A New Fatigue Life Model Based on the Family of Skew-elliptical Distributions

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ABSTRACT *Fatigue is structural damage produced by cyclic stress and tension. An important statistical model for fatigue life is the Birnbaum-Saunders distribution, which was developed to model ruptured lifetimes of metals that had been subjected to fatigue. This model has been previously generalized and in this article we extend it starting from a skew-elliptical distribution. In this work we found the probability density, reliability, and hazard functions; as well as its moments and variation, skewness, and kurtosis coefficients. In addition, some properties of this new distribution were found.*

KEYWORDS: Birnbaum-Saunders distribution, life distributions, material fatigue, reliability analysis, skew-elliptical distributions.

1 Background

Fatigue is structural damage which is produced when material has been subjected to cyclic stress and tension. Valluri [24, 1963] proposed a theory for material fatigue based on a three stage process: (i) the beginning of an imperceptible fissure, which, according to Murthy [20, 1974] occurs at the 5 to 10 % mark of material use life; (ii) the growth and propagation of the fissure, which provokes a crack in the material due to cyclic stress and tension; (iii) and the rupture or failure of material, whose stage, according to Saunders [23, 1976], occupies an negligible lifetime. For this reason, statistical models for fatigue processes are primarily concerned with describing the random variation of lifetimes associated with second stage (ii) of this process, through life distributions, whose parameters allow those materials subjected to fatigue to be characterized and at the same time predicting its behavior under different cyclic force patterns and tension (see for example, Galea, Leiva-Sánchez, and Paula [13, 2004]). Among the probability models that more popularly have been proposed to describe lifetime due to fatigue, we find these distributions: Weibull, lognormal, gamma, and inverse-Gaussian, all of which fit with great precision into the central zone of the life distribution. Nevertheless, it

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is important to concern ourselves with the very low or very high percentiles of distribution, zone precisely where little data is usually found. This in turn leads to a poor fit of the models previously mentioned.

An important lifetime model, differing from those previously mentioned, originating from a problem of material-fatigue, is the one developed by Birnbaum and Saunders [4, 1969]. This model fits very well within the extremes of the distribution, even when there is little data. Desmond [8, 1985] showed that the Birnbaum-Saunders (B-S) distribution describes the total time that passes until some type of damage has accumulated, produced by the development and growth of a dominant crack, surpassing a threshold, and causing the material to fail.

The B-S distribution is defined in terms of the normal distribution, by means of the random variable (r.v.)

$$T = \beta \left[\frac{\alpha}{2} Z + \sqrt{\left(\frac{\alpha}{2} Z \right)^2 + 1} \right]^2, \tag{1}$$

where $Z \sim N(0,1)$, $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter and the median of the distribution. This is denoted by $T \sim BS(\alpha, \beta)$. The probability density function (pdf) of T is

$$f_T(t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2\alpha^2} \left(\frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right] \frac{t^{-3/2}(t+\beta)}{2\alpha\sqrt{\beta}}, \ t > 0. \tag{2}$$

It is easy to show that if $T \sim BS(\alpha, \beta)$, then

$$Z = \alpha^{-1} \left[\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}} \right] \sim N(0,1). \tag{3}$$

On the other hand, some multivariate distributions that have been of great importance lately is the family of elliptical contour densities or simply elliptical distributions. This family includes distributions with a lesser or greater kurtosis than the normal distribution. Furthermore, the elliptical family has the normal distribution as a particular case.

The elliptical laws have been studied by numerous authors and the most important results obtained have been summarized in the books by Fang, Kotz, and Ng [12, 1990] and by Gupta and Varga [17, 1993], as well as in the most recent work by Díaz-García, Leiva-Sánchez and Galea [9, 2002] and [10, 2003]. The use of elliptical distributions as a generalization of normal distribution is not based on empirical arguments or on physical laws. In general, its reasoning is purely statistical and/or mathematical, in the sense that: (a) the theory developed under normal distribution is a particular case of the theory derived within elliptical distributions; (b) many of the properties of a normal distribution can be generalize to the case of elliptical distributions; (c) some important statistics in the theory of normal inference are invariant within the elliptical family. For these reasons, currently, a large part of normal theory is being reconstructed using elliptical distributions, allowing that any statistical analysis in which a normal distribution can be assumed, can be generalized to this whole family.

For a r.v. (one dimensional case), the elliptical distributions are symmetrical distributions in $\mathbb{R}$. A r.v. X with elliptical distribution is characterized by the parameters μ and σ, and is characterized by a function generator of densities, g, for which the notation $X \sim EC(\mu, \sigma^2; g)$ is used. In general, μ and σ are position and scale parameters respectively. $\mu = \mathbb{E}(X)$ when the first moment of the distribution exists. $\text{Var}(X) = c_0\sigma^2$ when the first two moments exists, where $c_0 = -2\phi'(0)$ and ϕ' is the derivative of the function ϕ associated with the characteristic function of X. In this article we consider only spherical distributions. That is $Z \sim EC(0,1;g)$. Thus, the characteristic function of Z is given by

$$\psi(s) = \phi(s^2);\ s \in \mathbb{R}, \tag{4}$$

with $\phi : \mathbb{R}^+ \to \mathbb{R}$, and the pdf of X is given by

$$f(z) = c\ g(z^2);\ z \in \mathbb{R}, \tag{5}$$

where g is the kernel of the pdf of Z and c the normalization constant.

Some specific distributions $EC(0,1;g)$ are presented next. Z follows the Pearson VII distribution with parameters $r > 0$ and $q > 1/2$ if its pdf is

$$f(z) = \frac{\Gamma(q)}{(r\pi)^{1/2}\Gamma(q-1/2)}\left(1+\frac{z^2}{r}\right)^{-q}$$

and follows the type Kotz distribution with parameters $r, s > 0$ and $q > 1/2$ if its pdf is

$$f(z) = \frac{sr^{(2q-1)/2s}}{\Gamma\left((2q-1)/2s\right)} z^{2(q-1)} \exp(-rz^{2s}).$$

Remark 1. *The following are particular cases of distributions previously mentioned.*

i) The distribution t (denoted by $t(\nu)$), where ν are its degrees of freedom, is a particular case of the Pearson type VII distribution when $q = (\nu+1)/2$ and $r = \nu$.

ii) The Cauchy distribution is a $t(\nu)$ with $\nu = 1$ and it does not have moments.

iii) The normal distribution is a particular case of the Kotz type when $q = s = 1$ and $r = 1/2$.

Recently, Díaz-García and Leiva-Sánchez [11, 2005] presented a generalization of the distribution given in (2) assuming now that

$$Z = \frac{1}{\alpha}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim EC(0,1;g). \tag{6}$$

Starting from (6), they got that

$$T = \beta\left[\frac{\alpha}{2}Z + \sqrt{\left(\frac{\alpha}{2}Z\right)^2 + 1}\right]^2 \tag{7}$$

follows a generalized Birnbaum-Saunders distribution, which is denoted by $T \sim GBS(\alpha, \beta; g)$. The pdf of T is

$$f_T(t) = c\ g\left(\frac{1}{\alpha^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}},\ t > 0, \tag{8}$$

where c and g are given in (5). So, if we define the function $a_t(\alpha,\beta)$ by

$$a_t(\alpha,\beta) = \frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right), \tag{9}$$

we get

$$\frac{d}{dt}a_t(\alpha,\beta) = \frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}, \tag{10}$$

thus being able to write (8) as

$$f_T(t) = f(a_t(\alpha,\beta))\frac{d}{dt}a_t(\alpha,\beta), \tag{11}$$

where f is given in (5). Several examples of the function f are considered in Díaz-García and Leiva-Sánchez [11, 2005], thereby obtaining a new and more general type of distributions that can be utilized as alternative models to Birnbaum and Saunders [4, 1969]. This generalization, in addition to the one mentioned earlier, are based on the search for life distributions that grow rapidly, and that have left tails heavier or lighter than the classic B-S distribution, among other interesting properties, such as for example the absence of moments in the life distribution. With this, the B-S distribution was generalized starting with an elliptical distribution. In this same article they also mention the importance of log-life distribution studies and the possible use of skew-distribution in the reliability analysis.

Now, outside the context of reliability and from a multivariate and elliptical perspective, the topic of log-distributions and skew-distributions have been posed, with the goal being to obtain greater degrees of generalization for problems which up until now have been resolved. On the one hand, the log-elliptical distribution have been defined as an extension of the log-normal distribution, which is dealt with in Fang, Kotz, and Ng [12, 1990]. Currently though, being treated similarly to the multivariate, skew-normal distribution, whose principle results are accredited to Azzalini, Dalla-Valle, and Capitanio [2, 1996] and [3, 1999]; and to Gupta, Farías-González, and Domínguez-Molina [15, 2004], among others, great strides have been made in the skew-elliptical distribution theory as seen in work done by Branco and Dey [5, 2001], and Arellano-Valle, Del Pino and San Martín [1, 2001] among others. All of which have been adequately summarized in Genton's [14, 2004] new book.

From a univariate perspective, a skew-elliptical distribution (SEC) is denoted by $Y \sim SEC(\mu,\sigma^2,\lambda;g)$, where μ, σ and λ are the position, scale, and skewness parameters respectively. In particular, if $\lambda = 0$, this distribution coincides with an elliptical distribution in $\mathbb{R}$. Thus, if $Y \sim SEC(0,1,\lambda;g)$, then the pdf of Y is

$$f_Y(y) = 2f(y)F(\lambda y);\ y \in \mathbb{R}. \tag{12}$$

Going back to the area of reliability, there are two aspects that foremost motivate this work. On the one hand, from a theoretical point of view and always in search of greater generalizations, we propose the use of skew-elliptical distributions in the generalization of a B-S distribution in order to get a new and wider type of life distributions due to fatigue. This new model has as particular cases the classic B-S distribution [4, 1969] and the generalization of Díaz-García and Leiva-Sánchez [11, 2005]. With this generalization, a more flexible distribution was obtained and one that is closer to reality. On the other hand, from an empirical point of view a second aspect that is highly motivating comes when we consider a system connected in parallel formed by two identical components both subjected to fatigue. Therefore, if we assume that these components have failure lifetimes independently and identically distributed (iid), then the failure lifetime of the system corresponds to the maximum failure lifetime of these components. The pdf of the distribution of this order statistic, surprisingly, coincides with that of the B-S distribution generated by SEC distributions, which we will show later. This interesting property of the B-S distribution can in the same way be applied to calculations of the reliability of the strain-stress model, given by $\mathbb{P}(X > Y)$ (see Gupta and Brown [16, 2001]), where X is the strain of the material and Y the stress that the material receives, which is also the reason for developing this new lifetime model.

This article has been divided into two sections. In the first section we have introduced the topic and done a bibliographical review. In second section we will present the new model, its density along with some graphs, its reliability and hazard functions; as well as its moments and variation, skewness, and kurtosis coefficients. In addition, some properties of this new distribution will be presented.

2 The New Model

In this section we present a new fatigue life model, obtaining greater generalization than what was obtained recently by Díaz-García and Leiva-Sánchez [11, 2005]. This double generalization is accounted for on the one hand by the extension of the normal distribution to the elliptical distribution and on the other by the parameter of skewness of the SEC distribution. Specifically, we found the pdf of the doubly generalized Birnbaum-Saunders (GBS$''$) distribution and some of its most important properties.

To specify some ideas, we now consider

$$Z = \frac{1}{\alpha}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim SEC(0, 1, \lambda; g). \tag{13}$$

Then, from (12), the pdf of Z given by (13) is

$$f_Z(z, \lambda) = 2f(z)F(\lambda z), \ z \in \mathbb{R}, \tag{14}$$

where f is given in (5), F is its respective cumulative distribution function (cdf) and $\lambda \in \mathbb{R}$ is the skewness parameter. Thus, if we follow the same procedure used in Díaz-García and Leiva-Sánchez [11, 2005],

$$T = \beta\left[\frac{\alpha}{2}Z + \sqrt{\left(\frac{\alpha}{2}Z\right)^2 + 1}\right]^2 \tag{15}$$

has the GBS$''$ distribution, denoted by $T \sim GBS''(\alpha, \beta; g, \lambda)$.

2.1 Density Function

Next we present the pdf of the r.v. given in (15) and other aspects associated with this.

Theorem 1. *Let $T \sim GBS''(\alpha, \beta; g, \lambda)$. Then, the pdf of T is given by*

$$f_T(t, \lambda) = 2f(a_t(\alpha, \beta))F(\lambda a_t(\alpha, \beta))\frac{d}{dt}a_t(\alpha, \beta), \ t > 0, \tag{16}$$

where $a_t(\alpha, \beta)$ is given in (9) and f, F in (14).

Proof. It is direct starting from (14). ■

Remark 2. *Note that (16) can be written in terms of the pdf given in (11), with which it is only necessary to calculate*

$$F(\lambda a_t(\alpha,\beta))) = \int_{-\infty}^{\lambda a_t(\alpha,\beta))} f(x)dx, \tag{17}$$

where f, F are given in (14), in order to find specific expressions for the pdf of T. Particular cases for the pdf given in (11) are presented in Díaz-García and Leiva-Sánchez [11, 2005], and some of which we summarize in Table 1.

Table 1. pdf of the generalized Birnbaum-Saunders distribution which are as follows.

Elliptical law	**pdf GBS**
Pearson VII	$\frac{\Gamma(q)}{(r\pi)^{1/2}\Gamma(q-1/2)}\left(1+\frac{1}{r\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{-q}\left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right);\ \alpha,\beta,r>0,\ q>1/2$
$t(\nu)$	$\frac{\Gamma((\nu+1)/2)}{(\nu\pi)^{1/2}\Gamma(\nu/2)}\left(1+\frac{1}{\nu\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{-(\nu+1)/2}\left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right);\ \alpha,\beta,\nu>0$
Cauchy	$\frac{1}{\pi}\left(1+\frac{1}{\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{-1}\left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right);\ \alpha,\beta>0$
Type Kotz	$\frac{s}{\Gamma((2q-1)/(2s))}\left[\frac{r}{\alpha^{2s}}\right]^{\frac{2q-1}{2s}}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)^{q-1}\exp\left(-\frac{r}{\alpha^{2s}}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]^{s}\right)\left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right);\alpha,\beta,r,s>0,q>\frac{1}{2}$
Normal	$\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)\left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right);\ \alpha,\beta>0$

2.2 Density Graphs

Next, we present various density graphs in which it is possible to observe how the skewness parameter influences the GBS$''$ distribution. In these graphs we also present a comparison with the generalized B-S distribution of Díaz-García and Leiva-Sánchez ([11, 2005]), that is, when $\lambda = 0$.

The graphs were done with a software program call Mathematica 5.0. In addition to the analysis based on the skewness parameter, a routine made in the same program was used to make an animation of the graphs. With this animation we made an analysis of how the parameters of each SEC distribution analyzed influenced the pdf of the GBS$''$ distribution.

In general, with respect to the skewness parameter (λ), based on the density graphs given in Figures 1-3, we can observe the following: the new life model which we proposed yields distributions with tails heavier than those generated by elliptical distributions. When λ is positive ($\lambda = 1$ in our analysis) this situation is accentuated for unimodal distributions. When λ is negative ($\lambda = -1$ in our analysis) we noticed that the unimodal distribution is concentrated around the mode in which case we also observed that the distribution fits better to observations of small magnitude.

Specifically and based on the graphical animation, we observed the following. For the Pearson type VII distribution (Figure 1), as r or q increases, the GBS$''$ distribution tends to become more platykurtic and symmetrical, the position moves to the right, and variability diminishes. For the Kotz type distribution (Figure 3), as q increases, the GBS$''$ distribution tends to become more leptokurtic, symmetrical, and bimodal, whereas the separation of the modes becomes greater. As r increases, the GBS$''$ distribution tends to become more leptokurtic and variability increases slightly. As s increases, the GBS$''$ distribution tends to become more leptokurtic and variability diminishes slightly.

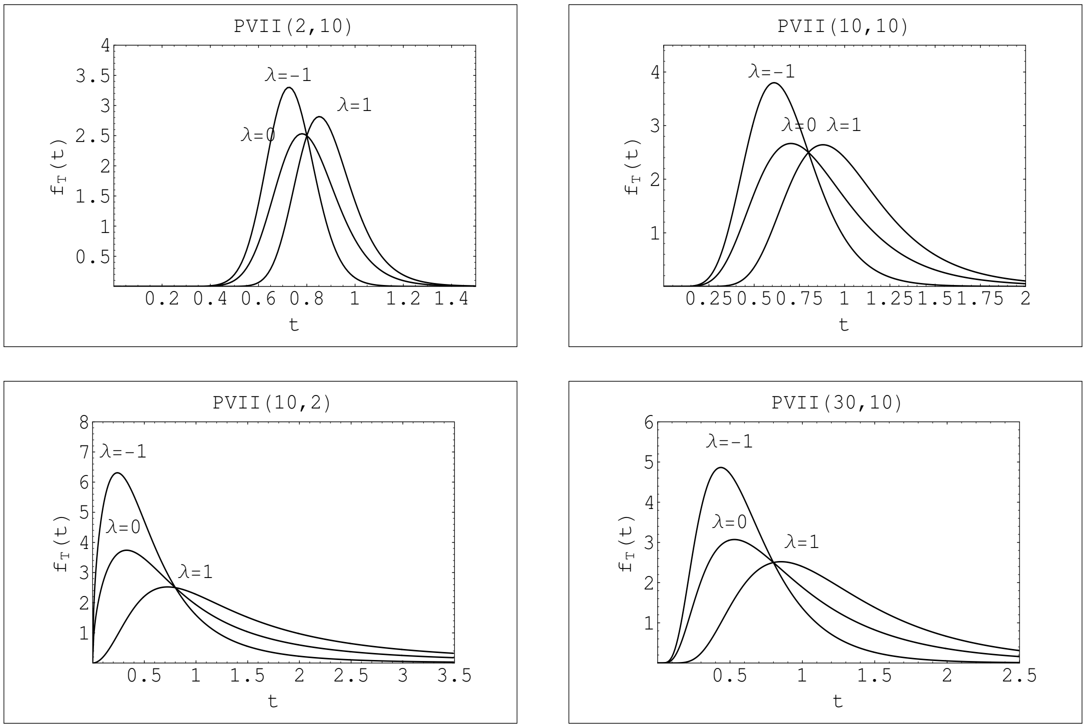

Figure 1: Density graphs of a GBS$''$ distribution for $\alpha = 0.5$ and $\beta = 0.8$ obtained from a Pearson VII distribution with parameters r and q (PVII(r, q)).

2.3 Properties and Characterizations

Next, we present some properties of the distribution of the r.v. given in (15), some of which were presented in Birnbaum and Saunders [4, 1969], when $Z \sim N(0,1)$ and in Díaz-García and Leiva-Sánchez [11, 2005]), when $Z \sim EC(0,1;g)$.

Theorem 2. *Let $T \sim GBS''(\alpha, \beta; g, \lambda)$. Then,*

i) $aT \sim GBS''(\alpha, a\beta; g, \lambda)$, $a > 0$,

ii) $T^{-1} \sim GBS''(\alpha, \beta^{-1}; g, \lambda)$.

Proof. The proof of i) and ii) are immediate from the theorem of the change of variable. ■

Remark 3. *Just like the B-S and GBS distributions, we see that the properties established in Theorem 2ii) show that the GBS$''$ distribution also belongs to the family of random variables closed under reciprocation (see Saunders [22, 1974]).*

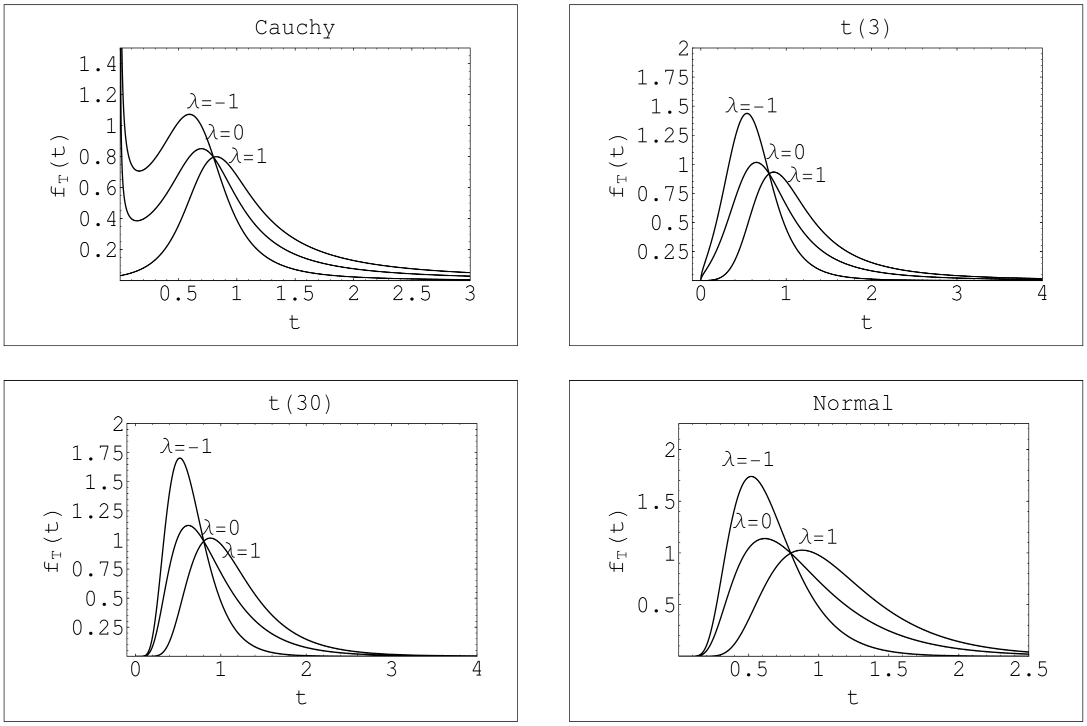

Figure 2: Density graphs of a GBS$''$ distribution for $\alpha = 0.5$ and $\beta = 0.8$ obtained from the distributions indicated.

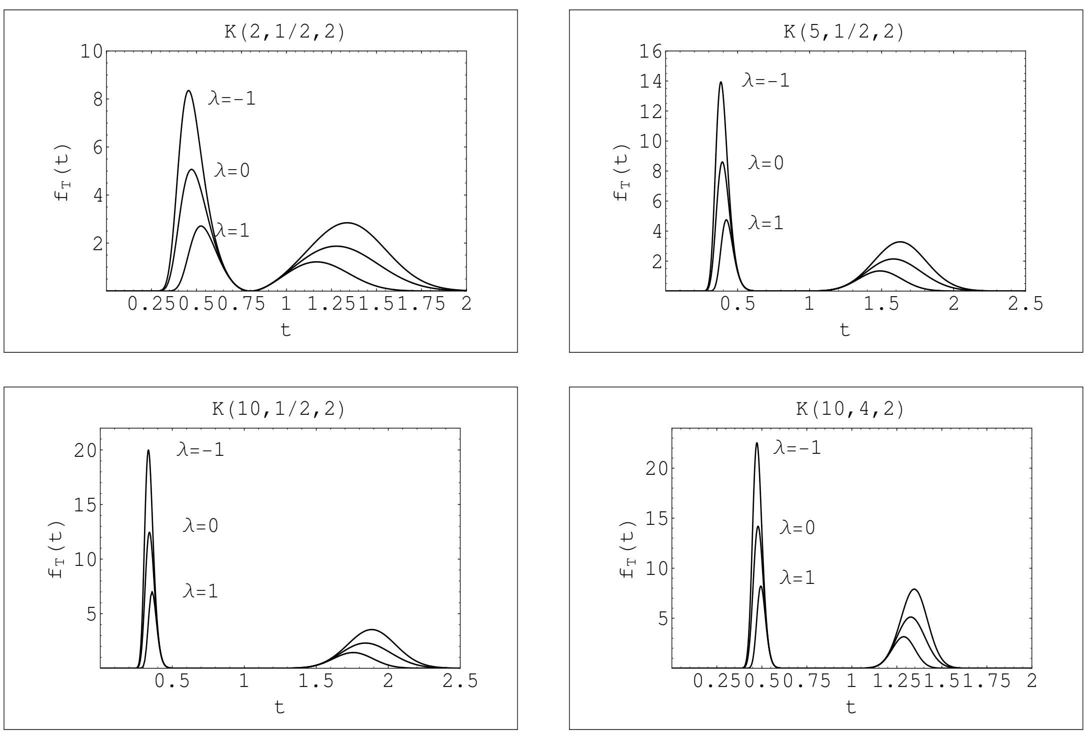

Figure 3: Density graphs of a GBS$''$ distribution for $\alpha = 0.5$ and $\beta = 0.8$ obtained from a Kotz type distribution with parameters q, r, and s (K(q,r,s)).

Theorem 3. *Let $T \sim GBS''(\alpha,\beta;g,\lambda)$ and F_T its cdf, F given in (14) and F_Z the cdf of Z given in (13). Then,*

i) $F_T(t;\lambda) = F_Z(a_t(\alpha,\beta),\lambda)$. *In particular, for* $\lambda = 0$, $F_T(t;\lambda) = F(a_t(\alpha,\beta))$.

ii) $F_T(t;-\lambda) = 2F_T(t;\lambda=0) - F_T(t;\lambda)$.

iii) *For* $\lambda = 1$, $F_T(t;\lambda) = [F(a_t(\alpha,\beta))]^2$.

Proof.

$$
\begin{aligned}
i) F_T(t;\lambda) &= \int_0^t 2f(a_x(\alpha,\beta))F(\lambda a_x(\alpha,\beta))\frac{d}{dx}a_x(\alpha,\beta)dx \\
&= \int_{-\infty}^{a_t(\alpha,\beta)} 2f(x)F(\lambda x)dx \\
&= F_Z(a_t(\alpha,\beta),\lambda) \blacksquare
\end{aligned}
$$

$$
\begin{aligned}
ii) F_T(t;-\lambda) &= \int_0^t 2f(a_x(\alpha,\beta))F(-\lambda a_x(\alpha,\beta))\frac{d}{dx}a_x(\alpha,\beta)dx \\
&= \int_0^t 2f(a_x(\alpha,\beta))(1-F(\lambda a_x(\alpha,\beta)))\frac{d}{dx}a_x(\alpha,\beta)dx \\
&= \int_0^t 2f(a_x(\alpha,\beta))\frac{d}{dx}a_x(\alpha,\beta)dx - \int_0^t 2f(a_x(\alpha,\beta))F(\lambda a_x(\alpha,\beta))\frac{d}{dx}a_x(\alpha,\beta)dx \\
&= 2F_T(t;\lambda=0) - F_T(t;\lambda) \blacksquare
\end{aligned}
$$

$$
\begin{aligned}
iii) F_T(t;\lambda=1) &= \int_{-\infty}^{a_t(\alpha,\beta)} 2f(x)F(\lambda x)dx \\
&= \int_{-\infty}^{a_t(\alpha,\beta)} \frac{d}{dx}F^2(x)dx \\
&= F^2(a_t(\alpha,\beta)) \blacksquare
\end{aligned}
$$

Remark 4. *From Theorem 3i) and following the same procedure from Chang and Tang [6, 1974], the p-th percentil of the distribution,* $t_p = F_T^{-1}(p;\lambda)$, *is given by*

$$
t_p = \frac{\beta}{4}\left(\alpha z_p + \sqrt{\alpha^2 z_p^2 + 4}\right)^2,
$$

where z_p is the p-th percentil of the SEC distribution. For $\lambda = 0$, that is when $Z \sim EC(0, 1; g)$, $t_{0.5} = \beta$. Also, if $Z \sim N(0, 1)$, t_p is given as in Jeng [18, 2003] in whose case z_p is the p-th percentil of the standard normal distribution.

Theorem 4. *Let $T \sim GBS''(\alpha, \beta; g, \lambda)$ and F_T its cdf, F given in (14) and F_Z the cdf of Z given in (13). Then, the reliability, and hazard functions of T are respectively,*

$$R_T(t; \alpha, \beta) = 1 - F_Z(a_t(\alpha, \beta), \lambda) \text{ and } h_T(t; \alpha, \beta) = 2f(a_t(\alpha, \beta)) \left(\frac{F(\lambda a_t(\alpha, \beta))}{1 - F_Z(a_t(\alpha, \beta), \lambda)} \right) \frac{d}{dt} a_t(\alpha, \beta).$$

Proof. It is immediate basing ourselves again in Theorem 3i) and in the definitions of the reliability and hazard functions (see for example, Meeker and Escobar [19, 1998] given by $R_T(t; \alpha, \beta) = 1 - F_T(t; \alpha, \beta)$ and $h_T(t; \alpha, \beta) = f_T(t, \alpha, \beta)/R_T(t, \alpha, \beta)$. ∎

As it was mentioned previously, a surprising property of the classic B-S distribution given in (2) is that the maximum of the two B-S iid random variables follow a GBS$''$ distribution whose formal aspects are stated in the following theorem.

Theorem 5. *Let T_1 and T_2 two r.v. iid of according to $T \sim BS(\alpha, \beta)$. Then,*

$$T_{(2)} = \max\{T_1, T_2\} \sim GBS''(\alpha, \beta; g, \lambda = 1).$$

Proof. It is direct from the maximum distribution. ∎

Remark 5. *Starting with Theorem 5 it is possible to calculate the reliability of the strain-stress model (see Gupta and Brown [16, 2001]), $\mathbb{P}(X > Y)$, where X is the strain of the material and Y is the stress that it receives.*

2.4 Moments

Most of the elliptical distributions have moments. The presence or absence of moments is transferred to SEC distributions. In order to determine the moments of $T \sim GBS''(\alpha, \beta; g, \lambda)$, we are going to use binomial representation $(a + b)^m = \sum_{k=0}^{m} a^k b^{m-k}$ and the properties of the distribution of Z, T and T^{-1}, given in (13), (15) and Theorem 2 respectively. We will show that the moments of the k-th order of T depends on the existence of the moments of the same order of Z^2.

Lemma 1. *Let $T \sim GBS''(\alpha, \beta, \lambda; g)$ and suppose that $\mathbb{E}(T^{2k})$ and $\mathbb{E}(T^{2k+1})$ exist. Then,*

i) $$\mathbb{E}\left(\frac{T}{\beta} + \frac{\beta}{T}\right)^{2k} = 2 \sum_{0 \le j < k} \binom{2k}{j} \beta^{-2(k-j)} \mathbb{E}(T^{2(k-j)}) + \binom{2k}{k}.$$

ii) $$\mathbb{E}\left(\frac{T}{\beta} + \frac{\beta}{T}\right)^{2k+1} = 2 \sum_{0 \le j \le k} \binom{2k+1}{j} \beta^{-(2k+1-2j)} \mathbb{E}(T^{2k+1-2j}).$$

Proof. From the binomial theorem, we have

$$\begin{aligned}
\left(\frac{T}{\beta} + \frac{\beta}{T}\right)^{2k} &= \sum_{j=0}^{n} \binom{2k}{j} \beta^{2(k-j)} T^{2(j-k)}) \\
&= \sum_{0 \le j < k} \binom{2k}{j} \beta^{2(k-j)} T^{2(j-k)} + \sum_{k < j \le 2k} \binom{2k}{j} \beta^{2(k-j)} T^{2(j-k)} + \binom{2k}{k}.
\end{aligned}$$

We can see from Theorem 2 that T and $\beta^2 T^{-1}$ have the same distribution as $GBS''(\alpha,\beta;g,\lambda)$. Consequently, for $j<k$, we have

$$\mathbb{E}(T^{2(j-k)}) = \beta^{-4(k-j)}\mathbb{E}(T^{2(k-j)}). \tag{18}$$

Thus, taking the expected value of $(T/\beta+\beta/T)^{2k}$ and substituting $\mathbb{E}(T^{2(j-k)})$ by (18), we have

$$\mathbb{E}\left(\frac{T}{\beta}+\frac{\beta}{T}\right)^{2k} = \sum_{0\le j<k}\binom{2k}{j}\beta^{-2(k-j)}\mathbb{E}(T^{2(k-j)}) + \sum_{k<j\le 2k}\binom{2k}{j}\beta^{2(k-j)}\mathbb{E}(T^{2(j-k)}) + \binom{2k}{k} \tag{19}$$

In (19) the two first terms on the right side are equal. So, finally,

$$\mathbb{E}\left(\frac{T}{\beta}+\frac{\beta}{T}\right)^{2k} = 2\sum_{0\le j<k}\binom{2k}{j}\beta^{-2(k-j)}\mathbb{E}(T^{2(k-j)}) + \binom{2k}{k} \blacksquare$$

For the proof of part ii), we consider the relationship

$$\left(\frac{T}{\beta}+\frac{\beta}{T}\right)^{2k+1} = \sum_{j=0}^{k}\binom{2k+1}{j}\beta^{2k+1-2j}T^{2j-2k-1} + \sum_{j=k+1}^{2k+1}\binom{2k+1}{j}\beta^{2k+1-2j}T^{2j-2k-1} \tag{20}$$

Now, if we calculate the expected value in (20) and we substitute $(T^{2j-2k-1})$ for

$$\mathbb{E}(T^{2j-2k-1}) = \beta^{-2(2k+1-2j)}\mathbb{E}(T^{2k+1-2j}),\ 0\le j\le k, \tag{21}$$

we get that the two terms from $\mathbb{E}\left(T/\beta+\beta/T\right)^{2k+1}$ are equal, proving part ii). $\blacksquare$

Theorem 6. *Let $T\sim GBS''(\alpha,\beta,\lambda;g)$ and Z given in (13). Then, $\mathbb{E}(Z^{2n})$ exists if and only if $\mathbb{E}(T^n)$ exists, and*

$$\mathbb{E}(Z^{2n}) = \frac{1}{\alpha^{2n}}[(-2)^n + A_n + B_n],$$

where

$$A_n = \sum_{1\le k\le n:\ even}\binom{n}{k}(-2)^{n-k}\left[2\sum_{0\le j<k/2}\binom{k}{j}\beta^{-(k-2j)}\mathbb{E}(T^{k-2j}) + \binom{k}{k/2}\right]$$

and

$$B_n = \sum_{1\le k\le n:\ odd}\binom{n}{k}(-2)^{n-k}\left[2\sum_{0\le j\le (k-1)/2}\binom{k}{j}\beta^{-(k-2j)}\mathbb{E}(T^{k-2j})\right].$$

Proof. Of Z given in (13), we have

$$Z^2 = \frac{1}{\alpha^2}\left(\frac{T}{\beta}+\frac{\beta}{T}-2\right). \tag{22}$$

Therefore, raising to the n the expression (22) and applying the binomial development, we have

$$\begin{aligned} Z^{2n} &= \frac{1}{\alpha^{2n}}\left(\frac{T}{\beta}+\frac{\beta}{T}-2\right)^{n} \\ &= \frac{1}{\alpha^{2n}}\left[(-2)^{n}+\sum_{k=1}^{n}\binom{n}{k}(-2)^{n-k}\left(\frac{T}{\beta}+\frac{\beta}{T}\right)^{k}\right] \qquad (23) \\ &= \frac{1}{\alpha^{2n}}\left[(-2)^{n}+\sum_{k:\ even}\binom{n}{k}(-2)^{n-k}\left(\frac{T}{\beta}+\frac{\beta}{T}\right)^{k}+\sum_{k:\ odd}\binom{n}{k}(-2)^{n-k}\left(\frac{T}{\beta}+\frac{\beta}{T}\right)^{k}\right]. \end{aligned}$$

In this way, if we take the expected value in (23), we note that $\mathbb{E}(Z^{2n})$ depends on $\mathbb{E}\left(T/\beta+\beta/T\right)^{k}$, $k \geq 1$. Also, applying Lemma 1 for the values of k evens and odds, we see that the expected values of the second and third term within the parenthesis are equal to A_n and B_n respectively, which the theorem proves. ■

Corollary 1. *Let $T \sim GBS''(\alpha,\beta,\lambda;g)$ and Z given in (13). Then,*

i) $\mathbb{E}(T)=\frac{\beta}{2}(2+\mathbb{E}(Z^2)\alpha^2)$;

ii) $\mathbb{E}(T^2)=\frac{\beta^2}{2}(2+4\mathbb{E}(Z^2)\alpha^2+\mathbb{E}(Z^4)\alpha^4)$;

iii) $\mathbb{E}(T^3)=\frac{\beta^3}{2}(2+9\mathbb{E}(Z^2)\alpha^2+6\mathbb{E}(Z^4)\alpha^4+\mathbb{E}(Z^6)\alpha^6)$;

iv) $\mathbb{E}(T^4)=\frac{\beta^4}{2}(2+16\mathbb{E}(Z^2)\alpha^2+20\mathbb{E}(Z^4)\alpha^4+8\mathbb{E}(Z^6)\alpha^6+\mathbb{E}(Z^8)\alpha^8)$.

Proof. From Theorem 6, we have that for $n=1,2,3,4$,

a) $\alpha^2\mathbb{E}(Z^2)=-2+2\beta^{-1}\mathbb{E}(T)$;

b) $\alpha^4\mathbb{E}(Z^4)=6-8\beta^{-1}\mathbb{E}(T)+2\beta^{-2}\mathbb{E}(T^2)$;

c) $\alpha^6E(Z^6)=-20+30\beta^{-1}E(T)-12\beta^{-2}E(T^2)+2\beta^{-3}E(T^3)$;

d) $\alpha^8\mathbb{E}(Z^8)=70-112\beta^{-1}\mathbb{E}(T)+56\beta^{-2}\mathbb{E}(T^2)-16\beta^{-3}\mathbb{E}(T^3)+2\beta^{-4}\mathbb{E}(T^4)$.

The proof of Corollary 1 follows from the results given in a)-d). For example, from a) we have that $\mathbb{E}(T)=\frac{\beta}{2}(2+\mathbb{E}(Z^2)\alpha^2)$. Now, substituting $\mathbb{E}(T)$ in b), the proof of ii) is proved. Similarly the proof of iii) and iv) is also proved. ■

Remark 6. *Let $T \sim GBS''(\alpha,\beta,\lambda;g)$ and Z given in (13). Then, from Corollary 1 it follows that*

$$Var(T)=\frac{(\alpha\beta)^2}{4}[4\mathbb{E}(Z^2)+(2\mathbb{E}(Z^4)-\mathbb{E}^2(Z^2))\alpha^2].$$

Furthermore, from Theorem 2 and Corollary 1, we have

$$\mathbb{E}(T^{-1})=\frac{1}{2\beta}(2+\mathbb{E}(Z^2)\alpha^2) \quad \text{and} \quad Var(T^{-1})=\frac{\alpha^2}{4\beta^2}[4\mathbb{E}(Z^2)+(2\mathbb{E}(Z^4)-\mathbb{E}^2(Z^2))\alpha^2].$$

In the following theorem we will present the variation, skewness, and kurtosis coefficients. For Z given in (13) and k integer positive, we denote $V_k = \mathbb{E}(Z^k)$ as the k-th moment of the SEC distribution.

Theorem 7. *Let $T \sim GBS''(\alpha, \beta; g, \lambda)$. Then, the variation, skewness, and kurtosis coefficients of T are respectively*

$$\gamma = \frac{\alpha\sqrt{[4V_2 + (2V_4 - V_2^2)\alpha^2]}}{(2 + V_2\alpha^2)},$$

$$\beta_1(T) = \frac{4\alpha^2[6(V_4 - V_2^2) + (2V_6 + V_2^3 - 3V_2V_4)\alpha^2]^2}{[4V_2 + (2V_4 - V_2^2)\alpha^2]^3}$$

and

$$\beta_2(T) = -\frac{(-8V_8 + 16V_6V_2 - 12V_4V_2^2 + 3V_2^4)\alpha^4 + (-32V_6 + 48V_2V_4 - 24V_2^3)\alpha^2 - 16V_4}{[4V_2 + (2V_4 - V_2^2)\alpha^2]^2}.$$

Proof. Remember that the variation, skewness, and kurtosis coefficients are defined by

$$\gamma = \frac{\sigma}{\mu}, \quad \beta_1(T) = \frac{(\mu_3)^2}{(\mu_2)^3} \quad \text{and} \quad \beta_2(T) = \frac{\mu_4}{(\mu_2)^2} \tag{24}$$

respectively, where $\sigma = \sqrt{\text{Var}(T)}$, $\mu = \mathbb{E}(T)$ and $\mu_k = \mathbb{E}[T - \mathbb{E}(T)]^k$, with $k = 2, 3, 4$.

Now, as $\text{Var}(T) = \frac{(\alpha\beta)^2}{4}[4V_2 + (2V_4 - V_2^2)\alpha^2] = \mu_2$ and $\mu = \frac{\beta}{2}(2 + V_2\alpha^2)$, due to Corollary 1 and Remark 6, we find the variation coefficient, γ, given in (24). ■

On the other hand, as $\mu_3 = \mathbb{E}(T^3) - 3\mathbb{E}(T)\mathbb{E}(T^2) + 2\mathbb{E}^3(T)$, basing ourselves in Corollary 1 and after some algebraic manipulations, we get

$$\mu_3 = \frac{\beta^3}{4}[6(V_4 - V_2^2)\alpha^4 + (2V_6 + V_2^3 - 3V_2V_4)\alpha^6].$$

Substituting μ_2 and μ_3 in $\beta_1(T)$ given in (24), we find the skewness coefficient. ■

Similarly, since $\mu_4 = \mathbb{E}(T^4) - 4\mathbb{E}(T^3)\mathbb{E}(T) + 6\mathbb{E}(T^2)\mathbb{E}^2(T) - 3\mathbb{E}^4(T)$, using the Corollary 1 and manipulating algebraically, it follows that

$$\mu_4 = -\frac{(\alpha\beta)^4}{16}[(-8V_8 + 16V_6V_2 - 12V_4V_2^2 + 3V_2^4)\alpha^4 + (-32V_6 + 48V_2V_4 - 24V_2^3)\alpha^2 - 16V_4].$$

To get the kurtosis coefficient you can substitute μ_2 and μ_4 in $\beta_2(T)$ given in (24). ■

Remark 7. *These three indicators are of great use in the reliability analysis, in fact a variety of uses of the variation coefficient in this area have been proposed (see for example, Chhikara and Folks [7, 1977], and Meeker and Escobar [19, 1998, pp. 81-82 and p. 110]). Similarly, the skewness and kurtosis coefficients have great relevance in the life distribution fit, since skewness coefficient shows the degree of asymmetry, and the kurtosis coefficient the degree of flatness (see Díaz-García and Leiva-Sánchez [11, 2005]). These two last indicators were also proposed in Ng, Kundub, and Balakrishnan's [21, 2003] work related to the topic.*

Concluding Remarks

In this work, we have discussed a wider generalization of the Birnbaum-Saunders life distribution due to fatigue starting with a skew-elliptical distribution, obtaining its density and some of its graphics to see how the skewness parameter influences on its behavior. We have also outlined some important properties of this new model, which has allowed us to outline its complete characterization. For those distributions that have moments, we also found its moments and using these as a starting point, obtained the variation, skewness, and kurtosis coefficients, all of which play an important role in the reliability analysis. Therefore, with this double generalization, we developed a new family of life distributions which can be used in wider and different situations.

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