf{B}(x)$ to describe $F(x)$ as

$$F(x) = \mathbf{B}(x) + \mathbf{E}(x) \wedge dt. \quad (19)$$

The electromagnetic excitation $G(x) \in \Omega^2_-(\mathbb{R}^{1,3})$ can also be considered as an odd 2-form field given by [23, 28]

$$G(x) = \mathbf{D}(x) - \mathbf{H}(x) \wedge dt. \quad (20)$$

Eqs.(1,2) can be summarized as

$$dF(x) = 0, \quad (21)$$

and eqs.(6,7) are synthetically written as

$$dG(x) = J(x), \quad (22)$$

when the odd 3-form current density field $J(x) = \rho(x) - \mathbf{j}(x) \wedge dt$ is defined [28]. If we admit primarily eq.(21), the electric and magnetic fields are only defined after a spacetime splitting.

1.6 Vacuum constitutive relations

Hereon it is assumed implicitly that the Hodge star operator *changes* the parity of the differential forms⁴.

Constitutive relations are written as

$$G(x) = \star F(x). \quad (23)$$

This relation can be expressed in the vacuum, after a spacetime splitting, as:

$$\mathbf{D}(x) = \varepsilon_0 \star \mathbf{E}(x), \quad \mathbf{B}(x) = \mu_0 \star \mathbf{H}(x), \quad (24)$$

where ε_0 denotes the vacuum electric permittivity and μ_0 denotes the vacuum magnetic permeability. From eq.(21) it is possible to find $A(x) \in \Omega^1(\mathbb{R}^{1,3})$ such that

$$F(x) = dA(x). \quad (25)$$

The even 1-form field $A(x)$ denotes the well-known electromagnetic potential. In components, eq.(25) is written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (26)$$

Eq.(25) is invariant under the maps $A(x) \mapsto A(x) + \omega(x)$, $\omega(x) \in \Omega^1(\mathbb{R}^{1,3})$ such that $d\omega(x) = 0$. In particular eq.(25) is invariant when $A(x) \mapsto A(x) + d\lambda(x)$, $\lambda(x) \in \Omega^1(\mathbb{R}^{1,3})$. The existence of form fields that are closed, but *not* exact, gives rise to the physical monopole and solitons in fluids, concerning paramount and striking applications such as superconductivity, topological defects and turbulent non-equilibrium thermodynamics of fluids, exhaustively investigated by Kiehn [32, 33, 34, 35, 36].

One [36] defines the odd 3-form field topological spin $S(x) = A(x) \wedge G(x) \in \Omega^3_-(\mathbb{R}^{1,3})$ and the even 3-form field topological torsion $T(x) = A(x) \wedge F(x) \in \Omega^3_+(\mathbb{R}^{1,3})$. It can be shown that optical activity is closely related to topological spin, while Faraday rotation is associated to topological torsion [36].

⁴This Hodge star operator is, *de facto*, the composition of the Hodge star operator with a pseudoscalar [47, 48, 49]. This new operator is then able to lead odd (even) form fields to even (odd) ones. (By abuse of notation we also denote this new operator by \star .)

Under a spacetime splitting it can be seen that

$$\begin{aligned} T(x) = A(x) \wedge F(x) &= (\mathbf{A}(x) - \phi(x)dt) \wedge (\mathbf{B}(x) + \mathbf{E}(x) \wedge dt) \\ &= \mathbf{A}(x) \wedge \mathbf{B}(x) + (\mathbf{A}(x) \wedge \mathbf{E}(x) - \phi(x)\mathbf{B}(x))dt \end{aligned} \quad (27)$$

and

$$\begin{aligned} S(x) = A(x) \wedge G(x) &= (\mathbf{A}(x) - \phi(x)dt) \wedge (\mathbf{D}(x) - \mathbf{H}(x) \wedge dt) \\ &= \mathbf{A}(x) \wedge \mathbf{D}(x) + (\mathbf{A}(x) \wedge \mathbf{H}(x) - \phi(x)\mathbf{D}(x))dt \end{aligned} \quad (28)$$

Kiehn [36] shows that $T(x)$ is related to the helicity, while $S(x)$ is associated to chirality of the electromagnetic fields. The 3-form field energy-momentum, is defined if an arbitrary direction \mathbf{e}_i is chosen:

$$U_i(x) = \frac{1}{2}[F(x) \wedge (\mathbf{e}_i \lrcorner G(x)) - G(x) \wedge (\mathbf{e}_i \lrcorner F(x))]. \quad (29)$$

The 3-form field energy-momentum is invariant under pseudodual maps $F(x) \mapsto \varphi(x)G(x)$ and $G(x) \mapsto -F(x)/\varphi(x)$, where $\varphi(x)$ is an arbitrary scalar field non-null in all points of $\mathbb{R}^{1,3}$.

2 Revisiting electromagnetism in non-homogeneous media

In the last decade, a lot of manuscripts have been concerning electrodynamics in material media via differential forms. For instance, see [3, 4, 5, 7, 27].

For permeability and permittivity tensors such that the product $\varepsilon^t \mu^{-t}$ is diagonalizable, the expression for the Green diadic [4, 5] is given by

$$g = \frac{\det \mu}{4\pi\tilde{r}} \begin{pmatrix} \exp(im_1\tilde{r}) & 0 & 0 \\ 0 & \exp(im_2\tilde{r}) & 0 \\ 0 & 0 & \exp(im_3\tilde{r}) \end{pmatrix}, \quad (30)$$

where $\tilde{r} = \sqrt{\det \mu} \left(\frac{x^2}{\mu_1} + \frac{y^2}{\mu_2} + \frac{z^2}{\mu_3} \right)^{\frac{1}{2}}$, and m_1, m_2, m_3 denote eigenvalues associated with the matrix $\varepsilon^t \mu^{-t}$, such that $Re m_i > 0, i=1,2,3$. μ_1, μ_2, μ_3 denote eigenvalues associated with μ .

The diadic given by eq.(30) can be immediately written as the tensor product [4, 5]:

$$g = \frac{\det \mu}{4\pi\tilde{r}} \left(e^{im_1\tilde{r}} dx^1 \otimes dx^2 + e^{im_2\tilde{r}} dy^1 \otimes dy^2 + e^{im_3\tilde{r}} dz^1 \otimes dz^2 \right). \quad (31)$$

2.1 Material media

A medium is completely isotropic if the electric permittivity ε and the magnetic permeability μ can be written, respectively, as $\mu = \text{diag}(\mu_1, \mu_1, \mu_1)$ and $\varepsilon = \text{diag}(\varepsilon_1, \varepsilon_1, \varepsilon_1)$. If the elements of μ and ε in an anisotropic medium are hermitian,

such medium is called electrically or magnetically girotropic. For instance, a plasma with static magnetic field in the z axis

$$\varepsilon = \begin{pmatrix} \varepsilon_1 & -i\varepsilon_p & 0 \\ i\varepsilon_p & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \quad (32)$$

is electrically girotropic. A magnetic girotropic medium is given by

$$\mu = \begin{pmatrix} \mu_1 & -i\mu_2 & 0 \\ i\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_z \end{pmatrix}. \quad (33)$$

2.2 Electrically anisotropic media

Suppose that, in a given material medium, $\varepsilon = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $\mu = \mu_1 I$, where I denotes the identity matrix. It is immediate that $\tilde{r} = \mu_1 r$ and $\det \mu = \mu_1^3$, from where it can be seen that $\varepsilon \mu^{-t} = \mu_1^{-1} \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$. Since the Green form g is given by [4]

$$g = [(\det \mu)^{-1} k^t \mu k I - \omega^2 \varepsilon \mu^{-t}]^{-1}, \quad (34)$$

then the diagonal components of g are given by

$$\begin{aligned} g_{jj}(k) &= \left[\frac{k^2}{\mu_1^2} - \omega^2 \frac{\varepsilon_j}{\mu_1} \right]^{-1} \\ &= \frac{\mu_1^2}{k^2 - \omega^2 \varepsilon_j \mu_1}. \end{aligned} \quad (35)$$

It must be emphasized that the Green form in eqs.(34,35) does *not* describe the physical space, since it is related to the reciprocal space. A physical solution of eq.(34) is given by

$$g = \frac{\mu_1^2}{4\pi r} \begin{pmatrix} \exp(i\omega\sqrt{\varepsilon_1\mu_1}r) & 0 & 0 \\ 0 & \exp(i\omega\sqrt{\varepsilon_2\mu_1}r) & 0 \\ 0 & 0 & \exp(i\omega\sqrt{\varepsilon_3\mu_1}r) \end{pmatrix} \quad (36)$$

and then the Green diadic is given by

$$\begin{aligned} g(\vec{r}_1, \vec{r}_2) &= \frac{\mu_1^2}{4\pi r} \left(\exp(i\omega\sqrt{\varepsilon_1\mu_1}r) dx_1 \otimes dx_2 + \exp(i\omega\sqrt{\varepsilon_2\mu_1}r) dy_1 \otimes dy_2 \right. \\ &\quad \left. + \exp(i\omega\sqrt{\varepsilon_3\mu_1}r) dz_1 \otimes dz_2 \right) \end{aligned} \quad (37)$$

where $r = \|\vec{r}_1 - \vec{r}_2\|$. This equation is originally obtained by Warnick [4].

In the particular case of an uniaxial medium, with $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $\varepsilon_3 = \varepsilon_z$, eq.(37) is led to

$$\begin{aligned} g(\vec{r}_1, \vec{r}_2) &= \frac{\mu_1^2}{4\pi r} \left(\exp(i\omega\sqrt{\varepsilon\mu_1}r) dx_1 \otimes dx_2 + \exp(i\omega\sqrt{\varepsilon\mu_1}r) dy_1 \otimes dy_2 \right. \\ &\quad \left. + \exp(i\omega\sqrt{\varepsilon_z\mu_1}r) dz_1 \otimes dz_2 \right) \end{aligned} \quad (38)$$

In any material isotropic medium we have $\varepsilon_z = \varepsilon$, and the vacuum is obtained when $\varepsilon \mapsto \varepsilon_0$ and $\mu \mapsto \mu_0$. In this case,

$$g = \frac{\mu_0^2}{4\pi r} I \quad (39)$$

2.3 Plane waves

Heretofore we denote $\mathbf{E} = \mathbf{E}(x), \mathbf{B} = \mathbf{B}(x), \dots$, in order to simplify the notation to be used. Suppose that a plane wave propagates in the s direction and let the electric field be expressed by

$$\mathbf{E} = \mathbf{E}_0 \exp(i(k\vec{s} \cdot \vec{r} - \omega t)). \quad (40)$$

Denoting $n = k/\omega = 1/v$, eqs.(1) and (7) gives

$$\mathbf{D} = -n\mathbf{H} \wedge s, \quad \mathbf{B} = n\mathbf{E} \wedge s. \quad (41)$$

Using the constitutive relation $\mathbf{B} = \star_h \mathbf{H}$, eqs.(41) give

$$\begin{aligned} \mathbf{D} &= \frac{n^2}{\mu} \star (s \wedge \mathbf{E}) \wedge s \\ &= \frac{n^2}{\mu} \star [\mathbf{E} - s(s \cdot \mathbf{E})]. \end{aligned} \quad (42)$$

By abuse of notation, here μ denotes the magnetic permeability, a real number, instead of the tensor μ . Define the component \mathbf{E}_\perp of \mathbf{E} such that $\mathbf{E}_\perp \cdot s = 0$. Then, since \mathbf{E}_\perp is in the plaquette defined by $\mathbf{E} \wedge s$, eq.(42) is written as

$$\mathbf{D} = \frac{n^2}{\mu} \mathbf{E}_\perp. \quad (43)$$

2.4 Fresnel equations

Now let $\varepsilon_1, \varepsilon_2$ and ε_3 be the eigenvalues of the matrix representation of ε . From eq.(42) we have

$$E_i = \frac{n^2 s_i}{n^2 - \mu \varepsilon_i} \mathbf{E} \cdot s, \quad (44)$$

from where we obtain

$$\frac{s_1^2}{n^2 - \mu \varepsilon_1} \mathbf{E} \cdot s + \frac{s_2^2}{n^2 - \mu \varepsilon_2} \mathbf{E} \cdot s + \frac{s_3^2}{n^2 - \mu \varepsilon_3} \mathbf{E} \cdot s = \frac{1}{n^2} \mathbf{E} \cdot s \quad (45)$$

$$\sum_{i=1}^3 \frac{s_i^2}{n^2 - \mu \varepsilon_i} = \frac{1}{n^2}. \quad (46)$$

Since $s \cdot s = 1$, then $\sum_{i=1}^3 s_i^2 = 1$, and from eq.(46) it follows that

$$\sum_{i=1}^3 \frac{s_i^2}{\frac{1}{n^2} - \frac{1}{\mu \varepsilon_i}} = 0. \quad (47)$$

If we define the so-called principal propagation velocity $v_i := (\mu \varepsilon_i^{-1/2}, 0, 0)$, eq.(47) is lead to

$$\frac{s_1^2}{v^2 - v_1^2} + \frac{s_2^2}{v^2 - v_2^2} + \frac{s_3^2}{v^2 - v_3^2} = 0. \quad (48)$$

Eqs.(46), (47) and (48) are called Fresnel wave equations [14].

2.5 Ferrite

Ferrite is a material medium defined by

$$\mu = \mu_0 \begin{pmatrix} \alpha & -i\beta & 0 \\ i\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix} =: \mu_0 \mu_r \quad (49)$$

From now on we consider $\beta < \alpha$, and it is easy to see that $\det \mu = \mu_0^3(\alpha^2 - \beta^2)\gamma$ and that the eigenvalues of μ_r are $\alpha + \beta$, $\alpha - \beta$ and γ . From the expression

$$\vec{r} := (\alpha^2 - \beta^2)^{\frac{1}{2}} \gamma^{\frac{1}{2}} \mu_0 \left(\frac{x}{\sqrt{\alpha + \beta}}, \frac{y}{\sqrt{\alpha - \beta}}, \frac{z}{\sqrt{\gamma}} \right),$$

it follows that

$$\tilde{r} := \|\vec{r}\| = \mu_0 (\alpha^2 - \beta^2)^{\frac{1}{2}} \gamma^{\frac{1}{2}} \left(\frac{x^2}{\alpha + \beta} + \frac{y^2}{\alpha - \beta} + \frac{z^2}{\gamma} \right)^{1/2}. \quad (50)$$

The square roots of the eigenvalues of $\omega^2 \varepsilon \mu^{-t}$ are given by

$$m_1 = \omega \sqrt{\frac{\varepsilon}{(\alpha + \beta)\mu_0}}, \quad m_2 = \omega \sqrt{\frac{\varepsilon}{(\alpha - \beta)\mu_0}}, \quad m_3 = \omega \sqrt{\frac{\varepsilon}{\gamma\mu_0}}, \quad (51)$$

where ε is the diagonal element of the ferrite permittivity tensor. g is given by

$$g = \frac{\mu_0^2 (\alpha^2 - \beta^2)^{\frac{1}{2}} \gamma^{\frac{1}{2}}}{4\pi r'} \text{diag}(e^{i\omega \sqrt{\varepsilon\mu_0(\alpha-\beta)\gamma} r'}, e^{i\omega \sqrt{\varepsilon\mu_0(\alpha+\beta)\gamma} r'}, e^{i\omega \sqrt{\varepsilon\mu_0(\alpha^2-\beta^2)\gamma} r'}), \quad (52)$$

where $\vec{r} = \sqrt{(\alpha^2 - \beta^2)\gamma}\mu_0 \vec{r}'$ and $\vec{r}' = \left(\frac{x}{\sqrt{\alpha+\beta}}, \frac{y}{\sqrt{\alpha-\beta}}, \frac{z}{\sqrt{\gamma}} \right)$. When $\beta = 0$ and $\alpha = \gamma$ in eq.(52) it follows that

$$g = \frac{\mu_0^2 \alpha^2}{4\pi r} \exp(i\omega \sqrt{\varepsilon\mu_0 \alpha} r) I, \quad (53)$$

where $r_i = \sqrt{x_i^2 + y_i^2 + z_i^2}$. Denoting $\mu = \alpha\mu_0$ eq.(53) can be written as

$$g = \frac{\mu^2}{4\pi r} \exp(i\omega \sqrt{\varepsilon\mu} r) I, \quad (54)$$

which is the well-known expression for an isotropic medium.

2.6 Faraday rotations

From eq.(7) and eq.(40), it follows that

$$\star d\mathbf{H} = -i\omega \varepsilon \mathbf{E}. \quad (55)$$

Taking the differential of the last equation we obtain

$$\begin{aligned} d \star d\mathbf{H} &= \omega^2 \varepsilon \star_h \mathbf{H} \\ &= \omega^2 \varepsilon \mu \mathbf{H}. \end{aligned} \quad (56)$$

Now, if we solve eqs.(41) for \mathbf{B} , we obtain

$$\mathbf{B} = \frac{n^2}{\varepsilon} \star_h \mathbf{H}_\perp, \quad (57)$$

where $\mathbf{H}_\perp = \mathbf{H} - (s \cdot \mathbf{H})s$. In components, the field given by eq.(57) is written as

$$B_i = \frac{n^2 \mu_i}{n^2 - \varepsilon \mu_i} (s \cdot H) s_i. \quad (58)$$

From eq.(58) the component of \mathbf{H} in the z -direction is zero, and if we make the assumption that $\mathbf{H} = (H_1 dx + H_2 dy) e^{i(kz - \omega t)}$, eq.(56) gives

$$\omega^2 \varepsilon \mu_0 \begin{pmatrix} \alpha & -i\beta \\ i\beta & \alpha \end{pmatrix} = k^2 \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \quad (59)$$

which solution is given by

$$k_+^2 = \omega^2 \varepsilon \mu_0 (\alpha + \beta), \quad k_-^2 = \omega^2 \varepsilon \mu_0 (\alpha - \beta), \quad (60)$$

describing two (left- and right-handed) circularly polarized plane waves. Now, substituting eqs.(60) in eq.(59) it follows that $H_1 = \pm iA$, if $H_2 = A$, $A \in \mathbb{C}$. The general solution of the system is

$$\mathbf{H} = \left[-ic_1 A e^{ik_+z} + ic_2 A e^{ik_-z} \right] e^{-i\omega t} dx + \left[-c_1 A e^{ik_-z} + c_2 A e^{ik_+z} \right] e^{-i\omega t} dy. \quad (61)$$

Choosing $c_1 = -c_2 = \frac{i}{2}$, we obtain

$$H_1(z) = \frac{A}{2} e^{ik_+z} + \frac{A}{2} e^{ik_-z}, \quad H_2(z) = i \frac{A}{2} e^{ik_+z} - i \frac{A}{2} e^{ik_-z}, \quad (62)$$

which can be written as

$$\begin{aligned} H_1(z) &= A \cos\left(\frac{k_+ - k_-}{2} z\right) \exp(-i(k_+ + k_-)z/2), \\ H_2(z) &= A \sin\left(\frac{k_+ - k_-}{2} z\right) \exp(-i(k_+ + k_-)z/2). \end{aligned} \quad (63)$$

Let $\theta \in \mathbb{R}$ such that

$$\tan \theta = \frac{H_2(z)}{H_1(z)} = \tan\left(\frac{k_+ - k_-}{2} z\right). \quad (64)$$

It is immediate that

$$\theta_k = \left(\frac{k_+ - k_-}{2} z\right) + 2k\pi, \quad k \text{ is an integer.} \quad (65)$$

Restricting $\theta \in [0, 2\pi)$ it is clear that the phase difference between the left- and right-handed components is $2\theta_0$, where

$$\theta_0 = \frac{1}{2} z \omega \sqrt{\varepsilon \mu_0 \alpha} \left(\sqrt{1 + \frac{\beta}{\alpha}} - \sqrt{1 - \frac{\beta}{\alpha}} \right). \quad (66)$$

Consider k_\pm in eq.(60) given by a second-order approximation, i.e.,

$$k_\pm = \omega \varepsilon \mu_0 \left(1 \pm \frac{\alpha}{2\beta} + \frac{1}{8} \frac{\alpha^2}{\beta^2} + \mathcal{O}\left(\frac{\alpha}{\beta}\right)^3 \right). \quad (67)$$

Substituting in eq.(66) we have

$$\theta_0 = \frac{\beta}{2\alpha} z \omega \sqrt{\alpha \mu_0 \varepsilon}. \quad (68)$$

It shows the well-known result asserting that ferrite is indeed a non-reciprocal medium.

3 Material media viewed as spacetime deformations in vacuum

In the formalism that describes the electromagnetism in linear media, e.g. [23, 39], the dual Hodge operator action is equivalent to the constitutive χ tensor action on 2-form fields:

$$\star\Delta = \chi\Delta, \quad \Delta \in \Omega_{\pm}^2(\mathbb{R}^{1,3}). \quad (69)$$

If cartesian coordinates are introduced, eq.(69) is equivalent to

$$\star\Delta = \frac{1}{4}\epsilon_{\mu\nu\sigma\kappa}\chi^{\sigma\kappa\tau\rho}\Delta_{\tau\rho}dx^{\mu} \wedge dx^{\nu}. \quad (70)$$

3.1 The constitutive tensor

In this subsection we present and discuss the main results in, e.g., [38, 39], concerning the relation between the constitutive and the Riemann curvature tensors. In linear media, the 2-form electromagnetic intensity $F(x) \in \Omega^2(\mathbb{R}^{1,3})$ is related to the electromagnetic excitation $G \in \Omega_+^2(\mathbb{R}^{1,3})$ by the equation

$$F(x) = \chi G(x). \quad (71)$$

Using cartesian coordinates, $F(x)$ and $G(x)$ are expressed as

$$G(x) = \frac{1}{2}G_{\mu\nu}(x)dx^{\mu} \wedge dx^{\nu}, \quad F(x) = \frac{1}{2}F_{\mu\nu}(x)dx^{\mu} \wedge dx^{\nu} \quad (72)$$

and $F_{\mu\nu}(x)$ and $G_{\mu\nu}(x)$ are related by

$$G_{\mu\nu}(x) = \frac{1}{4}\epsilon_{\mu\nu\alpha\beta}\chi^{\alpha\beta\sigma\lambda}F_{\sigma\lambda}(x), \quad (73)$$

where ϵ is the Levi-Civita tensor. The symmetry

$$\chi^{\lambda\nu\sigma\kappa} = -\chi^{\lambda\nu\kappa\sigma}, \quad \chi^{\lambda\nu\sigma\kappa} = -\chi^{\nu\lambda\sigma\kappa}. \quad (74)$$

arises, since $F_{\mu\nu}(x)$ and $G_{\mu\nu}(x)$ are antisymmetric. Besides, the lagrangian density $\mathcal{L}(x) = G(x) \wedge F(x)$ is written as [38, 39]

$$\mathcal{L}(x) = \frac{1}{4}\chi^{\lambda\nu\sigma\kappa}F_{\lambda\nu}(x)F_{\sigma\kappa}(x). \quad (75)$$

From the relation

$$G^{\lambda\nu}(x) = 2\frac{\partial\mathcal{L}(x)}{\partial F_{\lambda\nu}(x)} = \frac{1}{2}\chi^{\lambda\nu\sigma\kappa}F_{\sigma\kappa}(x). \quad (76)$$

there exists the relation

$$\begin{aligned} \chi^{\lambda\nu\sigma\kappa} &= 2\frac{\partial^2\mathcal{L}(x)}{\partial F_{\lambda\nu}(x)\partial F_{\sigma\kappa}(x)} = 2\frac{\partial^2\mathcal{L}(x)}{\partial F_{\sigma\kappa}(x)\partial F_{\lambda\nu}(x)} \\ &= \chi^{\sigma\kappa\lambda\nu} \end{aligned} \quad (77)$$

For more details, see [39]. The number of independent coordinates is 21, (using eqs.(74,77)), which comes from the analogy to the Riemann curvature tensor. Only

the antisymmetric combinations are non-trivial. An order two antisymmetric tensor has exactly six components and then there would exist 36 components. Expressing χ as a 6×6 matrix ($\chi \in \text{Hom}(\mathbb{R}^6, \mathbb{R}^6)$) and using the bivectorial notation, (where the indices $I, J, \dots = 01, 02, 03, 23, 31, 12$ are defined) we can see from eq.(77) that the matrix χ^{IJ} is symmetric ($\chi^{IJ} = \chi^{JI}$), and there exists $(6 \times 7)/2$ independent components in χ . Using physical arguments, one can show that in uniform media we have the relation [38, 39]

$$\chi^{[\lambda\nu\sigma\kappa]} = 0. \quad (78)$$

In vacuum χ can be written as

$$\chi^{\lambda\nu\sigma\kappa} = Y_0 \sqrt{|\hat{g}|} (g^{\lambda\sigma} g^{\nu\kappa} - g^{\lambda\kappa} g^{\nu\sigma}), \quad (79)$$

where \hat{g} denotes the determinant of the metric of spacetime with components $g^{\mu\nu}$ and Y_0 is the vacuum admittance.

From the constitutive tensor χ Post defines two invariants:

$$\chi_1 = \chi_{\lambda\nu}{}^{\lambda\nu}, \quad (80)$$

which is called scalar curvature of the medium described by χ , and

$$\chi_2 = \epsilon_{\lambda\nu\sigma\kappa} \chi^{\lambda\nu\rho\tau} \chi^{\sigma\kappa\mu\alpha} \epsilon_{\rho\tau\mu\alpha}. \quad (81)$$

Post [38] proves that χ_2 is non-zero for any medium and $\chi_1 = \chi^{[\lambda\nu\sigma\kappa]}$ is identically null in any medium possessing central symmetry. The constitutive tensor is explicitly represented by:

$$\chi = \begin{pmatrix} -\epsilon & \gamma \\ \gamma^\dagger & \mu^{-1} \end{pmatrix} = \begin{array}{c|ccc|ccc} \chi^{\lambda\nu\sigma\kappa} & 01 & 02 & 03 & 23 & 31 & 12 \\ \hline & -E_1 & -E_2 & -E_3 & B_1 & B_2 & B_3 \\ \hline 01 D_1 & -\epsilon_{11} & -\epsilon_{12} & -\epsilon_{13} & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ 02 D_2 & -\epsilon_{21}^* & -\epsilon_{22} & -\epsilon_{23} & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 03 D_3 & -\epsilon_{31}^* & -\epsilon_{32}^* & -\epsilon_{33} & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ \hline 23 H_1 & \gamma_{11}^* & \gamma_{21}^* & \gamma_{31}^* & \zeta_{11} & \zeta_{12} & \zeta_{13} \\ 31 H_2 & \gamma_{12}^* & \gamma_{22}^* & \gamma_{32}^* & \zeta_{21}^* & \zeta_{22} & \zeta_{23} \\ 12 H_3 & \gamma_{13}^* & \gamma_{23}^* & \gamma_{33}^* & \zeta_{31}^* & \zeta_{32}^* & \zeta_{33} \end{array}$$

The matrix μ_{lk} is the magnetic permeability matrix, ϵ_{lk} is the electric permittivity matrix and γ_{lk} is a matrix that describes the electric and magnetic polarization effects. One can prove that in media possessing central symmetry, the matrix γ_{lk} is null [38, 39]. In isotropic media the relations

$$\gamma_{lk} \equiv 0, \quad \epsilon_{lk} = \epsilon_0 \delta_{lk}, \quad \zeta_{lk} = \mu_0^{-1} \delta_{lk}, \quad (82)$$

are satisfied. In this case, $\chi_1 = 0$ and $\chi_2 = -12\epsilon_0/\mu_0$ [38].

We shall study the light propagation in crystalline media presenting optical activity, which are characterized by 18 of the 32 classes [38, 43] presented in the table below. Each class is represented by a symmetry represented in the table:

1	C	9	C, z_2, x_2	17	C, z_4	25	z_6
2	–	10	z_3, x_2	18	z_4	26	z_3, x_2, E_z
3	C, z_2	11	z_3, E_x	19	S_z, x_2	27	z_3, E_z
4	E_z	12	C, z_3	20	S_z	28	C, x_4, y_4
5	z_2	13	z_3	21	C, z_6, x_2	29	x_4, y_4
6	C, z_2, x_2	14	C, z_4, x_2	22	z_6, x_2	30	S_x, S_y
7	z_2, x_2	15	z_4, x_2	23	z_6, E_x	31	C, x_2, y_2, S
8	z_2, E_x	16	z_4, E_x	24	C, z_6	32	x_2, y_2, S

C denotes central symmetry, S is the cyclic permutation of the indices, E_x is a reflection with respect to the yz plane (analogous definition for E_y and E_z), S_x is a rotation using the x axis, followed by a reflection related to the yz plane (And analogous definitions for S_y and S_z).

All crystal classes described in the above table present natural optical activity, and the corresponding respective matrices γ_{kl} , composing the tensor χ , are described below (the number before the matrices indicates the class number above described):

$$\begin{aligned}
& 2 \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}; \quad 4 \begin{pmatrix} 0 & 0 & \gamma_{13} \\ 0 & 0 & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & 0 \end{pmatrix}; \quad 5 \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{21} & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix}; \\
& 7 \begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix}; \quad 8 \begin{pmatrix} 0 & \gamma_{12} & 0 \\ \gamma_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad 10, 15, 22 \begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{11} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix}, \\
& 11, 16, 23 \begin{pmatrix} 0 & \gamma_{12} & 0 \\ -\gamma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad 13, 18, 25 \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ -\gamma_{12} & \gamma_{11} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix}; \quad 19 \begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & -\gamma_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \\
& 20 \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{12} & -\gamma_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad 29, 32 \begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{11} & 0 \\ 0 & 0 & \gamma_{11} \end{pmatrix}
\end{aligned}$$

The matrices corresponding to the classes 29,32 describe the chiral vacuum [36].

4 Arbitrary constitutive tensors from the vacuum CT

From now on we adopt the notation $F = F(x), G = G(x), \dots$, omitting the argument x .

► **Spectral Theorem:** There always exists a conformal transformation that diagonalizes the constitutive tensor χ ◀

Considering the splitting $\mathbb{R}^{1,3} \simeq \mathbb{R}^3 \times \mathbb{R}$, we can write

$$G = \chi F \iff \begin{pmatrix} \mathbf{D} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\varepsilon & \gamma \\ \gamma^\dagger & \mu^{-1} \end{pmatrix}_{6 \times 6} \begin{pmatrix} -\mathbf{E} \\ \mathbf{B} \end{pmatrix} \quad (83)$$

where $\varepsilon = \varepsilon_0 I$ and $\mu^{-1} = \mu_0^{-1} I$. By the theorem above, there exists a matrix Γ composed by the eigenvectors of χ such that $\Gamma^{-1} \chi \Gamma = \Lambda$ is a diagonal matrix.

4.1 The chiral vacuum

In order to illustrate the general approach, we firstly consider the chiral vacuum, described by the matrix

$$\begin{pmatrix} -\varepsilon & \gamma_\odot \\ \gamma_\odot^\dagger & \mu^{-1} \end{pmatrix}_{6 \times 6} \quad (84)$$

where $\gamma_\odot = \gamma_{11} I$. The matrix (84) has eigenvalues σ_1, σ_2 and eigenvectors $\{(0, 0, \sigma_1, 0, 0, 1), (0, \sigma_1, 0, 0, 1, 0), (\sigma_1, 0, 0, 1, 0, 0), (0, 0, \sigma_2, 0, 0, 1), (0, \sigma_2, 0, 0, 1, 0), (\sigma_2, 0, 0, 1, 0, 0)\}$, where

$$\sigma_{1,2} = \mu_0^{-1} - \varepsilon_0 \pm \frac{\sqrt{(\mu_0^{-1} + \varepsilon_0)^2 - 4\gamma_{11}^2}}{2\gamma_{11}}. \quad (85)$$

Then

$$\Gamma^{-1} \chi \Gamma = \Lambda = \begin{pmatrix} -\Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}_{6 \times 6} \quad (86)$$

where $\Sigma_1 = -\sigma_1 I$ and $\Sigma_2 = \sigma_2 I$. Denoting $\mathring{F} = \Gamma^{-1} F$ and $\mathring{G} = \Gamma^{-1} G$, we obtain

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{H} \end{pmatrix} = \chi \begin{pmatrix} -\mathbf{E} \\ \mathbf{B} \end{pmatrix} \Rightarrow \Gamma \begin{pmatrix} \mathring{\mathbf{D}} \\ \mathring{\mathbf{H}} \end{pmatrix} = \chi \Gamma \begin{pmatrix} -\mathring{\mathbf{E}} \\ \mathring{\mathbf{B}} \end{pmatrix} \Rightarrow \begin{pmatrix} \mathring{\mathbf{D}} \\ \mathring{\mathbf{H}} \end{pmatrix} = \Gamma^{-1} \chi \Gamma \begin{pmatrix} -\mathring{\mathbf{E}} \\ \mathring{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} -\Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} -\mathring{\mathbf{E}} \\ \mathring{\mathbf{B}} \end{pmatrix}. \quad (87)$$

Define the odd form fields $\mathring{\mathbf{D}}$ and $\mathring{\mathbf{H}}$ as

$$\mathring{\mathbf{D}} = \Sigma_1^{-1} \mathring{\mathbf{D}}, \quad \mathring{\mathbf{H}} = \Sigma_2^{-1} \mathring{\mathbf{H}}. \quad (88)$$

It follows that

$$\begin{pmatrix} \mathring{\mathbf{D}} \\ \mathring{\mathbf{H}} \end{pmatrix} = \begin{pmatrix} \Sigma_1^{-1} & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} -\mathring{\mathbf{D}} \\ \mathring{\mathbf{H}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\mathring{\mathbf{E}} \\ \mathring{\mathbf{B}} \end{pmatrix}, \quad (89)$$

and we prove that

$$\mathring{G} = \chi_0 \mathring{F}, \quad (90)$$

where χ_0 vacuum constitutive tensor⁵. We obtain for the chiral vacuum, after doing the inverse maps, the constitutive relation

$$G = \chi^\odot F \quad (91)$$

where

$$\chi^\odot = \Gamma \Lambda \chi_0 \Gamma^{-1} \quad (92)$$

⁵modulo dilation of the axis $\mathbf{e}_4, \mathbf{e}_5$ and \mathbf{e}_6 by μ_0 and contraction of $\mathbf{e}_1, \mathbf{e}_2$ and \mathbf{e}_3 by ε_0 .

Then the constitutive tensor χ° , related to the chiral vacuum, is completely described by the matrix γ and the vacuum constitutive tensor. We only used conformal maps in $\mathbb{R}^{1,3}$, which are elements of the group. This kind of structure in electromagnetism was discovered by Bateman [44], who was the first to observe that the Maxwell equations are invariant under the conformal group [36, 37].

4.2 Arbitrary linear media: crystalline media, optical activity, magnetic and dielectric Faraday effects

The method is fundamentally analogous to the chiral vacuum case. Consider an arbitrary linear media described by the matrix

$$\chi = \begin{pmatrix} -\varepsilon & \gamma \\ \gamma^\dagger & \mu^{-1} \end{pmatrix}_{6 \times 6}, \quad (93)$$

where

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}. \quad (94)$$

The matrix (93) has eigenvalues σ_A ($A = 1, 2, \dots, 6$). Then

$$\Gamma^{-1}\chi\Gamma = \Lambda = \begin{pmatrix} -\Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}_{6 \times 6} \quad (95)$$

where $\Sigma_1 = -\text{diag}(\sigma_1, \sigma_2, \sigma_3)$ and $\Sigma_2 = -\text{diag}(\sigma_4, \sigma_5, \sigma_6)$. Denoting $\mathring{F} = \Gamma^{-1}F$ and $\mathring{G} = \Gamma^{-1}G$, we obtain

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{H} \end{pmatrix} = \chi \begin{pmatrix} -\mathbf{E} \\ \mathbf{B} \end{pmatrix} \Rightarrow \Gamma \begin{pmatrix} \mathring{\mathbf{D}} \\ \mathring{\mathbf{H}} \end{pmatrix} = \chi \Gamma \begin{pmatrix} -\mathring{\mathbf{E}} \\ \mathring{\mathbf{B}} \end{pmatrix} \Rightarrow \begin{pmatrix} \mathring{\mathbf{D}} \\ \mathring{\mathbf{H}} \end{pmatrix} = \Gamma^{-1}\chi\Gamma \begin{pmatrix} -\mathring{\mathbf{E}} \\ \mathring{\mathbf{B}} \end{pmatrix} = \begin{pmatrix} -\Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} -\mathring{\mathbf{E}} \\ \mathring{\mathbf{B}} \end{pmatrix}. \quad (96)$$

Defining the vectors $\mathring{\mathbf{D}} = \Sigma_1^{-1}\mathring{\mathbf{D}}$, $\mathring{\mathbf{H}} = \Sigma_2^{-1}\mathring{\mathbf{H}}$, it follows that

$$\begin{pmatrix} \mathring{\mathbf{D}} \\ \mathring{\mathbf{H}} \end{pmatrix} = \begin{pmatrix} \Sigma_1^{-1} & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} -\mathring{\mathbf{D}} \\ \mathring{\mathbf{H}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\mathring{\mathbf{E}} \\ \mathring{\mathbf{B}} \end{pmatrix}. \quad (97)$$

This implies that

$$\mathring{G} = \chi_0 \mathring{F}, \quad (98)$$

where χ_0 is the vacuum constitutive relations. Calculating the inverse maps, we obtain for any medium the constitutive relation

$$G = \chi F \quad (99)$$

where

$$\chi = \Gamma\Lambda\chi_0\Gamma^{-1} \quad (100)$$

The constitutive tensor associated to the 32 crystal classes presenting natural optical activity is described uniquely from χ_0 , i.e., from the spacetime metric, since

$$\chi_0^{\lambda\nu\sigma\kappa} = Y_0 \sqrt{|g|} (g^{\lambda\sigma} g^{\nu\kappa} - g^{\lambda\kappa} g^{\nu\sigma}) [38, 23]$$

Using coordinates we write

$$\boxed{G_{\mu\nu} = \frac{Y_0}{4} \sqrt{|g|} \epsilon_{\mu\nu\alpha\beta} (\Gamma^\dagger)_\delta^\alpha \Gamma_\rho^\beta \Lambda_\theta^\sigma (g^{\delta\theta} g^{\rho\lambda} - g^{\delta\lambda} g^{\rho\theta}) F_{\sigma\lambda}} \quad (101)$$

Note that the expression above is the constitutive relation for any crystalline material, and it depends only of the matrix γ (given at the end of Sec. 3, for all crystal classes), that describes optical natural activity. Then it can be seen as the deformation of the metric of Minkowski spacetime into a metric of curved riemannian spacetime, since in order to describe the constitutive relations of any crystalline medium we only need the metric of Minkowski spacetime.

In particular, it is also possible to express, from the Lorentzian metric of Minkowski spacetime, the constitutive tensor associated to the dielectric and magnetic Faraday rotations (plasma and ferrite respectively), and the natural optical activity in arbitrary rotational symmetric media [39]. It is respectively given by the following matrices:

$$\begin{pmatrix} -\varepsilon_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon & -i\varepsilon_{23} & 0 & 0 & 0 \\ 0 & i\varepsilon_{23} & -\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\mu \end{pmatrix}, \quad \begin{pmatrix} -\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^{-1} & i\zeta_{23} \\ 0 & 0 & 0 & 0 & -i\zeta_{23} & \mu^{-1} \end{pmatrix}, \quad (102)$$

$$\begin{pmatrix} -\varepsilon & 0 & 0 & i\gamma_{11} & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & i\gamma_{11} & 0 \\ 0 & 0 & -\varepsilon & 0 & 0 & i\gamma_{11} \\ i\gamma_{11} & 0 & 0 & 1/\mu & 0 & 0 \\ 0 & -i\gamma_{11} & 0 & 0 & 1/\mu & 0 \\ 0 & 0 & -i\gamma_{11} & 0 & 0 & 1/\mu \end{pmatrix}, \quad (103)$$

where $\zeta_{ij} = (\mu^{-1})_{ij}$. Post proves [39] that electromagnetic waves propagate with phase velocity u given by

$$u = \pm((\varepsilon \pm \varepsilon_{23})\mu)^{-1/2}, \quad (104)$$

(dielectric Faraday rotation),

$$u = \pm \sqrt{\frac{\zeta \pm \zeta_{23}}{\varepsilon}}, \quad (105)$$

(magnetic Faraday rotation) and

$$u = \pm \frac{\gamma_{11}}{\varepsilon} \pm \sqrt{\frac{1}{\varepsilon\mu} + \frac{\gamma_{11}^2}{\varepsilon^2}}, \quad (106)$$

(natural optical activity).

In the whole process described in this subsection, we only have accomplished conformal transformations in $\mathbb{R}^{1,3}$.

Concluding Remarks

We investigated the relation between electrodynamics in anisotropic material media and its analogous formulation in an spacetime, with non-null Riemann curvature tensor. The propagation of electromagnetic waves in material media is proved to be analogous to consider the electromagnetic wave propagation in the vacuum, now in a curved spacetime, which is obtained by a deformation of the Lorentzian metric of Minkowski spacetime. Such process of performing deformations of the metric of Minkowski spacetime can be rigorously described using extensors. Also, there exists a close relation between Maxwell equations in curved spacetime and in an anisotropic material medium, indicating that electromagnetism and spacetime properties are deeply related. Besides, the geometrical aspects of wave propagation can be described by an effective geometry which represents a modification of the Lorentzian metric of Minkowski spacetime.

We discussed the optical activity of a given material medium, closely related to topological spin, and the Faraday rotation, associated to topological torsion. Both quantities are defined in terms of the magnetic potential and the electric and magnetic fields and excitations. The existence of form fields that are closed, but *not* exact, gives rise to the monopole and solitons in fluids, concerning topological defects and turbulent non-equilibrium thermodynamics, exhaustively investigated by Kiehn [36]. In a forthcoming paper, since the integral over \mathbb{R}^3 of the topological torsion spatial component, introduced by eq.(27), is the writhe of a framed oriented link, it is possible to investigate link invariants in gauge theory, from the knot theory viewpoint.

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