Linear Groups of Isometries with Poset Structures

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Abstract

Let V be an n-dimensional vector space over a finite field $\mathbb{F}_q$ and $P = \{1, 2, \ldots, n\}$ a poset. We consider on V the poset-metric d_P. In this paper, we give a complete description of groups of linear isometries of the metric space (V, d_P), for any poset-metric d_P. We show that a linear isometry induces an automorphism of order in poset P, and consequently we show the existence of a pair of ordered bases of V relative to which every linear isometry is represented by an $n \times n$ upper triangular matrix.

Key words: Poset codes, poset metrics, linear isometries.

Coding theory takes place in finite dimensional linear spaces over finite fields. One of the main questions of the theory (classical problem) asks to find a k-dimensional subspace in $\mathbb{F}_q^n$, the space of n-tuples over the finite field $\mathbb{F}_q$, with the largest minimum distance possible. There are many possible metrics that can be defined in $\mathbb{F}_q^n$, the most common ones are the Hamming and Lee metrics.

In 1987 Harald Niederreiter generalized the classical problem of coding theory (see [7]). Brualdi, Graves and Lawrence (see [2]) also provided in 1995 a wider situation for the above problem: using partially ordered sets and defining the concept of poset-codes, they started to study codes with a poset-metric. This has been a fruitful approach, since many new perfect codes have been found with such poset metrics (see [1], [2], [3], [5] and [6]).

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We let P be a partially ordered set (abbreviated as *poset*) of cardinality n with order relation denoted, as usual, by $\leq$. An *ideal* of P is a subset $I \subseteq P$ with the property that $x \in I$ and $y \leq x$ implies that $y \in I$. Given $A \subseteq P$, we denote by $\langle A \rangle$ the smallest ideal of P containing A. Without loss of generality, we assume that $P = \{1, 2, \ldots, n\}$ and that the coordinates of vectors in $\mathbb{F}_q^n$ are in one-to-one correspondence with the elements of P.

Given $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$, the *support of* x is the set

$$\operatorname{supp}(x) := \{i \in P : x_i \neq 0\},$$

and we define the *P-weight* of x to be the cardinality of the smallest ideal containing supp(x):

$$w_P(x) = |\langle \operatorname{supp}(x) \rangle|.$$

The function

$$d_P : \mathbb{F}_q^n \times \mathbb{F}_q^n \to \mathbb{N}$$

defined by $d_P(x, y) = w_P(x - y)$ is a metric in $\mathbb{F}_q^n$ ([2, Lemma 1.1]), called a *poset-metric* or a *P-poset-metric*, when it is important to stress the order taken in consideration. We denote such a metric space by $(\mathbb{F}_q^n, d_P)$.

An $[n, k, \delta_P]_q$ *poset-code* is a k-dimensional subspace $C \subset \mathbb{F}_q^n$, where $\mathbb{F}_q^n$ is endowed with a poset-metric d_P and

$$\delta_P(C) = \min\{w_P(x) : \mathbf{0} \neq x \in C\}$$

is the *P-minimum distance* of the code C. If P is an *antichain* order, that is, an order with no comparable elements, P-weight, P-poset-metric and P-minimum distance become the Hamming weight, Hamming metric and minimum distance of classical coding theory.

A *linear isometry* T of metric space $(\mathbb{F}_q^n, d_P)$ is a linear transformation $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ that preserves P-poset-metric,

$$d_P(T(x), T(y)) = d_P(x, y),$$

for every $x, y \in \mathbb{F}_q^n$. Equivalently, a linear transformation T is an isometry if $w_P(T(x)) = w_P(x)$ for every $x \in \mathbb{F}_q^n$. A linear isometry of $(\mathbb{F}_q^n, d_P)$ is said to be a *P-isometry*. We denote by $GL_P(\mathbb{F}_q^n)$ the group of linear isometries of $(\mathbb{F}_q^n, d_P)$. In this work, we give a complete description of those groups, for any given poset-metric P.

The paper is arranged as follows:

In the first section of this work we give a full description of the linear isometries of the vector space $\mathbb{F}_q^n$ endowed with the most simple (non trivial) cases of poset-metrics: when the partial order is a total order (Theorem 1.1), one single chain of length n, or when it is a disjoint union of chains (Theorem 1.2). The interesting property of this initial case assures that every linear isometry permutes the subspaces having support in chains of the same cardinality (Proposition 1.1).

In the second section, we present the description of the linear isometries for an arbitrary partial order. The property of permuting chains of same length, showed in the first section, corresponds, in the case of a general poset P, to Theorem 2.1, which assures that every linear isometry T induces an automorphism of the poset P. The key-point for these proof is Proposition 2.1, which assures that $\langle \mathrm{supp}\,(T\,(u))\rangle \subseteq \langle \mathrm{supp}\,(T\,(v))\rangle$ if $\langle \mathrm{supp}\,(u)\rangle \subseteq \langle \mathrm{supp}\,(v)\rangle$, $u, v \in \mathbb{F}_q^n$. The characterization of linear isometries is given in Theorem 2.2: there is an ordered base β of $\mathbb{F}_q^n$ relative to which every $T \in GL_P\left(\mathbb{F}_q^n\right)$, is represented by the product $A \cdot U$ of matrices, where U is a monomial matrix corresponding to an isomorphism of the poset P and A is an upper-triangular matrix.

The third section is devoted to some examples, with a complete description of $GL_P\left(\mathbb{F}_q^n\right)$ where we give a detailed description of with some of the most commonly used poset-metrics: antichain-metric, weak-metric and crown-metric.

We will present only the concepts of the theory of partially ordered sets that are strictly necessary for this work, referring the reader to [8] for more details.

1 Initial Case: Disjoint Union of Chains

As said in the introduction, before we characterize the linear isometries of $GL_P\left(\mathbb{F}_q^n\right)$ for an arbitrary partial order $P = \{1, 2, \ldots, n\}$, we describe the linear isometries in the case P is a disjoint union of chains, since the arguments used here are both simple and instructive for a formulation of the results in the general case.

A *totally ordered set* (or *linearly ordered set*) is a poset P in which any two elements are comparable. A subset C of a poset P is called a *chain* if C is a totally ordered set when regarded as a subposet of P.

The following theorem characterizes $GL_P\left(\mathbb{F}_q^n\right)$ when P is a totally ordered set.

Theorem 1.1 *Let $P = \{1, 2, \ldots, n\}$ be a totally ordered set. Then, there is an ordered base β of $\mathbb{F}_q^n$ relative to which every linear isometry $T \in GL_P\left(\mathbb{F}_q^n\right)$*

is represented by the $n \times n$ upper triangular matrix with $x_{ii} \neq 0$ for every $i \in \{1, 2, \ldots, n\}$.

Proof. Let $i_1 < i_2 < \ldots < i_n$ be the order in the poset P. Let $\beta' = \{e_1, e_2, \ldots, e_n\}$ be the canonical base of $\mathbb{F}_q^n$. Consider the ordered base $\beta = \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$. Given $x = \sum_{j=1}^n x_j e_{i_j}$, we have that

$$w_P(x) = \max_{j \in \{1,2,\ldots,n\}} \{j : x_j \neq 0\},$$

that is, $w_P(x)$ is the order of the highest non-zero coordinate of x relative to the base β.

Let T be a linear P-isometry. For each $k \in \{1, 2, \ldots, n\}$, we express

$$T(e_{i_k}) = \sum_{j=1}^{n} x_{jk} e_{i_j}.$$

Since T is an isometry, we must have

$$\begin{aligned} w_P(T(e_{i_k})) &= \max_{j \in \{1,2,\ldots,n\}} \{j : x_{jk} \neq 0\} \\ &= w_P(e_{i_k}) \\ &= k \end{aligned}$$

and it follows that $T(e_{i_k}) = \sum_{j=1}^{k} x_{jk} e_{i_j}$, with $x_{kk} \neq 0$ and we find that the matrix of T relative to the base β is given by

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & x_{22} & x_{23} & \cdots & x_{2n} \\ 0 & 0 & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_{nn} \end{pmatrix}$$

with $x_{kk} \neq 0$ for every $k \in \{1, 2, \ldots, n\}$. $\square$

Two posets P and Q are *isomorphic* if there exists an *order-preserving bijection* $\phi : P \to Q$, called of *isomorphism*, whose inverse is order preserving; that is,

$$x \leq y \text{ in } P \text{ if and only if } \phi(x) \leq \phi(y) \text{ in } Q.$$

An isomorphism $\phi : P \to P$ is called *automorphism*.

We also recall that given a poset P, we can say that a subset $P' \subseteq P$, with the induced order, is *connected* if, for every $x, y \in P'$, there is a sequence of elements $x = x_0, x_1, \ldots, x_k = y \in P'$ such that $x_i \geq x_{i-1}$ or $x_i \leq x_{i-1}$ for every $i \in \{1, 2, \ldots, k\}$. A *connected component* of P is a maximal connected subposet. It is clear that every poset P can be described as a disjoint union

$$P = P_1 \overset{\circ}{\cup} P_2 \overset{\circ}{\cup} \ldots \overset{\circ}{\cup} P_r,$$

where each P_j is a connected component.

For each subset $P' \subseteq P$ we denoted by $[P']$ the subspace of $\mathbb{F}_q^n$ generated by the base $\{e_i\}_{i \in P'}$, e_i the canonical vector of $\mathbb{F}_q^n$. So, if $P = \bigcup_{j=1}^{r} P_j$ is a disjoint union of connected components, we have that $\mathbb{F}_q^n$ is a direct sum

$$[P_1] \oplus \ldots \oplus [P_r] .$$

If β_i is a base of $[P_i]$, for $i \in \{1, 2, \ldots, r\}$, we have that $\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_r$ is a base of $\mathbb{F}_q^n = [P]$. Moreover, if β is the canonical base of $\mathbb{F}_q^n$, then $\beta \cap [P_i]$ is a base of $[P_i]$, *the canonical base of* $[P_i]$, for every $i \in \{1, 2, \ldots, r\}$. In this situation, if $x \in P_i$ we have that $w_P(x) = w_{P_i}(x)$, where w_{P_i} is the induced weight on P_i. Moreover, if $x = \sum_{i=1}^{r} y_i$, with $y_i \in P_i$ for every $i \in \{1, 2, \ldots, r\}$, we have that

$$w_P(x) = \sum_{i=1}^{r} w_P(y_i) = \sum_{i=1}^{r} w_{P_i}(y_i) .$$

Let $P = \{1, 2, \ldots, n\}$ be a disjoint union of r chains $P_1, P_2, \ldots, P_r$, and let us write $P_i = \left\{i_{k_{i-1}+1} < \ldots < i_{k_i}\right\}$, for $i \in \{1, 2, \ldots, r\}$. For the sake of simplicity, we remove the indices and assume, without loss of generality, that such poset P is given by $P_i = \{k_{i-1} + 1 < \ldots < k_i\}$, for $i \in \{1, 2, \ldots, n\}$. Let $x = \sum_{i=k_s+1}^{k_{s+1}} x_i$ and $y = \sum_{j=k_t+1}^{k_{t+1}} y_j$ be vectors spanned by the coordinates corresponding to P_{k_s} and P_{k_t} respectively. Since each of those are totally ordered, we have that

$$w_P(x) = \max\{i = k_s + 1, \ldots, k_{s+1} : x_i \neq 0\}$$

and

$$w_P(y) = \max\{j = k_t + 1, \ldots, k_{t+1} : y_j \neq 0\} .$$

If $s \neq t$ we have that $w_P(x+y) = w_P(x) + w_P(y)$ while $s = t$ implies $w_P(x+y) \leq \max\{w_P(x), w_P(y)\}$.

We consider on $\mathbb{F}_q^n$ the canonical base $\{e_1, e_2, \ldots, e_n\}$. The vectors with P-weight equal to 1 are exactly the multiples of the vectors $e_{k_0+1}, e_{k_1+1}, ..., e_{k_{r-1}+1}$, the vectors that have only one non-zero coordinate and such a coordinate is a minimal vector in some chain. So, if T is a linear isometry, we must have that $T(e_{k_s+1}) = \alpha e_{k_t+1}$, for some $0 \neq \alpha \in \mathbb{F}_q^n$ and some $t \in \{0, 1, \ldots, r-1\}$. If $x \in \mathbb{F}_q^n$ is a vector with $w_P(x) = 2$, we must have that either $x = \alpha_1 e_{k_{r_1}+1} + \alpha_2 e_{k_{r_2}+1}$ for some $0 \neq \alpha_1, \alpha_2 \in \mathbb{F}_q$ and $r_1 \neq r_2$ (it means, x has exactly two non zero coordinates, both of them corresponding to minimal elements in different chains) or $x = \alpha e_{k_t+1} + \beta e_{k_t+2}$, for some $\alpha, \beta \in \mathbb{F}_q$, with $\beta \neq 0$ and some r such that $k_t + 2 < k_{t+1}$.

So, let e_i be a vector in the base β with $w_P(e_i) = 2$. Then $e_i = e_{k_s+2}$ for some $s \in \{0, 1, \ldots, r-1\}$. If $T(e_{k_s+2}) = \alpha_1 e_{k_{r_1}+1} + \alpha_2 e_{k_{r_2}+1}$, with $0 \neq \alpha_1, \alpha_2 \in \mathbb{F}_q$ and $r_1 \neq r_2$, we find that the vectors

$$y = e_{k_s+2} - \alpha_1 T^{-1}\left(e_{k_{r_1}+1}\right) \text{ and } z = e_{k_s+2} - \alpha_2 T^{-1}\left(e_{k_{r_2}+1}\right)$$

satisfy

$$w_P(T(y)) = w_P(T(z)) = 1.$$

But $w_P(y), w_P(z) \geq 2$, unless

$$e_{k_s+2} = \alpha_1 T^{-1}\left(e_{k_{r_1}+1}\right)$$

or

$$e_{k_s+2} = \alpha_2 T^{-1}\left(e_{k_{r2}+1}\right).$$

So, the only possibility left is to have $T(e_{k_s+2}) = \alpha e_{k_t+1} + \beta e_{k_t+2}$, for some $\alpha, \beta \in \mathbb{F}_q$, $\beta \neq 0$ and t is such that $T(e_{k_s+1}) = \gamma e_{k_t+1}$, for some $\gamma \neq 0$.

So, we find that if $w_P(x) = 1$, $x \in [P_r]$ and $T(x) \in [P_t]$, then, for every $y \in [P_r]$ with $w_P(y) = 2$, we have that $T(y) \in [P_t]$.

Proceeding in this manner we can show that for every k, given $x \in [P_r]$ such that $w_P(x) = k$, if $T(x) \in [P_t]$, it follows that, for every $y \in [P_r]$ such that $w_P(y) = k+1$, then also its image is in the same subspace, i.e., $T(y) \in [P_t]$. But this implies that $T([P_r]) \subseteq [P_t]$. Since there are finitely many such subspaces, we find that actually $T([P_r]) = [P_t]$. We observe that, in this case, we must have that $\mathrm{supp}([P_r])$ and $\mathrm{supp}(T([P_r]))$ are isomorphic sub-posets of P, and we have proved the following:

Proposition 1.1 *Let $P = P_1 \mathring{\cup} P_2 \mathring{\cup} \ldots \mathring{\cup} P_r$ be a partial order on the set $\{1, 2, \ldots, n\}$ consisting of disjoint chains. Let T be a linear isometry of $\mathbb{F}_q^n$, endowed with the metric $d_P(\cdot, \cdot)$. Then, for every chain $P_i \subset P$, $i \in \{1, 2, \ldots, r\}$, there is a chain P_j, with* $\mathrm{supp}([P_i])$ *isomorphic to* $\mathrm{supp}([P_j])$*, such that $T([P_i]) = [P_j]$.*

In other words, a linear isometry permutes the subspaces having support in chains of the same cardinality.

As in [9], a *monomial matrix* is a matrix with exactly one nonzero entry in each row and column. Thus a monomial matrix over $\mathbb{F}_2$ is a permutation matrix, and a monomial matrix over an arbitrary finite field is a permutation matrix times an invertible diagonal matrix.

We now describe the main result of this section:

Theorem 1.2 *Let $P = P_1 \overset{\circ}{\cup} P_2 \overset{\circ}{\cup} \ldots \overset{\circ}{\cup} P_s$ be a poset consisting of a disjoint union of r chains. Denoted by μ_i the cardinality of the i-th chain, $i \in \{1, 2, \ldots, s\}$. For every $j \in \{1, 2, \ldots, n\}$ let $\nu_j = |\{P_i : |P_i| = j\}|$, where $|\cdot|$ is the cardinality of the given set. Then, there is an ordered base β of $\mathbb{F}_q^n$ relative to which every linear isometry $T \in GL_P\left(\mathbb{F}_q^n\right)$ is represented by the product $A \cdot U$ of $n \times n$ matrices, where U is a monomial matrix that acts exchanging coordinate subspaces with isomorphic supports and*

$$A = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_s \end{pmatrix},$$

where each A_i is a $\mu_i \times \mu_i$ upper triangular matrix with non zero diagonal entries.

Proof. Let $P_1 \overset{\circ}{\cup} P_2 \overset{\circ}{\cup} \ldots \overset{\circ}{\cup} P_s$ be the order in the poset P with

$$P_j = \{i_{\mu_{j-1}+1} < \ldots < i_{\mu_{j-1}+\mu_j}\},$$

where $\mu_0 := 0$. Let $\beta' = \{e_1, e_2, \ldots, e_n\}$ be the canonical base of $\mathbb{F}_q^n$. Consider the ordered base $\beta = \beta_{\mu_1} \cup \ldots \cup \beta_{\mu_s}$ where $\beta_{\mu_j} = \left\{e_{i_{\mu_{j-1}+1}}, e_{i_{\mu_{j-1}+2}}, \ldots, e_{i_{\mu_{j-1}+\mu_j}}\right\}$ is the canonical base of $[P_j]$. Let T be a linear P-isometry. For each $j \in \{1, 2, \ldots, s\}$ and each $t \in \{1, 2, \ldots, \mu_j\}$, we express

$$T\left(e_{i_{\mu_{j-1}+t}}\right) = \sum_{l=1}^{n} x_{l,(\mu_{j-1}+t)} e_{i_l}.$$

By Proposition 1.1, we know that for every $j \in \{1, 2, \ldots, s\}$ there is an $j' \in \{1, 2, \ldots, s\}$ such that $T([P_j]) = [P_{j'}]$, where $\mathrm{supp}([P_j])$ and $\mathrm{supp}([P_{j'}])$ are isomorphic posets. We let σ be the permutation of $P = \{1, 2, \ldots, n\}$ such that

$$\sigma(\mu_{j-1} + t) = \mu_{j'-1} + t, \ j \in \{1, 2, \ldots, s\}, \ t \in \{1, 2, \ldots, \mu_j\}.$$

Considering the base β and the canonical action of the symmetric group on $\mathbb{F}_q^n$ (permutating the coordinates), we get that σ is represented by a monomial matrix U. Let S be the linear transformation defined by U (relative to the base β). This is clearly a linear P-isometry and we get that $T \circ S^{-1}$ is a linear P-isometry under which every subspace P_j is invariant, for every $j \in \{1, 2, \ldots, s\}$. So, the matrix of $T \circ S^{-1}$ relative to the base β is given by

$$A := \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_s \end{pmatrix}.$$

But $\mathrm{supp}([P_j])$ is a chain for every $j \in \{1, 2, \ldots, s\}$. So, if we assume that

$$i_{\mu_j+1} < i_{\mu_j+2} < \cdots < i_{\mu_j+\mu_{j+1}}$$

for every $j \in \{1, 2, \ldots, s\}$, $t \in \{1, 2, \ldots, \mu_{j+1}\}$, Theorem 1.1 assures that each A_j is an upper triangular matrix with non-zero diagonal entries. Since $T = (T \circ S) \circ S^{-1}$, we get that the matrix of T relative to the base β is given by the product $A \cdot U$. $\square$

Corollary 1.1 *With the hypotheses of the theorem above, the cardinality of $GL_P(\mathbb{F}_q^n)$ is given by*

$$\left|GL_P\left(\mathbb{F}_q^n\right)\right| = (q-1)^n \cdot \left(\prod_{k=1}^{n} \nu_k!\right) \cdot \left(\prod_{j=1}^{s} q^{\frac{\mu_j(\mu_j-1)}{2}}\right),$$

where ν_j is the number of maximal chains in P of order j.

Proof. Every block-matrix A_j obtained in the previous Theorem is a $\mu_j \times \mu_j$ triangular matrix with non zero diagonal entries. Moreover, every choice of such matrices defines a P-isometry. So, there are $q-1$ possible choices for each of the μ_j diagonal entries of A_j and q possible choices for each of the $\mu_j(\mu_j-1)/2$ entries of A_j above the diagonal. It follows that, in the decomposition given in the previous Theorem, there are exactly

$$\prod_{j=1}^{s}(q-1)^{\mu_j} q^{\frac{\mu_j(\mu_j-1)}{2}} = (q-1)^{\Sigma_{j=1}^{s}\mu_j} \prod_{j=1}^{s} q^{\frac{\mu_j(\mu_j-1)}{2}} = (q-1)^n \prod_{j=1}^{s} q^{\frac{\mu_j(\mu_j-1)}{2}}$$

different possibilities for the matrix A. The monomial component U permutes chains with the same cardinality, and consequently, if there are ν_k maximal chains of cardinality k contained in P, there are $\nu_k!$ possibilities to permute them, so that the monomial component found in Theorem 1.2 may be chosen in exactly $\prod_{k=1}^{n} \nu_k!$ different ways and we find that

$$\left|GL_P\left(\mathbb{F}_q^n\right)\right| = (q-1)^n \cdot \left(\prod_{k=1}^{n} \nu_k!\right) \cdot \left(\prod_{j=1}^{s} q^{\frac{\mu_j(\mu_j-1)}{2}}\right).$$

□

2 Linear Isometries for a General Poset Structures

¿From here on, we denote by $\{e_1, e_2, \ldots, e_n\}$ the canonical base of $\mathbb{F}_q^n$.

Given $x, y \in P$, we say that y *covers* x if $x < y$ and if no element $z \in P$ satisfies $x < z < y$. A chain $x_1 < x_2 < \ldots < x_k$ in a finite poset P is called *saturated* if x_i covers x_{i-1} for $i \in \{1, 2, \ldots, k\}$.

Given an order automorphism $\phi : P \to P$, we define the *canonical linear P-isometry T_ϕ induced by ϕ* as $T_\phi(\sum_{i=1}^{n} a_i e_i) := \sum_{i=1}^{n} a_i e_{\phi(i)}$.

We will show that a linear isometry $T \in GL_P\left(\mathbb{F}_q^n\right)$ induces an automorphism of the poset P in the following way: given $i \in \{1, 2, \ldots, n\}$ we consider any saturated chain $i_1 < i_2 < \ldots < i_k$ containing i. Then there are $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$, with j_{s+1} covering j_s for all $s \in \{1, 2, \ldots, k-1\}$, such that $\langle \mathrm{supp}\,(e_{j_l})\rangle = \langle \mathrm{supp}\,(T\,(e_{i_l}))\rangle$ for any $l \in \{1, 2, \ldots, k\}$. So, if $i = i_l$, we can define the order automorphism ϕ by $\phi\,(i_l) = j_l$.

The key to prove this is to show that $\langle \mathrm{supp}\,(T\,(u))\rangle \subseteq \langle \mathrm{supp}\,(T\,(v))\rangle$ if $\langle \mathrm{supp}\,(u)\rangle \subseteq \langle \mathrm{supp}\,(v)\rangle$, for every $T \in GL_P\left(\mathbb{F}_q^n\right)$.

We will start with some preliminary lemmas.

Lemma 2.1 *Let $P = \{1, 2, \ldots, n\}$ be a poset, $\{e_1, e_2, \ldots, e_n\}$ the canonical base of $\mathbb{F}_q^n$ and $T \in GL_P(\mathbb{F}_q^n)$. If $\langle \mathrm{supp}\,(e_i)\rangle \subseteq \langle \mathrm{supp}\,(e_j)\rangle$, then*

$$\langle \mathrm{supp}\,(T\,(e_i))\rangle \subseteq \langle \mathrm{supp}\,(T\,(e_j))\rangle\,.$$

Proof. We observe that, for any vectors $u, v \in \mathbb{F}_q^n$, if $\mathrm{supp}\,(u) \subseteq \mathrm{supp}\,(v)$ then $w_P\,(u) \leq w_P\,(v)$. Moreover, the inequality is strict if and only if $\langle \mathrm{supp}\,(u)\rangle \subsetneq$

$\langle \operatorname{supp}(v) \rangle$. We remember that T is a linear isometry, so that $w_P(v) = w_P(T(v))$, for every vector v.

We prove the lemma by contradiction, assuming that $\langle \operatorname{supp}(T(e_i)) \rangle \nsubseteq \langle \operatorname{supp}(T(e_j)) \rangle$.

Suppose $\langle \operatorname{supp}(T(e_i)) \rangle \cap \langle \operatorname{supp}(T(e_j)) \rangle = \varnothing$. Since T is linear,

$$w_P(T(e_i + e_j)) = w_P(T(e_i) + T(e_j))$$

and since the ideals do not intersect, we have that

$$w_P(T(e_i) + T(e_j)) = w_P(T(e_i)) + w_P(T(e_j)).$$

Since T is an isometry, we find that

$$\begin{aligned} w_P(T(e_i)) + w_P(T(e_j)) &= w_P(e_i) + w_P(e_j) > w_P(e_j) \\ w_P(T(e_i + e_j)) &= w_P(e_i + e_j). \end{aligned}$$

However, we are assuming that $\langle \operatorname{supp}(e_i) \rangle \subseteq \langle \operatorname{supp}(e_j) \rangle$, so that $w_P(e_i + e_j) = w_P(e_j)$, a contradiction.

Now we can assume that $\langle \operatorname{supp}(T(e_i)) \rangle \cap \langle \operatorname{supp}(T(e_j)) \rangle \neq \varnothing$. If we put $\operatorname{supp}(T(e_i)) \cap \operatorname{supp}(T(e_j)) = \{k_1, \ldots, k_r\}$, we have two cases to consider.

Case 1: $\{k_1, \ldots, k_r\} \neq \varnothing$.

In this case, we can write

$$\operatorname{supp}(T(e_i)) = \{k_1, \ldots, k_r\} \cup \{i_1, \ldots, i_s\}$$

and

$$T(e_i) = \alpha_{k_1} e_{k_1} + \ldots + \alpha_{k_r} e_{k_r} + \beta_{i_1} e_{i_1} + \ldots + \beta_{i_s} e_{i_s}.$$

Let

$$y = e_i - \beta_{i_1} T^{-1}(e_{i_1}) - \ldots - \beta_{i_s} T^{-1}(e_{i_s}).$$

Then

$$w_P(y) \geq w_P(e_i),$$

unless

$$e_i = \beta_{i_1} T^{-1}(e_{i_1}) + \ldots + \beta_{i_s} T^{-1}(e_{i_s}) = T^{-1}(\beta_{i_1} e_{i_1} + ... + \beta_{i_s} e_{i_s}),$$

contradicting the hypothesis that $\{k_1, \ldots, k_r\} \neq \varnothing$. But $T(y) = \alpha_{k_1} e_{k_1} + \ldots + \alpha_{k_r} e_{k_r}$, and since there is $i_l \in \{i_1, \ldots, i_s\} \subseteq \operatorname{supp}(T(e_i))$ such that $i_l \notin \operatorname{supp}(T(e_j))$, we find that $w_P(T(y)) < w_P(T(e_i)) = w_P(e_i)$. So

$$w_P(T(y)) < w_P(y),$$

a contradiction.

Case 2: $\{k_1, \ldots, k_r\} = \varnothing$.

This means that $\mathrm{supp}\,(T\,(e_i)) \cap \mathrm{supp}\,(T\,(e_j)) = \varnothing$. Put $T\,(e_i) = \alpha_{i_1} e_{i_1} + \ldots + \alpha_{i_t} e_{i_t}$. Then there is an

$$l \in \langle \mathrm{supp}\,(T\,(e_i)) \rangle \setminus \mathrm{supp}\,(T\,(e_i))\,. \tag{1}$$

Let

$$y = e_i - \alpha_{i_1} T^{-1}\,(e_{i_1}) - \ldots - \alpha_{i_t} T^{-1}\,(e_{i_t}) + T^{-1}\,(e_l)\,.$$

Then

$$w_P\,(y) \geq w_P\,(e_i)\,,$$

unless $e_i = T^{-1}\,(e_l)$, and this contradicts (1). But, $T\,(y) = e_l$ and hence

$$w_P\,(T\,(y)) = w_P\,(e_l) < w_P\,(e_i) \leq w_P\,(y)\,,$$

again a contradiction. $\square$

Lemma 2.2 *Let $P = \{1, 2, \ldots, n\}$ be a poset, $T \in GL_P(\mathbb{F}_q^n)$ and $\{e_1, e_2, \ldots, e_n\}$ the canonical base of $\mathbb{F}_q^n$. Then,*

$$\bigcup_{i=1}^{s} \langle \mathrm{supp}\,(T\,(e_{j_i})) \rangle = \left\langle \mathrm{supp}\left(\sum_{i=1}^{s} T\,(e_{j_i}) \right) \right\rangle,$$

for every $s \in \{1, 2, \ldots, n\}$ and $j_1, \ldots, j_s \in \{1, \ldots, n\}$.

Proof. If $j \in \langle \mathrm{supp}\,(\sum_{i=1}^{s} T\,(e_{j_i})) \rangle$, there is an i such that $j \in \langle \mathrm{supp}\,(T\,(e_{j_i})) \rangle$, so that

$$\left\langle \mathrm{supp}\left(\sum_{i=1}^{s} T\,(e_{j_i}) \right) \right\rangle \subseteq \bigcup_{i=1}^{s} \langle \mathrm{supp}\,(T\,(e_{j_i})) \rangle\,.$$

We will prove the other inclusion by induction on s. The case $s = 1$ is trivial and we can assume, as the induction hypothesis that

$$\left\langle \mathrm{supp}\left(\sum_{i=1}^{s-1} T\,(e_{j_i}) \right) \right\rangle = \bigcup_{i=1}^{s-1} \langle \mathrm{supp}\,(T\,(e_{j_i})) \rangle\,,$$

for every subset $\{j_1, \ldots, j_{s-1}\} \subseteq \{1, \ldots, n\}$.

Given $J = \{j_1, \ldots, j_s\} \subseteq \{1, \ldots, n\}$ and $t \in \{1, 2, \ldots, s\}$, we can define

$$\Theta_{J,t} = \langle \operatorname{supp}(T(e_{j_t})) \rangle \setminus \left(\bigcup_{i=1, i\neq t}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle \right).$$

But $\Theta_{J,t} = \varnothing$ means that every $j \in \langle \operatorname{supp}(T(e_{j_t})) \rangle$ we have

$$j \in \bigcup_{i=1, i\neq t}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle$$

so that

$$\bigcup_{i=1}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle = \bigcup_{i=1, i\neq t}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle$$

and by the induction hypothesis we have that

$$\bigcup_{i=1}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle = \left\langle \operatorname{supp}\left(\sum_{i=1, i\neq t}^{s} T(e_{j_i}) \right) \right\rangle. \tag{2}$$

Since

$$\left\langle \operatorname{supp}\left(\sum_{i=1}^{s} T(e_{j_i}) \right) \right\rangle \subseteq \bigcup_{i=1}^{s} \langle \operatorname{supp}(T(e_{j_i})) \rangle$$

we have that

$$\left\langle \operatorname{supp}\left(\sum_{i=1}^{s} T(e_{j_i}) \right) \right\rangle \subseteq \left\langle \operatorname{supp}\left(\sum_{i=1, i\neq t}^{s} T(e_{j_i}) \right) \right\rangle. \tag{3}$$

Since T is a linear isometry, we have that

$$w_P\left(\sum_{i=1}^{s} T(e_{j_i}) \right) = w_P\left(T\left(\sum_{i=1}^{s} e_{j_i} \right) \right) = w_P\left(\sum_{i=1}^{s} e_{j_i} \right),$$

$$w_P\left(\sum_{i=1, i\neq t}^{s} T(e_{j_i}) \right) = w_P\left(T\left(\sum_{i=1, i\neq t}^{s} e_{j_i} \right) \right) = w_P\left(\sum_{i=1, i\neq t}^{s} e_{j_i} \right).$$

But

$$w_P\left(\sum_{i=1}^{s} e_{j_i} \right) \geq w_P\left(\sum_{i=1, i\neq t}^{s} e_{j_i} \right) \tag{4}$$

and since by definition, we have that $w_P(v) = |\langle \text{supp}(v) \rangle|$, considering inequality (4) in (3) we find that

$$\left\langle \text{supp}\left(\sum_{i=1}^{s} T(e_{j_i})\right)\right\rangle = \left\langle \text{supp}\left(\sum_{i=1, i\neq t}^{s} T(e_{j_i})\right)\right\rangle$$

and from (2) we get that

$$\left\langle \text{supp}\left(\sum_{i=1}^{s} T(e_{j_i})\right)\right\rangle = \bigcup_{i=1}^{s} \langle \text{supp}(T(e_{j_i}))\rangle,$$

so that the lemma holds if for every $s \geq 2$, there is $J = \{j_1, \ldots, j_s\}$ and $t \in \{1, 2, \ldots, s\}$ such that $\Theta_{J,t} = \varnothing$.

The case of an antichain P is trivial, so we can assume that the poset P is not an antichain order, and hence there are $l_1, l_2 \in \{1, 2, \ldots, n\}$ such that l_2 covers l_1. So, given $s \geq 2$, for every $J = \{l_1, l_2, j_3, \ldots, j_s\}$ we have that $\Theta_{J,l_1} = \varnothing$, since

$$\langle \text{supp}(e_{l_1})\rangle = \langle l_1 \rangle \subseteq \langle l_2 \rangle = \langle \text{supp}(e_{l_2})\rangle.$$

□

Now we can state and prove the proposition that extends Lemma 2.1 to general vectors.

Proposition 2.1 *Let $P = \{1, 2, \ldots, n\}$ be a poset, $T \in GL_P(\mathbb{F}_q^n)$. Then, for every $u, v \in \mathbb{F}_q^n$,*

$$\langle \text{supp}(T(u))\rangle \subseteq \langle \text{supp}(T(v))\rangle,$$

if $\langle \text{supp}(u)\rangle \subseteq \langle \text{supp}(v)\rangle$.

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be the canonical base of $\mathbb{F}_q^n$ and express u and v as a linear combination of this base:

$$\begin{aligned} u &= \alpha_1 e_{u_1} + \alpha_2 e_{u_2} + \ldots + \alpha_r e_{u_r} \\ v &= \beta_1 e_{v_1} + \beta_2 e_{v_2} + \ldots + \beta_s e_{v_s} \end{aligned}$$

with $\text{supp}(u) = \{u_1, \ldots, u_r\}$ and $\text{supp}(v) = \{v_1, \ldots, v_s\}$. Since $\langle \text{supp}(u)\rangle \subseteq \langle \text{supp}(v)\rangle$ we have that $\langle \text{supp}(e_{u_i})\rangle \subseteq \langle \text{supp}(v)\rangle$ for every $i \in \{1, 2, \ldots, r\}$, so there is an $j \in \{1, 2, \ldots, s\}$ such that $\langle \operatorname{supp}(e_{u_i}) \rangle \subseteq \langle \operatorname{supp}(e_{v_j}) \rangle$. But Lemma 2.1 assures that $\langle \operatorname{supp}(T(e_{u_i})) \rangle \subseteq \langle \operatorname{supp}(T(e_{v_j})) \rangle$. It follows that

$$\begin{aligned} \langle \operatorname{supp}(T(u)) \rangle &= \left\langle \operatorname{supp}\left(\sum_{i=1}^{r} T(e_{u_i}) \right) \right\rangle \\ &\subseteq \bigcup_{i=1}^{r} \langle \operatorname{supp}(T(e_{u_i})) \rangle \\ &\subseteq \bigcup_{j=1}^{s} \langle \operatorname{supp}(T(e_{v_j})) \rangle \end{aligned}$$

and by Lemma 2.2 we have that

$$\begin{aligned} \langle \operatorname{supp}(T(v)) \rangle &= \left\langle \operatorname{supp}\left(\sum_{j=1}^{s} T(\beta_j e_{v_j}) \right) \right\rangle \\ &= \bigcup_{j=1}^{s} \langle \operatorname{supp}(T(\beta_j e_{v_j})) \rangle \\ &= \bigcup_{j=1}^{s} \langle \operatorname{supp}(T(e_{v_j})) \rangle \end{aligned}$$

and we find

$$\langle \operatorname{supp}(T(u)) \rangle \subseteq \langle \operatorname{supp}(T(v)) \rangle .$$

□

An ideal I of a poset P is said to be a *prime ideal* if it contains an unique maximal element.

Lemma 2.3 *Let $P = \{1, 2, ..., n\}$ be a poset, $\beta = \{e_1, e_2, \ldots, e_n\}$ be the canonical base of $\mathbb{F}_q^n$ and $T \in GL_P(\mathbb{F}_q^n)$. Then, for every $r \in \{1, 2, \ldots, n\}$, we have that $\langle \operatorname{supp}(T(e_r)) \rangle$ is a prime ideal.*

Proof. We want to prove that the ideal $\langle \operatorname{supp}(T(e_r)) \rangle$ is generated by a single greatest element (greater than every other element), or alternatively, it has only one maximal element (no one greater than it). Let $\{j_1, j_2, \ldots, j_k\}$ be a set of maximal elements in $\langle \operatorname{supp}(T(e_r)) \rangle$. Then we have that

$$\begin{aligned}
\langle \operatorname{supp}(T(e_r)) \rangle &= \bigcup_{i=1}^{k} \langle j_i \rangle \\
&= \bigcup_{i=1}^{k} \langle \operatorname{supp}(e_{j_i}) \rangle \\
&= \left\langle \operatorname{supp}\left(\sum_{i=1}^{r} e_{j_i} \right) \right\rangle .
\end{aligned}$$

But Proposition 2.1 assures that we can apply T^{-1} to both sides of the equation above preserving the equality, so that

$$\langle \operatorname{supp}(e_r) \rangle = \left\langle \operatorname{supp}\left(T^{-1}T(e_r)\right) \right\rangle = \left\langle \operatorname{supp}\left(T^{-1}\left(\sum_{i=1}^{r} e_{j_i} \right)\right) \right\rangle . \tag{5}$$

Since T^{-1} is linear, we have that

$$\left\langle \operatorname{supp}\left(T^{-1}\left(\sum_{i=1}^{r} e_{j_i} \right)\right) \right\rangle = \left\langle \operatorname{supp}\left(\sum_{i=1}^{r} T^{-1}(e_{j_i}) \right) \right\rangle$$

and by Lemma 2.2, we have that

$$\left\langle \operatorname{supp}\left(\sum_{i=1}^{r} T^{-1}(e_{j_i}) \right) \right\rangle = \bigcup_{i=1}^{k} \left\langle \operatorname{supp}\left(T^{-1}(e_{j_i})\right) \right\rangle . \tag{6}$$

But looking at equations (5) and (6) we find that $\bigcup_{i=1}^{k} \langle \operatorname{supp}(T^{-1}(e_{j_i})) \rangle$ is the prime ideal $\langle \operatorname{supp}(e_r) \rangle$. Since we are expressing a prime ideal as the union of ideals, one of them, let us say $\langle \operatorname{supp}(T^{-1}(e_{j_s})) \rangle$ for some $s \in \{1, 2, \ldots, r\}$, must contain the maximal element r and hence $\langle \operatorname{supp}(T^{-1}(e_{j_s})) \rangle = \langle \operatorname{supp}(e_r) \rangle$. Using again Proposition 2.1, we find that

$$\langle \operatorname{supp}(e_{j_s}) \rangle = \langle \operatorname{supp}(T(e_r)) \rangle$$

so that $\langle \operatorname{supp} T(e_r) \rangle$ is a prime ideal and consequently $\{j_1, j_2, \ldots, j_k\} = \{j_s\}$. $\square$

Now we can state and prove the proposition that extends Lemma 2.3 to the general case.

Proposition 2.2 *Let $P = \{1, 2, \ldots, n\}$ be a poset and $T \in GL_P\left(\mathbb{F}_q^n\right)$. Then, for every $v \in \mathbb{F}_q^n$ such that $\langle \operatorname{supp}(v) \rangle$ is a prime ideal, $\langle \operatorname{supp}(T(v)) \rangle$ is also a prime ideal.*

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ the canonical base of $\mathbb{F}_q^n$ and $v \in \mathbb{F}_q^n$. Suppose that $v = \alpha_1 e_{i_1} + \ldots + \alpha_s e_{i_s}$. Then

$$\begin{aligned} \langle \operatorname{supp}(v) \rangle &= \langle \operatorname{supp}(\alpha_1 e_{i_1} + \ldots + \alpha_s e_{i_s}) \rangle \\ &= \langle \operatorname{supp}(e_{i_1}) \rangle \cup \ldots \cup \langle \operatorname{supp}(e_{i_s}) \rangle , \end{aligned}$$

and since $\langle \operatorname{supp}(v) \rangle$ is a prime ideal, it follows there is an $k \in \{1, 2, \ldots, s\}$ such that

$$\langle \operatorname{supp}(e_{i_1}) \rangle \cup \ldots \cup \langle \operatorname{supp}(e_{i_s}) \rangle = \langle \operatorname{supp}(e_{i_k}) \rangle$$

so that $\langle \operatorname{supp}(v) \rangle = \langle \operatorname{supp}(e_{i_k}) \rangle$. Lemma 2.1 assures that

$$\langle \operatorname{supp}(T(v)) \rangle = \langle \operatorname{supp}(T(e_{i_k})) \rangle ,$$

and as $\langle \operatorname{supp}(T(e_{i_k})) \rangle$ is a prime ideal (by Lemma 2.3), and we conclude that $\langle \operatorname{supp}(T(v)) \rangle$ is a prime ideal. $\square$

Lemma 2.4 *If k covers i and J is an ideal such that $\langle i \rangle \subseteq J \subseteq \langle k \rangle$, then $J = \langle i \rangle$ or $J = \langle k \rangle$.*

Proof. If $\langle i \rangle = J$, there is nothing to be proved. So, we assume that $\langle i \rangle \subsetneq J \subseteq \langle k \rangle$. Then, there is an $j \in J$ such that $j \gneqq i$. Since $J \subseteq \langle k \rangle$ it follows that $j \leq k$. So $i \lneqq j \leq k$, and since k covers i, we have that $j = k$ and hence $J = \langle k \rangle$. $\square$

Theorem 2.1 *Let $P = \{1, 2, \ldots, n\}$ be a poset, $\{e_1, e_2, \ldots, e_n\}$ be the canonical base of $\mathbb{F}_q^n$ and $T \in GL_P\left(\mathbb{F}_q^n\right)$ linear isometry. Then, for every saturated chain with a minimal element $i_i < i_2 < \ldots < i_r$ there is an unique saturated sequence of prime ideals*

$$\langle \operatorname{supp}(e_{j_1}) \rangle \subset \langle \operatorname{supp}(e_{j_2}) \rangle \subset \ldots \subset \langle \operatorname{supp}(e_{j_r}) \rangle .$$

such that

$$\langle \operatorname{supp}(T(e_{i_k})) \rangle = \langle \operatorname{supp}(e_{j_k}) \rangle$$

for every $k \in \{1, 2, \ldots, r\}$ and

$$\begin{array}{rccc} \phi : & P & \longrightarrow & P \\ & i_k & \longmapsto & \phi(i_k) := j_k \end{array}$$

is a well defined poset automorphism.

Proof. Proposition 2.2 assures us that $\langle \mathrm{supp}\,(T\,(e_{i_k}))\rangle$ is a prime for all $k \in \{1, 2, \ldots, r\}$, since $\langle \mathrm{supp}\,(e_{i_k})\rangle$ is a prime ideal. Then for each $k \in \{1, 2, \ldots, r\}$ there is just one maximal element $j_k \in \langle \mathrm{supp}\,(T\,(e_{i_k}))\rangle$. So $\langle \mathrm{supp}\,(T\,(e_{i_k}))\rangle = \langle \mathrm{supp}\,(e_{j_k})\rangle$ for all $k \in \{1, 2, \ldots, r\}$. Since

$$\langle \mathrm{supp}\,(e_{i_1})\rangle \subset \langle \mathrm{supp}\,(e_{i_2})\rangle \subset \ldots \subset \langle \mathrm{supp}\,(e_{i_r})\rangle\,,$$

it follows, from Proposition 2.1, that

$$\langle \mathrm{supp}\,(e_{j_1})\rangle \subset \langle \mathrm{supp}\,(e_{j_2})\rangle \subset \ldots \subset \langle \mathrm{supp}\,(e_{j_r})\rangle\,.$$

We affirm now that the sequence above is saturated. Suppose that for some $k \in \{1, 2, \ldots, r\}$ there is j' such that

$$\langle j_k\rangle \subsetneqq \langle j'\rangle \subsetneqq \langle j_{k+1}\rangle\,.$$

Since

$$\begin{array}{l} \langle j_k\rangle = \langle \mathrm{supp}\,(e_{j_k})\rangle = \langle \mathrm{supp}\,(T\,(e_{i_k}))\rangle\,, \\ \langle j_{k+1}\rangle = \left\langle \mathrm{supp}\left(e_{j_{k+1}}\right)\right\rangle = \left\langle \mathrm{supp}\left(T\left(e_{i_{k+1}}\right)\right)\right\rangle\,, \end{array}$$

it follows, applying Proposition 2.1) to the linear P-isometry T^{-1}, that

$$\begin{array}{rcl} \langle i_k\rangle & = & \left\langle \mathrm{supp}\left(T^{-1}T\,(e_{i_k})\right)\right\rangle \\ & \subsetneqq & \left\langle \mathrm{supp}\left(T^{-1}\,(e_{j'})\right)\right\rangle \\ & \subsetneqq & \left\langle \mathrm{supp}\left(T^{-1}T\left(e_{i_{k+1}}\right)\right)\right\rangle = \langle i_{k+1}\rangle\,, \end{array}$$

what contradicts, by Lemma 2.4, the hypothesis that $i_1 < \ldots < i_r$ is a saturated chain.

Let us now define $\phi : P \to P$ by $\phi\,(i_l) = j_l$. Since j_l is uniquely defined and does not depends on the choice of the saturated chain containing i_l (but only on $T\,(e_{i_l})$), we have that ϕ is well defined. Moreover, let us suppose that $x < y$ in P, and let

$$i_1 < \ldots < i_{k-1} < x < i_{k+1} < \ldots < i_{l-1} < y < i_{l+1} < \ldots < i_r$$

be a saturated chain containing x and y. Then there is only one saturated chain

$$j_1 < \ldots < j_{k-1} < j_k < j_{k+1} < \ldots < j_{l-1} < j_l < j_{l+1} < \ldots < j_r$$

such that $\phi\,(x) = j_k$ and $\phi\,(y) = j_l$. Since $j_k < j_l$ we get that $\phi\,(x) < \phi\,(y)$. Therefore ϕ is an application that preserve the order on P.

Finally, we affirm that ϕ is one-to-one. In fact, suppose that $\phi(x) = \phi(y)$. As $\phi(x) = \max \langle \operatorname{supp}(T(e_x)) \rangle$ and $\phi(y) = \max \langle \operatorname{supp}(T(e_y)) \rangle$ then

$$\langle \operatorname{supp}(T(e_x)) \rangle = \langle \operatorname{supp}(T(e_y)) \rangle ,$$

and from Proposition 2.1 follows that

$$\langle \operatorname{supp}(e_x) \rangle = \left\langle \operatorname{supp}\left(T^{-1}T(e_x)\right) \right\rangle = \left\langle \operatorname{supp}\left(T^{-1}T(e_y)\right) \right\rangle = \langle \operatorname{supp}(e_y) \rangle .$$

As both ideals $\langle \operatorname{supp}(e_x) \rangle$ and $\langle \operatorname{supp}(e_y) \rangle$ are primes, we must have $x = y$. Being ϕ one-to-one and P finite, we find that ϕ is a bijection that preserves the order and we conclude that ϕ is an automorphism of order. $\square$

The *m-th level* $\Gamma^{(m)}(P)$ is the set of elements of P that generates a prime ideal with cardinality m:

$$\Gamma^{(m)}(P) = \{i \in P : |\langle i \rangle| = m\} = \{i \in P : w_P(e_i) = m\} .$$

We now describe the main result of this work:

Theorem 2.2 *Let $P = \{1, 2, \ldots, n\}$ be a poset and $\{e_1, e_2, \ldots, e_n\}$ be the canonical base of $\mathbb{F}_q^n$. Then $T \in GL_P(\mathbb{F}_q^n)$ if and only if*

$$T(e_j) = \sum_{i \in \langle j \rangle} x_{ij} e_{\phi(i)}$$

where $\phi : P \to P$ is an automorphism of order and $x_{jj} \neq 0$, for any $j \in \{1, 2, \ldots, n\}$. Moreover, there is a pair of ordered bases β and β' of $\mathbb{F}_q^n$ relative to which every linear isometry $T \in GL_P\left(\mathbb{F}_q^n\right)$ is represented by an $n \times n$ upper triangular matrix $(a_{ij})_{1 \leq i,j \leq n}$ with $a_{ii} \neq 0$ for every $i \in \{1, 2, \ldots, n\}$.

Proof. Since $\langle \operatorname{supp}(e_j) \rangle$ is a prime ideal, it follows from Proposition 2.2 that $\langle \operatorname{supp}(T(e_j)) \rangle$ is also a prime ideal, for every $j \in \{1, 2, \ldots, n\}$. Given $j \in \{1, 2, \ldots, n\}$, let $j' = \phi(j)$ be the unique maximal element of the ideal $\langle \operatorname{supp}(T(e_j)) \rangle$, where $\phi : P \to P$ is the automorphism of order induced by the isometry T (see Theorem 2.1). Then

$$\langle \operatorname{supp}(T(e_j)) \rangle = \langle \operatorname{supp}(e_{j'}) \rangle = \left\langle \operatorname{supp}\left(e_{\phi(j)}\right) \right\rangle ,$$

and since ϕ is a automorphism of order we have that

$$\left\langle \operatorname{supp}\left(e_{\phi(j)}\right) \right\rangle = \{\phi(i) : i \in \langle j \rangle\} .$$

Therefore $\langle \text{supp}\,(T\,(e_j)) \rangle = \{\phi\,(i)\,| i \in \langle j \rangle\}$. Being $\phi\,(j) = \max\{\phi\,(i) : i \in \langle j \rangle\}$, we conclude that

$$T(e_j) = \sum_{i \in \langle j \rangle} x_{ij} e_{\phi(i)} \tag{7}$$

with $x_{jj} \neq 0$. It is straightforward to verify that every given an order automorphism $\phi : P \to P$, then, any linear map defined as in (7) is a P-isometry.

Let $\beta_m = \left\{e_i : i \in \Gamma^{(m)}\,(P)\right\}$ and

$$\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k.$$

be a decomposition of the canonical base of $\mathbb{F}_q^n$ as a disjoint union, where $k = \max\{w_P\,(e_i) : i = 1, 2, \ldots, n\}$. We order this base $\beta = \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$ in the following way (and denoted this total order by $\leq_\beta$): if $e_{i_r} \in \beta_{j_r}$ and $e_{i_s} \in \beta_{j_s}$ with $r \neq s$ then, $e_{i_r} \leq_\beta e_{i_s}$ if and only $j_r \leq j_s$. In other words, we begin enumerating the the vectors of β_1 and after exhausting them, we enumerate the vectors of β_2 and so on.

We define another ordered base β' as the base induced by the order automorphism ϕ,

$$\beta' := \left\{e_{\phi(i_1)}, e_{\phi(i_2)}, \ldots, e_{\phi(i_n)}\right\}$$

and let A be the matrix of T relative to the basis β and β':

$$[T]_{\beta,\beta'} = A = (a_{kl})_{1 \leq k,l \leq n}\,.$$

We find by the construction of the bases β and β' that $a_{kl} \neq 0$ implies $i_l \in \langle \phi\,(i_k) \rangle$. But $i_l \in \langle \phi\,(i_k) \rangle$ and $\langle i_l \rangle \neq \langle \phi\,(i_k) \rangle$ implies that $l < k$ so that A is upper triangular. Since A is invertible and upper triangular, we must have $\det(A) = \prod_{i=1}^n a_{ii} \neq 0$ so that $a_{ii} \neq 0$, for every $i \in \{1, 2, \ldots, n\}$. □

The upper triangular matrix obtained in the previous theorem is called a *canonical form of* T. We note that the ordered bases chosen in the theorem is unique up to re-ordination within the linearly independent sets β_i, $i = 1, 2, \ldots, k$.

Corollary 2.1 *Given* $T \in GL_P\left(\mathbb{F}_q^n\right)$ *there is an ordering* $\beta = \{e_{i_1}, e_{i_2}, \ldots, e_{i_n}\}$ *of the canonical base such that* $[T]_{\beta,\beta}$ *is given by the product* $A \cdot U$ *where* A *is an invertible upper triangular matrix and* U *is a monomial matrix obtained from the identity matrix by permutation of the columns, corresponding to the automorphism of order induced by* T *.*

Proof. Let ϕ be the automorphism of order induced by T. Let $T_{\phi^{-1}}$ be the linear isometry defined as $T_{\phi^{-1}}(e_j) = e_{\phi^{-1}(j)}$, for $j \in \{1, 2, \dots, n\}$. As we saw in Theorem 2.2,

$$T(e_j) = \sum_{i \in \langle j \rangle} x_{ij} e_{\phi(i)}.$$

So,

$$\begin{aligned}
T \circ T_{\phi^{-1}}(e_j) &= T\left(e_{\phi^{-1}(j)}\right) \\
&= \sum_{i \in \langle \phi^{-1}(j) \rangle} x_{i\phi^{-1}(j)} e_{\phi(i)} \\
&= x_{i\phi^{-1}(j)} e_j + \sum_{i \in \langle \phi^{-1}(j) \rangle, i \neq \phi^{-1}(j)} x_{i\phi^{-1}(j)} e_{\phi(i)}.
\end{aligned}$$

It follows that the automorphism of order induced by $T \circ T_{\phi^{-1}}$ is the identity, so, when taking the base β' as in the Theorem 2.2, we find that $\beta' = \beta$ and the matrix of $T \circ T_{\phi^{-1}}$ relative to this base is an upper triangular matrix $A = [T \circ T_{\phi^{-1}}]_\beta$. But $T_{\phi^{-1}}$ acts on $\mathbb{F}_q^n$ as a permutation of the vectors in β, so that in any ordered base containing those vectors, $U^{-1} = [T_{\phi^{-1}}]$ is obtained from the identity matrix by permutation of the columns. We note that $T_\phi = (T_{\phi^{-1}})^{-1}$ and it follows that

$$\begin{aligned}
[T]_\beta &= [T \circ T_{\phi^{-1}} \circ T_\phi]_\beta \\
&= [T \circ T_{\phi^{-1}}]_\beta [T_\phi]_\beta \\
&= A \cdot U.
\end{aligned}$$

□

Given a poset $P = \{1, 2, \dots, n\}$, we denote by $Aut(P)$ the group of the order-automorphisms of P.

Corollary 2.2 *Let $P = \{1, \dots, n\}$ be a poset and $k = \max\left\{m | \Gamma^{(m)}(P) \neq \varnothing\right\}$. Then*

$$\left|GL_P\left(\mathbb{F}_q^n\right)\right| = (q-1)^n \cdot \left(\prod_{i=1}^{k} q^{(i-1)\left|\Gamma^{(i)}(P)\right|}\right) \cdot |Aut(P)|.$$

Proof. ¿From Corollary 2.1, if $T \in GL_P\left(\mathbb{F}_q^n\right)$ there is an ordered base $\beta = \{e_{i_1}, e_{i_2}, \dots, e_{i_n}\}$ of the canonical base of $\mathbb{F}_q^n$ such that $|\langle i_l \rangle| \leq l$ for all $l \in \{1, 2, \dots, n\}$ and $[T]_\beta = A \cdot U$, being $A = (a_{kl})_{1 \leq k, l \leq n}$ an upper triangular matrix with $a_{kl} = 0$ if $i_k \notin \langle i_l \rangle$ and $U = [T_\phi]_\beta$ the matrix representing the automorphism ϕ induced by linear isometry T (see Theorem 2.2). Moreover, such base β depends only on ϕ and for every $\phi \in Aut\,(P)$, any matrix A as in the previous Corollary defines a linear P-isometry.

Given $l \in \{1, 2, \ldots, n\}$, there are $(q-1)$ possible different entries for a_{ll} (since $a_{ll} \neq 0$). But A is upper triangular, given $1 \leq i < j \leq n$ we have that $a_{ij} \neq 0$ only if $i \in \langle j \rangle$, so there are at most $|\langle j \rangle| - 1$ possible nonzero indices (i, j) with $1 \leq i < j \leq n$, and for each of those there are q possible different entries. Since there are exactly $\left|\Gamma^{(|\langle j \rangle|)}(P)\right|$ such indices, we find that, up to considering the order automorphism induced by the isometry, there are

$$(q-1)^n \cdot \left(\prod_{i=1}^{k} q^{(i-1)\left|\Gamma^{(i)}(P)\right|} \right)$$

linear P-isometries and we conclude counting the elements of $Aut(P)$. $\square$

3 Examples

We started this work with the particular case of totaly ordered posets or posets that are disjoint union of chains. In this section, we illustrate the results of this paper with three examples, the main classes of poset-metrics: the anti-chain order (which induces the classical Hamming weight), the weak order and the crown order.

Example 3.1 *If $A = \{1, 2, \ldots, n\}$ is antichain, then we have that w_A and d_A become the Hamming weight and Hamming metric of classical coding theory: if $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are vectors in $\mathbb{F}_q^n$, then*

$$\begin{aligned} &w_A(x) = |\{i : x_i \neq 0\}|, \\ &d_A(x, y) = w_A(x - y) = |\{i : x_i \neq y_i\}|. \end{aligned}$$

Since A has no comparable elements, the group of order automorphism $Aut\,(A)$ is isomorphic to the symmetric group $\mathbf{S}_n$. We also have that $\Gamma^{(1)}(A) = A$ and $\Gamma^{(m)}(A) = \varnothing$ if $m > 1$. It follows from Corollary 2.2 that

$$\left|GL_A\left(\mathbb{F}_q^n\right)\right| = (q-1)^n \cdot n!,$$

and from Corollary 2.1 we conclude that $GL_A\left(\mathbb{F}_q^n\right)$ is just the group of monomial matrices. Both conclusions agree perfectly with the fact that $GL_A\left(\mathbb{F}_q^n\right)$

is isomorphic to the semi-direct product $\left(\mathbb{F}_q^*\right)^n \rtimes \mathbf{S}_n$ *of the multiplicative group* $\left(\mathbb{F}_q^*\right)^n$ *with the symmetric group* $\mathbf{S}_n$*, as in [9]. The number* $(q-1)^n \cdot n!$ *can also be obtained from Corollary 1.1 (in this case we have* $\mu_1 = \ldots = \mu_n = 1$, $\nu_1 = n$ *and* $\nu_j = 0$ *if* $j > 1$*).*

Example 3.2 *Let* $n_1, \ldots, n_t$ *be positive integers with* $n_1 + \ldots + n_t = n$*. Then* $W = n_1\mathbf{1} \oplus \ldots \oplus n_t\mathbf{1}$ *will denote the* weak order *given by the ordinal sum of the antichains* $n_i\mathbf{1}$ *with* n_i *elements (see [4]). Explicitly,* $W = n_1\mathbf{1} \oplus \ldots \oplus n_t\mathbf{1}$ *is the poset whose underlying set and order relation are given by*

$$\{1, 2, \ldots, n\} = n_1\mathbf{1} \cup n_2\mathbf{1} \cup \ldots \cup n_t\mathbf{1},$$

$$n_i\mathbf{1} = \{n_1 + \ldots + n_{i-1} + 1, n_1 + \ldots + n_{i-1} + 2, \ldots, n_1 + \ldots + n_{i-1} + n_i\}$$

and

$$x < y \text{ if and only if } x \in n_i\mathbf{1},\ y \in n_j\mathbf{1} \text{ for some } i, j \text{ with } i < j.$$

Notice that if $n_1 = \ldots = n_t = 1$*, then* $W = 1\mathbf{1} \oplus \ldots \oplus 1\mathbf{1}$ *is totally ordered with* $1 < 2 < \ldots < t$ *and if* $t = 1$ *then* $W = n\mathbf{1}$ *is antichain.*

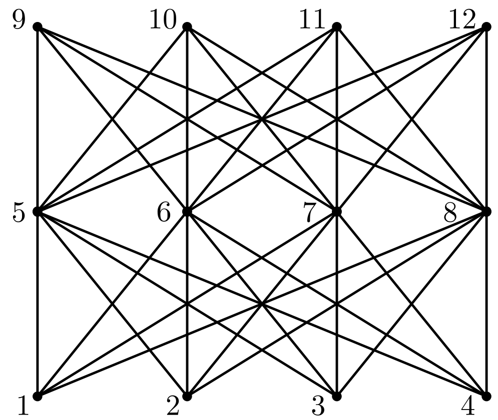

Figure 1: Weak order $W = 4\mathbf{1} \oplus 4\mathbf{1} \oplus 4\mathbf{1}$.

For a weak order $W = n_1\mathbf{1} \oplus \ldots \oplus n_t\mathbf{1}$ *we have that* $\Gamma^{(m)}(W) = n_s\mathbf{1}$ *if* $m = n_1 + n_2 + \ldots + n_{s-1} + 1$*, for any* $s \in \{1, 2, \ldots, t\}$ *and* $\Gamma^{(m)}(W) = \varnothing$ *otherwise. The group of the automorphism of order* $Aut(W)$ *is isomorphic to the cartesian product* $\mathbf{S}_{n_1} \times \mathbf{S}_{n_2} \times \ldots \times \mathbf{S}_{n_t}$ *(*$Aut(W)$ *is just the group of the applications* ϕ *that permutes only the elements of each* m*-th level). Corollary 2.2 assures us then that*

$$\left|GL_W\left(\mathbb{F}_q^n\right)\right| = (q-1)^n \cdot \left(\prod_{i=2}^{t} q^{n_i(n_1+n_2+\ldots+n_{i-1}+1)}\right) \cdot n_1! \cdot n_2! \cdot \ldots \cdot n_t!.$$

¿*From Theorem 2.2 follows that there are bases* β *and* β' *of* $\mathbb{F}_q^n$ *such that the matrix* $[T]_{\beta,\beta'}$ *is equal*

$$\begin{pmatrix} D_{n_1\times n_1} & * & * & \cdots & * \\ 0 & D_{n_2\times n_2} & * & \cdots & * \\ 0 & 0 & D_{n_3\times n_3} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_{n_t\times n_t} \end{pmatrix},$$

where

$$D_{n_s\times n_s} = diag\left(a_{\Sigma n_{s-1}+1,\Sigma n_{s-1}+1}, a_{\Sigma n_{s-1}+2,\Sigma n_{s-1}+2}, \ldots, a_{\Sigma n_{s-1}+n_s,\Sigma n_{s-1}+n_s}\right)$$

is a diagonal matrix for each $s = 1, 2, \ldots, t$*, and* $\Sigma n_{j-1} := n_1 + n_2 + \ldots + n_{j-1}$.

Considering the particular weak order $W = 4\mathbf{1}\oplus 4\mathbf{1}\oplus 4\mathbf{1}$ *(Hasse diagram illustrated in Figure* 1*), the matrix of a linear P-isometry* $[T]_{\beta,\beta'}$ *of* $T \in GL_W\left(\mathbb{F}_q^{12}\right)$ *is an upper triangular matrix as bellow:*

$$\left(\begin{array}{cccc|cccc|cccc}
a_{1,1} & 0 & 0 & 0 & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & a_{1,9} & a_{1,10} & a_{1,11} & a_{1,12} \\
0 & a_{2,2} & 0 & 0 & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & a_{2,9} & a_{2,10} & a_{2,11} & a_{2,12} \\
0 & 0 & a_{3,3} & 0 & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & a_{3,9} & a_{3,10} & a_{3,11} & a_{3,12} \\
0 & 0 & 0 & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} & a_{4,8} & a_{4,9} & a_{4,10} & a_{4,11} & a_{4,12} \\
\hline
0 & 0 & 0 & 0 & a_{5,5} & 0 & 0 & 0 & a_{5,9} & a_{5,10} & a_{5,11} & a_{5,12} \\
0 & 0 & 0 & 0 & 0 & a_{6,6} & 0 & 0 & a_{6,9} & a_{6,10} & a_{6,11} & a_{6,12} \\
0 & 0 & 0 & 0 & 0 & 0 & a_{7,7} & 0 & a_{7,9} & a_{7,10} & a_{7,11} & a_{7,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{8,8} & a_{8,9} & a_{8,10} & a_{8,11} & a_{8,12} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{9,9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10,10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{11,11} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{12,12}
\end{array}\right)$$

Example 3.3 *The* crown *is a poset with elements* $C = \{1, 2, \ldots, 2n\}$*,* $n > 1$*, in which* $i < n + i$*,* $i + 1 < n + i$ *for each* $i \in \{1, 2, \ldots, n-1\}$*, and* $1 < 2n$*,* $n < 2n$ *and these are the only strict comparabilities ([1]). The Hasse diagram of crown poset* P *with* $n = 4$ *is illustrated in Figure 2.*

Given a crown $C = \{1, 2, \ldots, 2n\}$*, we have that* $Aut(C)$ *is isomorphic to the dihedral group* D_n*, consisting of the orthogonal transformations which preserve a regular* n*-sided polygon centered at the origin of the euclidian plane. Considering the usual inclusion* $\iota : D_n \to \mathbf{S}_n$*, given* $g \in D_n$*, its action on* C *is defined by*

$$g(k) = \begin{cases} (\iota(g))(k) & \text{for } k = 1, 2, \ldots, n \\ (\iota(g))(k-n) & \text{for } k = n+1, \ldots, 2n \end{cases}$$

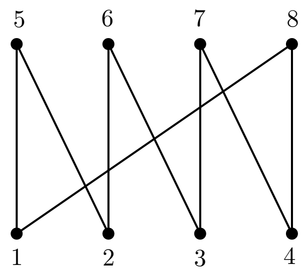

Figure 2: Crown poset $P = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

We note that $\Gamma^{(1)}(C) = \{1, 2, \dots, n\}$, $\Gamma^{(3)}(C) = \{n+1, \dots, 2n\}$, *and* $\Gamma^{(k)}(C) = \varnothing$, *for* $k \neq 1, 3$. *So, it follows from Corollary 2.2 that*

$$\left|GL_C\left(\mathbb{F}_q^{2n}\right)\right| = (q-1)^{2n} \cdot q^{2n} \cdot 2n.$$

Theorem 2.2 assures there is a pair of ordered bases β *and* β' *of* $\mathbb{F}_q^n$ *relative to which every linear isometry* $T \in GL_P\left(\mathbb{F}_q^n\right)$ *is represented by the* $[T]_{\beta,\beta'}$ $n \times n$ *upper triangular matrix*

$$\begin{pmatrix}
a_{1,1} & 0 & 0 & \cdots & 0 & a_{1,n+1} & 0 & \cdots & 0 & a_{1,2n} \\
0 & a_{2,2} & 0 & \cdots & 0 & a_{2,n+1} & a_{2,n+2} & \cdots & 0 & 0 \\
0 & 0 & a_{3,3} & \cdots & 0 & 0 & a_{3,n+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n,n} & 0 & 0 & \cdots & a_{n,2n-1} & a_{n,2n} \\
0 & 0 & 0 & \cdots & 0 & a_{n+1,n+1} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & a_{n+2,n+2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{2n-1,2n-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{2n,2n}
\end{pmatrix}.$$

In the particular case when $W = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (*see Figure 2*), *the canonical form of a linear* P*-isometry is*

$$\begin{pmatrix}
a_{1,1} & 0 & 0 & 0 & a_{1,5} & 0 & 0 & a_{1,8} \\
0 & a_{2,2} & 0 & 0 & a_{2,5} & a_{2,6} & 0 & 0 \\
0 & 0 & a_{3,3} & 0 & 0 & a_{3,6} & a_{3,7} & 0 \\
0 & 0 & 0 & a_{4,4} & 0 & 0 & a_{4,7} & a_{4,8} \\
0 & 0 & 0 & 0 & a_{5,5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{6,6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{7,7} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{8,8}
\end{pmatrix}.$$

The computations done in all the examples of this work is summarize in the tables bellow. We recall we are denoting by A, W, C, D and T respectively the antichain, weak, crown, disjoint union of chain and total orders. We recall that ν_j is the number of the components in D with cardinality equal to j (see Theorem 1.2).

Table 1: $Aut\left(P\right)$ and $\left|Aut\left(P\right)\right|$.

$P \diagdown Aut(P), \lvert Aut(P) \rvert$	$Aut\left(P\right)$	$\left\lvert Aut\left(P\right)\right\rvert$
T	$\{id\}$	1
D	$\mathbf{S}_{\nu_1} \times \mathbf{S}_{\nu_2} \times \ldots \times \mathbf{S}_{\nu_n}$	$\nu_1! \cdot \nu_2! \cdot \ldots \cdot \nu_t!$
A	$\mathbf{S}_n$	$n!$
W	$\mathbf{S}_{n_1} \times \mathbf{S}_{n_2} \times \ldots \times \mathbf{S}_{n_t}$	$n_1! \cdot n_2! \cdot \ldots \cdot n_t!$
C	D_n	$2n$

Table 2: $\Gamma^{(m)}\left(P\right) \neq \varnothing$ and $\left|\Gamma^{(m)}\left(P\right)\right|$.

$P \diagdown \Gamma^{(m)}(P) \neq \varnothing, \lvert \Gamma^{(m)}(P) \rvert$	$\Gamma^{(m)}\left(P\right) \neq \varnothing$	$\left\lvert\Gamma^{(m)}\left(P\right)\right\rvert$
T	$\Gamma^{(m)}\left(T\right) = \{1, 2, \ldots, m\}$	m
D	$\Gamma^{(m)}\left(D\right) = \left\{i_m, i_{\Sigma\mu_1+m}, \ldots, i_{\Sigma\mu_{s-1}+m}\right\}$	$\left\lvert\Gamma^{(m)}\left(D\right)\right\rvert \leq s$
A	$\Gamma^{(1)}\left(A\right) = A$	n
W	$\Gamma^{(\Sigma n_{s-1}+1)}\left(W\right) = n_s\mathbf{1}$	n_s
C	$\Gamma^{(1)}\left(C\right) = \{1, 2, \ldots, n\}$ $\Gamma^{(3)}\left(C\right) = \{n+1, n+2, \ldots, 2n\}$	n

Table 3: $\left|GL_P\left(\mathbb{F}_q^n\right)\right|$.

$P \diagdown \lvert GL_P(\mathbb{F}_q^n) \rvert$	$\left\lvert GL_P\left(\mathbb{F}_q^n\right)\right\rvert$
T	$(q-1)^n \cdot \left(\prod_{i=2}^{n} q^{i-1}\right)$
D	$(q-1)^n \cdot \left(\prod_{j=1}^{s} \nu_j!\right) \cdot \left(\prod_{k=1}^{s} q^{\frac{\mu_k(\mu_k-1)}{2}}\right)$
A	$(q-1)^n \cdot n!$
W	$(q-1)^n \cdot \left(\prod_{i=2}^{t} q^{n_i(\Sigma n_{i-1}+1)}\right) \cdot \left(\prod_{j=1}^{t} n_j!\right)$
C	$(q-1)^n \cdot q^n \cdot n$ if n is even

In the table bellow we compute $\left|GL_P\left(\mathbb{F}_q^n\right)\right|$ for T, A and C with $q=2$ and $n=2,3,\ldots,10$:

Table 4: Numbers of linear isometries of $|GL_P\left(\mathbb{F}_2^n\right)|$.

$n \diagdown \left|GL_P\left(\mathbb{F}_2^n\right)\right|$	$\left|GL_T\left(\mathbb{F}_2^n\right)\right|$	$\left|GL_A\left(\mathbb{F}_2^n\right)\right|$	$\left|GL_C\left(\mathbb{F}_2^n\right)\right|$
2	2	2	8
3	8	6	$*$
4	64	24	64
5	1024	120	$*$
6	32768	720	384
7	2097152	5040	$*$
8	268435456	40320	2048
9	$6.871947674 \cdot 10^{10}$	362880	$*$
10	$3.518437209 \cdot 10^{13}$	3628800	10240

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