Periodic Solutions in Unbounded Domains to the Boussinesq Equations

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Abstract

We study the existence and the uniqueness of strong periodic solutions for the Boussinesq equations in unbounded domains for the prescribed external forces.

1 Introduction

Let a viscous incompressible fluid filling a domain Ω of $\mathbb{R}^n,\ n \geq 2$. The evolution Boussinesq equations describe the evolution of the temperature and the velocity field of viscous incompressible Newtonian fluid. Due to the Boussinesq approximation (Chandrasekhar [7]), density variations are neglected except in the gravitational term (bouyancy term), in which they are assumed to be proportional to temperature variations. Then, in nondimensional form, the relationship among the velocity field $u(x,t) \in \mathbb{R}^n$, the pressure $p(x,t) \in \mathbb{R}$ and the temperature $\theta(x,t) \in \mathbb{R}$ can be described by the following initial value problem

$$\frac{\partial u}{\partial t} - \nu\Delta u + (u.\nabla u) + \frac{1}{\rho}\nabla p = \beta\theta g + f_1, \quad x \in \Omega,\ t \in \mathbb{R} \tag{1}$$

$$\nabla \cdot u = 0, \quad x \in \Omega,\ t \in \mathbb{R} \tag{2}$$

$$\frac{\partial \theta}{\partial t} - \chi\Delta\theta + (u \cdot \nabla\theta) = f, \quad x \in \Omega,\ t \in \mathbb{R} \tag{3}$$

$$u = 0 \text{ on } \partial\Omega \tag{4}$$

$$\theta = 0 \text{ on } \partial\Omega, \tag{5}$$

where g is the gravitational field at x, f is the reference temperature and f_1 is an external force. ρ, ν, β, χ are positive physical constants which represent respectively, the density, the kinematic viscosity, the coefficient of volume expansion and the thermal conductance. Without loss of generality, we have taken the constants ρ, ν, β, χ to be one. To avoid some technical complexly in the study of (1)-(5) we assume $f_1 = 0$ throughout paper. For general f_1, our arguments remain valid with a slight modification.

Considerable progress has been made in the mathematical analysis of system (1)-(5); see, for instance, Cannon[6], Fife and Joseph [9], Foias, Manley and Temam [10], Ôeda[27], Hishida [15], [16], [17], and papers cited there. Around the L^2-Theory, the paper [10] investigated strong solutions with initial data in L^2 (resp. $W^{1,2}$) for the case $n = 2$ (resp. $n = 3$) and discussed the existence of global attractors. In the framework of L^p-Theory, the paper [6] constructed solutions of class $L^p(0, T; L^q(\mathbb{R}^n))$ with suitable exponents p and q by using singular integral operators. The author of [15] using the semigroup approach, has studied the existence and uniqueness of strong solutions with values in $L^p(\Omega) \times L^q(\Omega)$ (Ω bounded domain of $\mathbb{R}^n$, $n \geq 2$) when the initial data are not necessarily smooth. Properties of exponential stability of these solutions are analyzed by this same author in [16]. In the framework of weak-L^p Theory, the author of [17] has studied the convection problem in an exterior domain of $\mathbb{R}^3$ and a class of stable steady convection flow is given.

However, the study of periodic solutions to system (1)-(5) was not investigated in unbounded domains. The purpose of the present paper is prove the existence and uniqueness of strong periodic solutions, in some class of unbounded domains, for the problem (1)-(5) in the framework of semigroups Theory; more explicitly, in the Theory of $Weak - L^p$ spaces. The existence of strong periodic solutions, to the Navier-Stokes equations in unbounded domains have been investigated by Kozono and Nakao [21], Maremonti ([25],[26]) and Taniuchi [30]. In particular, Moremonti [25] proved the unique existence of time periodic solution on the whole space $\mathbb{R}^3$ for small external force. The some problem, in the half-space $\mathbb{R}^3_+$, was considered in [26]. Kozono and Nakao [21], making use of $L^p - L^r$ estimates for the semigroup generated by the Stokes operator, constructed time-periodic solutions for small time-periodic forces and the stability of these solutions was considered in [30]. Yamasaki [31] analyzed the same problem of [21] in Morrey spaces. More complete references, including results for bounded domains, are found in [25], [26], [21].
This paper is organized as follows. Section §2, after some preliminaries, we state the main results. Section §3 is devoted to the prove the existence and the uniqueness of strong periodic solutions.

2 Preliminaries and Results

We first introduce some preliminaries about the lorentz spaces; the reader interested in the Lorentz spaces $L^{(p,q)}(\Omega)$ and their properties is referred, for instance, to [18], [28], [2]. For each Lebesgue mensurable function f defined on a domain Ω of $\mathbb{R}^n$, we define the distribution function of f by

$$\lambda_f(s) = m(\{x \in R^n : |f(x)| > s\}),\ s > 0,$$

where m is the Lebesgue measure in $\mathbb{R}^n$. With each function $\lambda_f(s)$ we associate the function

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \le t\} \ , \ t > 0.$$

It easy to check that λ_f and f^* are non-negative and non-increasing functions. Moreover, if λ_f is continuous and strictly decreasing, f^* is the inverse function of λ_f. The Lorentz space $L^{(p,q)} = L^{(p,q)}(\Omega)$ is the collection of all f such that $\|f\|^*_{pq} < \infty$, where

$$\|f\|^*_{pq} = \begin{cases} \left(\frac{p}{q}\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q dt/t\right)^{\frac{1}{q}} , \ 0 < p < \infty, \ 0 < q < \infty. \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t), \ 0 < p \le \infty, \ q = \infty. \end{cases}$$

We observe that $L^p(\Omega) = L^{(p,p)}(\Omega)$. In case $q = \infty$, $L^{(p,\infty)}(\Omega)$ are called the Marcinkiewicz spaces or weak-L^p spaces. Moreover, $L^{(p,q_1)}(\Omega) \subset L^p(\Omega) \subset L^{(p,q_2)}(\Omega) \subset L^{(p,\infty)}(\Omega)$ for $0 < q_1 \le p \le q_2 \le \infty$.

The quantity $\|f\|^*_{(p,q)}$ gives a natural topology for $L^{(p,q)}(\Omega)$ such that $L^{(p,q)}(\Omega)$ is a topological vector space. However, the triangle inequality is not true for $\|f\|^*_{(p,q)}$. A natural way of metrizing the space $L^{(p,q)}(\Omega)$ is to define

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\ ds, \quad \text{for } t > 0.$$

which can be computed [18] as

$$f^{**}(t) = \sup_{m(E)\ge t}\left\{\frac{1}{m(E)}\int_E |f(x)|\ dx\right\}.$$

Hence, we define the norm $\|f\|_{(p,q)}$ as

$$\|f\|_{(p,q)} = \begin{cases} \left(\frac{p}{q}\int_0^\infty [t^{\frac{1}{p}} f^{**}(t)]^q dt/t\right)^{\frac{1}{q}} & , \text{ if } \quad 1 < p < \infty, \ 1 \le q < \infty \\ \\ \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & , \text{ if } \quad 1 < p \le \infty, \ q = \infty \end{cases} .$$

The spaces $L^{(p,q)}$ endowed with the norm $\|f\|_{(p,q)}$ are Banach spaces and

$$\|f\|^*_{(p,q)} \le \|f\|_{(p,q)} \le \frac{p}{p-1}\|f\|^*_{(p,q)}$$

holds for $1 < p \le \infty$ and $1 \le q \le \infty$. We observe that in definition of $\|f\|_{(p,q)}$, the case $p = 1$ has been excluded; although both expressions make sense, they do not define a norm. An alternative definition [1] of the norm $\|f\|_{(p,\infty)}$ is

$$\|f\|_{(p,\infty)} = \sup_{m(E)<\infty}\left\{m(E)^{-1/p'}\int_E |f(x)|\ dx\right\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.
Duals of Lorentz spaces [18] are natural extensions of the property of duality

of L^p spaces. The conjugate space of $L^{(p,1)}(\Omega)$ is $L^{(p',\infty)}(\Omega)$ and the conjugate space of $L^{(p,q)}(\Omega)$ is $L^{(p',q')}(\Omega)$, $1 < p < \infty$, $1 < q < \infty$, and hence, these spaces are reflexive. For the same reasons that $L^1(\Omega)$ is not the conjugate space of $L^\infty(\Omega)$, $L^{(p,1)}(\Omega)$ is not the conjugate space of $L^{(p',\infty)}(\Omega)$ [18]. Note that $C_0^\infty(\Omega)$, which consists of all smooth functions with compact supports, is not dense in $L^{(p,\infty)}(\Omega)$.

Proposition 1 *(Generalized Holder's inequality). Let* $1 < p_1,\ p_2,\ r < \infty$. *Let* $f \in L^{(p_1,q_1)}(\Omega)$ *and* $g \in L^{(p_2,q_2)}(\Omega)$ *where* $\frac{1}{p_1} + \frac{1}{p_2} < 1$, *then the product* $h = fg$ *belongs to* $L^{(r,s)}(\Omega)$ *where* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, *and* $s \geq 1$ *is any number such that* $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}$. *Moreover,*

$$\|h\|_{(r,s)} \leq C(r)\|f\|_{(p_1,q_1)}\|g\|_{(p_2,q_2)}\ . \tag{6}$$

We next introduce the solenoidal function spaces. Let $C_{0,\sigma}^\infty(\Omega)$ represent the set of all solenoidal vector fields whose components are in $C_0^\infty(\Omega)$. By $L_\sigma^r(\Omega), 1 < r < \infty$, denote the closure of $C_{0,\sigma}^\infty$ with respect to norm L^r. If X represent a Banach Space, we denote by $BC^m([t_1,t_2);X)$ the set of all function $u \in C^m([t_1,t_2);X)$ such that $\sup_{t_1<t<t_2}\|d^m u(t)/dt^m\|_X < \infty$. Let us recall the Helmholtz decomposition:

$$L^r(\Omega) = L_\sigma^r(\Omega) \oplus G^r(\Omega),\ 1 < r < \infty,$$

where $G^r(\Omega) = \{\nabla p \in L^r(\Omega) : p \in L_{loc}^r(\overline{\Omega})\}$ [11]. P_r denote the projection operator from L^r onto L_σ^r. The Stokes operator $A_r = -Pr\Delta$ with domain $D(A_r) = \{u \in H^{2,r}(\Omega) : u|_{\partial\Omega} = 0\} \cap L_\sigma^r$. The adjoint of L_σ^r and A_r are $L^{r'}$ and $A_{r'}$, respectively, where $1/r + 1/r' = 1$.
We denote by B_q the Laplace operator in $L^q(\Omega), 1 < q < \infty$, with boundary condition of Dirichlet type being null: $B_q = -\Delta$ with domain $D(B_q) = W^{2,q}(\Omega)\bigcap W_0^{1,q}(\Omega)$. We know that $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}\}_{t\geq 0}$ of class C_0 in L_σ^r [12]. We recall that $-B_q$ generates a uniformly bounded holomorphic semigroup $\{e^{-tB_q}\}_{t\geq 0}$ in $L^q(\Omega)$ [29] of class C_0. Borchers and Miyakawa [5] established the following Helmholtz decomposition of the Lorentz spaces. We can extend P_r to a bounded operator on $L^{(r,d)}(\Omega)$, which we denote by $P_{r,d}$. Set $L_\sigma^{(r,d)}(\Omega) = Range(P_{r,d})$ and $G^{(r,d)}(\Omega) = Kernel(P_{r,d})$. Then,

$$L^{(r,d)}(\Omega) = L_\sigma^{(r,d)}(\Omega) \oplus G^{(r,d)}(\Omega), \tag{7}$$

and

$$L^{(r,d)}(\Omega) = \{u \in L^{(r,d)}(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial\Omega=0}\}, \tag{8}$$

$$G^{(r,d)}(\Omega) = \{\nabla v \in L^{(r,d)}(\Omega) : v \in L_{loc}^{(r,d)}(\bar{\Omega})\}. \tag{9}$$

For simplicity, we shall abbreviate the projection operator and the Stokes and Laplace operators on Lorenz spaces by P, A, B, respectively. In view of [5], the operator A and B define, respectively, closed operators in $L_\sigma^{(r,d)}(\Omega)$ and $L^{(r,d)}(\Omega)$ with domains

$$\begin{aligned} D_{r,d}(A) &= \{u \in L_\sigma^{(r,d)}(\Omega) : \nabla u, D^2u \in L^{(r,d)}(\Omega), u \mid_{\partial\Omega}= 0\}, \\ D_{r,d}(B) &= \{w \in L^{(r,d)}(\Omega) : \nabla w, D^2w \in L^{(r,d)}(\Omega), w \mid_{\partial\Omega}= 0\}, \end{aligned}$$

and $-A, -B$ generate bounded analytic semigroups on $L_\sigma^{(p,q)}(\Omega)$ and $L^{(p,q)}(\Omega)$, respectively. However, notice that this semigroups are not strongly continuous at $t = 0$ if $q = \infty$. Throughout this paper we impose the following assumption on the domain.

Assumption 1.
(CASE 1). Ω is the whole space $\mathbb{R}^n$ or the half-space $\mathbb{R}^n_+$, where $n \geq 3$.
(CASE 2). Ω is an exterior domain in $\mathbb{R}^n$ with boundary of class $C^{2+\mu}(\mu > 0)$, where $n \geq 4$.

Applying the operator projection in the (1)-(2) equations we can treat the problem (1)-(5) in suitable Lorentz spaces as the following Cauchy Problem for a semi linear evolution system of parabolic type:

$$\begin{aligned} u_t + Au + P(u.\nabla u) &= P(\theta g), \quad t \in \mathbb{R} & (10) \\ \theta_t + B\theta + (u \cdot \nabla\theta) &= f, \quad t \in \mathbb{R}. & (11) \end{aligned}$$

The system (10),(11) have associated the following system of integral equations in $L_\sigma^{(r,\infty)}(\Omega) \times L^{(\tilde{r},\infty)}(\Omega)$

$$\begin{aligned} u(t) &= -\int_{-\infty}^t e^{-(t-s)A}P(u \cdot \nabla u)ds + \int_{-\infty}^t e^{-(t-s)A}P(\theta g)ds & (12) \\ \theta(t) &= -\int_{-\infty}^t e^{-(t-s)B}(u \cdot \nabla\theta)ds + \int_{-\infty}^t e^{-(t-s)B}fds. & (13) \end{aligned}$$

Throughout this paper we impose the following assumption on the external force f and the field g:

Assumption 2.
The exponents $r, \tilde{r}$ and $q, \tilde{q}$ in concordance with the assumption 1, satisfy:
(CASE 1). $2 < r, \tilde{r} < n,\ \frac{n}{2} < q, \tilde{q} < n,\ \frac{1}{r} - \frac{1}{\tilde{r}} < \min\{\frac{2}{n} - \frac{1}{q}, \frac{2}{n} - \frac{1}{\tilde{q}}\}$.
(CASE 2.) $\frac{2n}{(n-1)} \leq r, \tilde{r} < n,\ \frac{n}{2} < q, \tilde{q} < n,\ \ \frac{1}{r} - \frac{1}{\tilde{r}} < \min\{\frac{2}{n} - \frac{1}{q}, \frac{2}{n} - \frac{1}{\tilde{q}}\}$.
For each $r, \tilde{r}$ and $q, \tilde{q}$ we assume that f is an element of

$$BC(\mathbb{R}, L^{(\tilde{p},\infty)}(\Omega) \cap L^{(\tilde{l},\infty)}(\Omega)), \tag{14}$$

for $1 < \tilde{p}, \tilde{l} < \infty$ with $1/\tilde{r} + 2/n < 1/\tilde{p},\ 1/\tilde{q} < 1/\tilde{l} < 1/\tilde{q} + 1/n$ provided $n \geq 4$ in both CASES (1,2). (Note that as $n < 2\tilde{q}, \tilde{r} < n$, the inequality $1/\tilde{q} < 1/\tilde{l} < 1/\tilde{q} + 1/n$ imply that $1/\tilde{l} < 2/n + 1/\tilde{r}$).
If $n = 3$, in the CASE 1, we assume that f satisfies

$$f \in BC(\mathbb{R}, L^{(\tilde{l},\infty)}(\Omega)) \text{such that } f(s) = B^\delta_{\tilde{p},\infty}h(s) \text{ for some } h \in BC(\mathbb{R}, D(B^\delta_{\tilde{p},\infty})), \tag{15}$$

for $1 < \tilde{p} < min\{\tilde{r}, \tilde{q}\}$, and $\delta > 0$ satisfying $3/2\tilde{p} + \delta > 1 + max\{1 + 3/2\tilde{r}, 1/2 + 3/2\tilde{q}\}$ and $1/\tilde{q} < 1/\tilde{l} < 1/\tilde{q} + 1/3$. With respect to the field g we make the following assumptions: $g \in L^{(a,\infty)}(\Omega)^n \cap L^{(b,\infty)}(\Omega)^n$ where a and b are such that $1/a > 2/n + 1/r - 1/\tilde{r},\ 1/b < 1/n + 1/q - 1/\tilde{r},\ (b > 1)$.

Remark 2 *The condition (15) can be replaced by* $f(s) = \nabla \cdot G(s),\ G(s) = (G_1, ..., G_n) \in BC(\mathbb{R}; L^{(\tilde{p},\infty)}(\Omega))^n$ *with* $\nabla G(t) \in BC(\mathbb{R}; L^{(\tilde{p},\infty)}(\Omega))^{n\times n}$ *for* $1 < \tilde{p} < \infty$ *with* $1/\tilde{r} + 1/3 < 1/\tilde{p}$. *This implies that* $f(s) = \Delta h(s)$ *for some* $h \in BC(\mathbb{R}; D(B_{\tilde{p},\infty}))$.

Remark 3 *In the CASE 2, we need assume* $n \geq 4$ *when* Ω *is an exterior domain because the restriction on gradient bounds for the semigroup generated by the Stokes operator in* $L^{(p,\infty)}$. *(See Lemma 7).*

Our results are stated as follows:

Theorem 4 *Let* Ω *satisfying the assumption 1 above and* f *being periodic function with period* $\tau > 0$, *(i.e, for all* $t \in \mathbb{R},\ f(t) = f(t+\tau)$*) satisfying the assumption 2. Then there exists positive constants* β_1, β_2 *such that if*

$$\sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{p},\infty)} + \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)} \leq \beta_1, \quad n \geq 4, \text{ in the CASES 1 y 2,}$$

$$\sup_{s\in\mathbb{R}} \|h(s)\|_{(\tilde{p},\infty)} + \sup_{s\in\mathbb{R}} \|P_{\tilde{l}} f(s)\|_{(l,\infty)} \leq \beta_1, \quad n = 3, \text{ in the CASE 1,}$$

$$\|g\|_{(b,\infty)} + \|g\|_{(a,\infty)} \leq \beta_2, \text{ in the CASES 1 y 2,}$$

there exist a periodic solution (u, θ) *of (12),(13), with the same period* τ *of the external forces, such that* $u \in BC(\mathbb{R}; L^{(r,\infty)}_\sigma)^n,\ \theta \in BC(\mathbb{R}; L^{(\tilde{r},\infty)})$, *with* $\nabla u \in BC(\mathbb{R}; L^{(q,\infty)})^{n\times n},\ \nabla\theta \in BC(\mathbb{R}; L^{(\tilde{q},\infty)})^n$.
If $\sup_{s\in\mathbb{R}} \|u(s)\|_{(r,\infty)} + \sup_{s\in\mathbb{R}} \|\nabla u(s)\|_{(q,\infty)}, \sup_{s\in\mathbb{R}} \|\theta(s)\|_{(\tilde{r},\infty)} + \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)}$ *are sufficiently small, then the solution is unique at this class of function spaces.*

Theorem 5 *Under the assumptions of Theorem 4, if* f *is Holder continuous function in* $\mathbb{R}$ *with values at* $L^{(n,\infty)}(\Omega)$ *and* $g \in L^{(n,\infty)}(\Omega)^n$, *then the periodic solution given by the Theorem 4 satisfies*

1. $u \in BC(\mathbb{R}; L^{(n,\infty)}_\sigma)^n \cap C^1(\mathbb{R}; L^{(n,\infty)}_\sigma)^n,\ \theta \in BC(\mathbb{R}; L^{(n,\infty)}) \cap C^1(\mathbb{R}; L^{(n,\infty)})$.
2. $u(t) \in D(A_{n,\infty}),\ \theta(t) \in D(B_{n,\infty}),\ A_n u \in C(\mathbb{R}; L^{(n,\infty)}_\sigma)^n,\ B_n\theta \in C(\mathbb{R}; L^{(n,\infty)})$, *for* $t \in \mathbb{R}$.
3. *The equations (10),(11) are satisfied in* $(L^{(n,\infty)}_\sigma)^n, L^{(n,\infty)}$, *respectively, for all* $t \in \mathbb{R}$.

Remark 6 *Our results also hold when* Ω *is a bounded domain and we can relax the assumption on the external force.*

3 Existence Uniqueness and Regularity of Periodic Solutions

At this section we prove the Theorem 4 and Theorem 5. Throughout this paper, we shall denote by c, C various constants. In particular, $C = C(*, ..., *)$ will denote the constants which depend only on the quantities appearing in parentheses. Let us first recall the following estimates $L^{(r,\infty)} - L^{(p,\infty)}$ for the semigroups $\{e^{-tA}\}_{t\geq 0}, \{e^{-tB}\}_{t\geq 0}$

Lemma 7 *[5],[17] (1). Let Ω as the CASE 1 of assumption 1. Then*

$$\begin{aligned} \|e^{-tA}a\|_{(r,\infty)} &\leq ct^{-n/2(1/p-1/r)}\|a\|_{(p,\infty)},\ 1 < p \leq r < \infty, \\ \|\nabla e^{-tA}a\|_{(r,\infty)} &\leq ct^{-n/2(1/p-1/r)-1/2}\|a\|_{(p,\infty)},\ 1 < p \leq r < \infty, \end{aligned}$$

for all $a \in L^{(p,\infty)}_\sigma$ and all $t > 0$, where $c = c(n, p, r)$.

(2). Let Ω as the CASE 2 of assumption 1. Then

$$\begin{aligned} \|e^{-tA}a\|_{(r,\infty)} &\leq ct^{-n/2(1/p-1/r)}\|a\|_{(p,\infty)},\ 1 < p \leq r < \infty, \\ \|\nabla e^{-tA}a\|_{(r,\infty)} &\leq ct^{-n/2(1/p-1/r)-1/2}\|a\|_{(p,\infty)},\ 1 < p \leq r \leq n, \end{aligned}$$

for all $a \in L^{(p,\infty)}_\sigma$ and all $t > 0$, where $c = c(n, p, r)$.

Remark 8 *The similar estimates hold true for the semigroup $\{e^{-tB}\}_{t\geq 0}$.*

Remark 9 *The estimates before hold in the particular case of L^p spaces. See, for instance [20], [19], [14],[4], [3].*

Lemma 10 *[5],[17] Let Ω satisfying the assumption 1 and suppose that $n \geq 2, 1 < q < n, 1 \leq d \leq \infty$ and $q^* = nq/(n-q)$. If $\phi \in L^{(p,\infty)}(\Omega)$ for some $p < \infty$ and $\nabla\phi \in L^{(q,d)}(\Omega)^n$, then $\phi \in L^{(q^*,d)}(\Omega)$ and the estimate*

$$\|\phi\|_{(q^*,d)} \leq C\|\nabla\phi\|_{(q,d)}$$

holds with $C > 0$ independent of ϕ.

We denote by X the space of scalar functions $\{u \in BC(\mathbb{R}; L^{(\tilde{r},\infty)}) : \nabla u \in BC(\mathbb{R}; L^{(\tilde{q},\infty)})^n\}$ with the norm $\|\cdot\|_X$ defined as

$$\|u\|_X \equiv \sup_{s\in\mathbb{R}} \|u(s)\|_{(\tilde{r},\infty)} + \sup_{s\in\mathbb{R}} \|\nabla u(s)\|_{(\tilde{q},\infty)}.$$

We also defined by Y the space of vector functions $\{u \in BC(\mathbb{R}; L^{(r,\infty)}_\sigma)^n : \nabla u \in BC(\mathbb{R}; L^{(q,\infty)}))^{n\times n}\}$ with the norm $\|\cdot\|_Y$ defined as

$$\|u\|_Y \equiv \sup_{s\in\mathbb{R}} \|u(s)\|_{(r,\infty)} + \sup_{s\in\mathbb{R}} \|\nabla u(s)\|_{(q,\infty)}.$$

These spaces are Banach spaces. We define the following operators F_1 and G on $Y \times Y$ and $Y \times X$, respectively, by

$$
\begin{aligned}
F_1(u,v)(t) &= -\int_{-\infty}^{t} e^{-(t-s)A} P(u\cdot\nabla v)(s)ds, &(16)\\
G(u,\theta)(t) &= -\int_{-\infty}^{t} e^{-(t-s)B} (u\cdot\nabla \theta)(s)ds. &(17)
\end{aligned}
$$

3.1 Proof of Theorem 4

We construct a periodic solution of integral problem (12),(13) according to the following scheme:

$$
\begin{aligned}
u_{m+1}(t) &= F(u_m,\theta_m)(t), &(18)\\
\theta_{m+1}(t) &= \theta_0(t) + G(u_m,\theta_m)(t), &(19)
\end{aligned}
$$

where $\theta_0(t) = \int_{-\infty}^{t} e^{-(t-s)B} f(s)ds,\ u_0(t) = \int_{-\infty}^{t} e^{-(t-s)A} P(\theta_0 g)ds,$

$$
\begin{aligned}
F(u_m,\theta_m)(t) &= -\int_{-\infty}^{t} e^{-(t-s)A} P(u_m\cdot\nabla u_m)(s)ds + \int_{-\infty}^{t} e^{-(t-s)A} P(g\theta_m)(s)ds,\\
G(u_m,\theta_m)(t) &= -\int_{-\infty}^{t} e^{-(t-s)B} (u_m\cdot\nabla \theta_m)(s)ds.
\end{aligned}
$$

Remark 11 *When f_1 is not null, in the scheme above we consider $u_0(t) = \int_{-\infty}^{t} e^{-(t-s)A} P(\theta f_1)ds$ and $u_{m+1} = u_0(t) + F(u_m,\theta_m)(t)$.*

Let us first encounter some estimates to approximations above; we shall need the following Lemmas.

Lemma 12 *Let $r, \tilde{r}, q$ and $\tilde{q}$ as Theorem 4. Then we have that*

$$
\begin{aligned}
\sup_{s\in\mathbb{R}} \|F_1(u,v)\|_{(r,\infty)} &\le c_1\Big(\sup_{s\in\mathbb{R}}\|u(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}}\|v(s)\|_{(r,\infty)} + \sup_{s\in\mathbb{R}}\|u(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}}\|\nabla v(s)\|_{(q,\infty)}\Big) &(20)\\
\sup_{s\in\mathbb{R}} \|\nabla F_1(u,v)\|_{(q,\infty)} &\le c_1\Big(\sup_{s\in\mathbb{R}}\|u(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}}\|\nabla v(s)\|_{(q,\infty)} + \sup_{s\in\mathbb{R}}\|\nabla u(s)\|_{(q,\infty)} \sup_{s\in\mathbb{R}}\|\nabla v(s)\|_{(q,\infty)}\Big) &(21)\\
\sup_{s\in\mathbb{R}} \|G(u,\theta)\|_{(\tilde{r},\infty)} &\le c_2\Big(\sup_{s\in\mathbb{R}}\|u(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}}\|\theta(s)\|_{(\tilde{r},\infty)} + \sup_{s\in\mathbb{R}}\|u(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}}\|\nabla \theta(s)\|_{(\tilde{q},\infty)}\Big) &(22)\\
\sup_{s\in\mathbb{R}} \|\nabla G(u,\theta)\|_{(\tilde{q},\infty)} &\le c_2\Big(\sup_{s\in\mathbb{R}}\|u(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}}\|\nabla \theta(s)\|_{(\tilde{q},\infty)} + \sup_{s\in\mathbb{R}}\|\nabla u(s)\|_{(q,\infty)} \sup_{s\in\mathbb{R}}\|\nabla \theta(s)\|_{(\tilde{q},\infty)}\Big) &(23)
\end{aligned}
$$

for all $u, v \in Y,\ \theta \in X$ where $c_1 = c_1(n,r,q), c_2 = c_2(n,\tilde{r},\tilde{q})$.

Proof. The proof is an application of lemma 7. In fact,

$$
\begin{aligned}
G(u,\theta)(t) &= -\int_{-\infty}^{t-1} e^{-(t-s)B}(u\cdot\nabla\theta)(s)ds - \int_{t-1}^{t} e^{-(t-s)B}(u\cdot\nabla\theta)(s)ds\\
&= G_1(t) + G_2(t).
\end{aligned}
$$

Then for all $\psi \in C_0^\infty$ and for all $t \in \mathbb{R}$, we have

$$\begin{aligned}
|(G_1(t),\psi)| &\leq \int_{-\infty}^{t-1} \|\nabla e^{-(t-s)B}\psi\|_{((r\tilde{r}/(r+\tilde{r}))',1)} \|\theta u\|_{(r\tilde{r}/(r+\tilde{r}),\infty)} ds \\
&\leq c\int_{-\infty}^{t-1} (t-s)^{-n/2r-1/2} \|\psi\|_{(\tilde{r}',1)} \|\theta(s)\|_{(\tilde{r},\infty)} \|u(s)\|_{(r,\infty)} ds \\
&\leq c\sup_{s\in\mathbb{R}} \|\theta(s)\|_{(\tilde{r},\infty)} \sup_{s\in\mathbb{R}} \|u(s)\|_{(r,\infty)} \int_{-\infty}^{t-1} (t-s)^{-n/2r-1/2} \|\psi\|_{(\tilde{r}',1)}.
\end{aligned}$$

By duality, for all $t \in \mathbb{R}$, $\|G_1(t)\|_{(\tilde{r},\infty)} \leq c\sup_{s\in\mathbb{R}} \|\theta(s)\|_{(\tilde{r},\infty)} \sup_{s\in\mathbb{R}} \|u(s)\|_{(r,\infty)}$.

$$\begin{aligned}
\|G_2(t)\|_{(\tilde{r},\infty)} &\leq \int_{t-1}^{t} (t-s)^{-n/2(1/r+1/\tilde{q}-1/\tilde{r})} \|u(s)\|_{(r,\infty)} \|\nabla\theta(s)\|_{(\tilde{q},\infty)} ds \\
&\leq c\sup_{s\in\mathbb{R}} \|u(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)}.
\end{aligned}$$

Now, using the Lemma 7 and Lemma 10 (with $d=\infty$), we obtain

$$\begin{aligned}
\|\nabla G(u,\theta)\|_{(\tilde{q},\infty)} &\leq \int_{-\infty}^{t-1} \|\nabla e^{-(t-s)B}(u\cdot\nabla\theta)(s)\|_{(\tilde{q},\infty)} ds + \int_{t-1}^{t} \|e^{-(t-s)B}(u\cdot\nabla\theta)(s)\|_{(\tilde{q},\infty)} ds \\
&\leq c\int_{-\infty}^{t-1} (t-s)^{-n/2r-1/2} \|u(s)\|_{r,\infty} \|\nabla\theta(s)\|_{(\tilde{q},\infty)} + \\
&+ c\int_{t-1}^{t} (t-s)^{-n/2q^*} \|u(s)\|_{(q^*,\infty)} \|\nabla\theta(s)\|_{(\tilde{q},\infty)} \\
&\leq c\sup_{s\in\mathbb{R}} \|u(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)} \int_{-\infty}^{t-1} (t-s)^{-n/2r-1/2} ds + \\
&+ \sup_{s\in\mathbb{R}} \|u(s)\|_{(q,\infty)} \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)} \int_{t-1}^{t} (t-s)^{-n/2q^*} ds \\
&\leq c(\sup_{s\in\mathbb{R}} \|u(s)\|_{(r,\infty)} \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)} + \sup_{s\in\mathbb{R}} \|\nabla u(s)\|_{(q,\infty)} \sup_{s\in\mathbb{R}} \|\nabla\theta(s)\|_{(\tilde{q},\infty)}),
\end{aligned}$$

for all $t \in \mathbb{R}$ and $c = c(n,\tilde{r},\tilde{q},r,q)$. This complete the estimates (22) and (23) of Lemma. The estimates (20),(21) are encounter similarly.

Lemma 13 *Let θ_0 defined as in (19). Then $\theta_0 \in X$.*

Proof. If f satisfies (14) then using the Lemma 7 we obtain

$$\begin{aligned}
\|\theta_0(t)\|_{(\tilde{r},\infty)} &\leq \int_{-\infty}^{t-1} \|e^{-(t-s)B} f(s)\|_{(\tilde{r},\infty)} ds + \int_{t-1}^{t} \|e^{-(t-s)B} f(s)\|_{(\tilde{r},\infty)} ds \\
&\leq c\sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{p},\infty)} \int_{-\infty}^{t-1} (t-s)^{-n/2(1/\tilde{p}-1/\tilde{r})} ds \\
&+ c\sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)} \int_{t-1}^{t} (t-s)^{-n/2(1/\tilde{l}-1/\tilde{r})} ds.
\end{aligned}$$

This is valid for all $t \in \mathbb{R}$. The constant $c = c(n, \tilde{r}, \tilde{q}, \tilde{p}, \tilde{l})$. From the assumptions (14), i.e, $1/\tilde{r} + 2/n < 1/\tilde{p}$ and $1/\tilde{l} < 2/n + 1/\tilde{r}$, we concluded that each integral above is finite and consequently, $\|\theta_0(t)\|_{(\tilde{r},\infty)} \leq c \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{p},\infty)} + c \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)}$.
A similar analysis prove that

$$\begin{aligned}\|\nabla\theta_0(t)\|_{(\tilde{q},\infty)} &\leq c \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{p},\infty)} \int_{-\infty}^{t-1} (t-s)^{-n/2(1/\tilde{p}-1/\tilde{q})-1/2} ds \\ &+ c \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)} \int_{t-1}^{t} (t-s)^{-n/2(1/\tilde{l}-1/\tilde{q})-1/2} ds,\end{aligned}$$

for all $t \in \mathbb{R}$ and $c = c(n, q, r, p, l)$. As $1/\tilde{p} > 1/\tilde{r} + 2/n > 1/n + 1/\tilde{q}$ and $1/\tilde{l} < 1/\tilde{q} + 1/n$, the two integrals above converge.
Now, if $n = 3$ the anterior analyze is wrong because will be necessary $3/2(1/\tilde{p} - 1/\tilde{r}) > 1$, with $\tilde{p} > 1$ and this its not happy. Consequently we assume a new condition; in fact, if f satisfies (15), using the following estimate (which is a consequence of the analytic properties of semigroup)

$$\|B^\delta e^{-tB} a\|_{(\tilde{p},\infty)} \leq Ct^{-\delta} \|a\|_{(\tilde{p},\infty)}, \ \forall a \in L^{(\tilde{p},\infty)}, \ t > 0, \ c = c(\tilde{p}, \delta), \ \delta \geq 0,$$

and the lemma 7, we obtain

$$\begin{aligned}\|\theta_0(t)\|_{(\tilde{r},\infty)} &\leq \int_{-\infty}^{t-1} \|e^{-(t-s)B} B^\delta h(s)\|_{(\tilde{r},\infty)} ds + \int_{t-1}^{t} \|e^{-(t-s)B} f(s)\|_{(\tilde{r},\infty)} ds \\ &\leq c \int_{-\infty}^{t-1} (t-s)^{-3/2(1/\tilde{p}-1/\tilde{r})} \|B^\delta e^{-(t-s)B/2} h(s) ds\|_{(\tilde{p},\infty)} \\ &+ c \int_{t-1}^{t} (t-s)^{-3/2(1/\tilde{l}-1/\tilde{r})} \|f(s)\|_{(\tilde{l},\infty)} ds \\ &\leq c \int_{-\infty}^{t-1} (t-s)^{-3/2(1/\tilde{p}-1/\tilde{r})-\delta} \|h(s)\|_{(\tilde{p},\infty)} ds \\ &+ c \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)} \int_{t-1}^{t} (t-s)^{-3/2(1/\tilde{l}-1/\tilde{r})} ds \\ &\leq c\Big(\sup_{s\in\mathbb{R}} \|h(s)\|_{(\tilde{p},\infty)} + \sup_{s\in\mathbb{R}} \|f(s)\|_{(\tilde{l},\infty)} \Big),\end{aligned}$$

for all $t \in \mathbb{R}$ with $c = c(n, \tilde{r}, \tilde{p}, \tilde{q}, \tilde{l}, \delta)$. A similar estimate can be obtained to $\|\nabla\theta_0\|_{(\tilde{q},\infty)}$, $(n = 3)$. This prove the Lemma.

Now we will estimate the terms $F(u_m, \theta_m)$ and $G(u_m, \theta_m)$. We start with the following Lemma.

Lemma 14 *The terms $\|F(u_m, \theta_m)\|_Y$, $\|G(u_m, \theta_m)\|_X$ give by (18),(19) satisfy the following estimates*

$$\|F(u_m, \theta_m)\|_Y \leq 2c_1 \|u_m\|_Y^2 + c_3 \|\theta_m\|_X, \tag{24}$$

$$\|G(u_m,\theta_m)\|_X \le 2c_2\|u_m\|_Y\|\theta_m\|_X, \tag{25}$$

where c_1, c_2 are as in the Lemma 12 and c_3 depending on g but is independent of m.

Proof. Note that

$$\Big\|\int_{-\infty}^{t} e^{-(t-s)A}P(g\theta_m)ds\Big\|_Y \le c_3\|\theta_m\|_X. \tag{26}$$

In fact,

$$\Big\|\int_{-\infty}^{t} e^{-(t-s)A}P(g\theta_m)\Big\|_{(r,\infty)} \le \int_{-\infty}^{t-1}\|e^{-(t-s)A}P(g\theta_m)\|_{(r,\infty)} + \int_{t-1}^{t}\|e^{-(t-s)A}P(g\theta_m)\|_{(r,\infty)}.$$

$$\begin{aligned}\int_{-\infty}^{t-1}\|e^{-(t-s)A}P(g\theta_m)\|_{(r,\infty)} &\le c\int_{-\infty}^{t-1}(t-s)^{\gamma}\|g\|_{(a,\infty)}\|\theta_m(s)\|_{(\tilde r,\infty)}\\ &\le c\|g\|_{(a,\infty)}\sup_{s\in\mathbb{R}}\|\theta_m(s)\|_{(\tilde r,\infty)}\int_{-\infty}^{t-1}(t-s)^{\gamma}ds,\end{aligned}$$

where $\gamma=-n/2(1/a+1/\tilde r-1/r)$. As $1/a>2/n-1/\tilde r+1/r$, the less integral converges.
Now,

$$\begin{aligned}\int_{t-1}^{t}\|e^{-(t-s)A}P(g\theta_m)\|_{(r,\infty)} &\le c\int_{t-1}^{t}(t-s)^{\xi}\|g\|_{(b,\infty)}\|\theta_m(s)\|_{(\tilde r,\infty)}\\ &\le c\|g\|_{(b,\infty)}\sup_{s\in\mathbb{R}}\|\theta_m(s)\|_{(\tilde r,\infty)}\int_{t-1}^{t}(t-s)^{\xi}ds,\end{aligned}$$

where $\xi=-n/2(1/b+1/\tilde r-1/r)$. By the assumption 2, $g\in L^{(b,\infty)}(\Omega)^n$ with $b>1$ and $1/b<1/n+1/q-1/\tilde r$. By the condition 2, as $r<n, n/2<q$, we have that $1/r+1/n>2/n>1/q$; then $1/b<1/n+1/q-1/\tilde r<1/n+1/n=2/n$ implying that $1/b<2/n+1/r-1/\tilde r$ and therefore the integral above converges.

$$\begin{aligned}\Big\|\nabla\int_{-\infty}^{t} e^{-(t-s)A}P(g\theta_m)\Big\|_{(q,\infty)} &\le \int_{-\infty}^{t-1}\|\nabla e^{-(t-s)A}P(g\theta_m)\|_{(q,\infty)} +\\ &+ \int_{t-1}^{t}\|\nabla e^{-(t-s)A}P(g\theta_m)\|_{(q,\infty)}.\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{t-1}\|\nabla e^{-(t-s)A}P(g\theta_m)\|_{(q,\infty)} &\le c\int_{\infty}^{t-1}(t-s)^{-n/2(1/a+1/\tilde r-1/q)-1/2}\|g\|_{(a,\infty)}\|\theta_m(s)\|_{(\tilde r,\infty)}\\ &\le c\|g\|_{(a,\infty)}\sup_{s\in\mathbb{R}}\|\theta_m(s)\|_{(\tilde r,\infty)}\times\\ &\times \int_{\infty}^{t-1}(t-s)^{-n/2(1/a+1/\tilde r-1/q)-1/2}.\end{aligned}$$

As $n < q$ and $1/\tilde{r} - 1/r > 2/n - 1/a$, we conclude that $1/a + 1/\tilde{r} - 1/q > 2/n + 1/r - 1/q > 1/n + 1/r > 1/n$. and hence the less integral converges. Analogously we can show that

$$\int_{t-1}^{t} \|\nabla e^{-(t-s)A} P(g\theta_m)\|_{(q,\infty)} \le c\|g\|_{(b,\infty)} \sup_{s\in\mathbb{R}} \|\theta_m(s)\|_{(\tilde{r},\infty)}.$$

Hence we prove the inequality (26) with

$$c_3 = c\Big(\|g\|_{(b,\infty)} + \|g\|_{(a,\infty)}\Big) \tag{27}$$

and c independent of m. Therefore, of lemma 12 and (26) we obtain (24). The inequality (25) is obtained applying directly the inequalities (22)-(23) of Lemma 12.

Consequently, of the Lemmas 13 y 14 we obtain

$$\|u_{m+1}\|_Y \le 2c_1\|u_m\|_Y^2 + c_3\|\theta_m\|_X, \tag{28}$$

$$\|\theta_{m+1}\|_X \le \|\theta_0\|_X + 2c_2\|u_m\|_Y\|\theta_m\|_X. \tag{29}$$

Let

$$a_m = \text{Max}\{\|u_m\|_Y, \|\theta_m\|_X\},\ m = 1, 2, ...\ \ a_0 = \|\theta_0\|_X.$$

Therefore, it follows from (28) and (29) that

$$a_{m+1} \le a_0 + \widetilde{c}a_m^2 + c_3 a_m,\ \widetilde{c} = max(2c_1, 2c_2).$$

Hence, if

$$c_3 < 1,\ 4a_0\widetilde{c} < (1-c_3)^2, \tag{30}$$

then the sequence $\{a_m\}_{m=0}^{\infty}$ is bounded with

$$a_m \le \frac{(1-c_3) - \sqrt{(1-c_3)^2 - 4\widetilde{c}a_0}}{2\widetilde{c}} \equiv k,\ \forall m = 0, 1, 2, ... \Rightarrow a_m \le k < 1/2\widetilde{c}. \tag{31}$$

We assume (30) (Note that this condition implies a small condition of f). Making $w_m = u_m - u_{m-1}$ $(u_{-1} \equiv 0)$, $\Theta_m = \theta_m - \theta_{m-1}$, $(\theta_{-1} \equiv 0)$, we have

$$\begin{aligned} w_{m+1}(t) &= -\int_{-\infty}^{t} e^{-(t-s)A} P(w_m \cdot \nabla u_m)(s)ds - \int_{-\infty}^{t} e^{-(t-s)A} P(u_{m-1} \cdot \nabla w_m)(s)ds \\ &+ \int_{-\infty}^{t} e^{-(t-s)A} P(g\Theta_m)(s)ds, \\ \Theta_{m+1}(t) &= -\int_{-\infty}^{t} e^{-(t-s)B} P(w_m \cdot \nabla\theta_m)(s)ds - \int_{-\infty}^{t} e^{-(t-s)B}(u_{m-1} \cdot \nabla\Theta_m)(s)ds. \end{aligned}$$

This equality imply that

$$\begin{aligned} \|w_{m+1}\|_Y &\le 2c_1(\|w_m\|_Y\|u_m\|_Y + \|u_{m-1}\|_Y\|w_m\|_Y) + c_3\|\Theta_m\|_X \\ &\le 2c_1k\|w_m\|_Y + c_3\|\Theta_m\|_X \\ &\le \widetilde{c}k(\|w_m\|_Y + \|\Theta_m\|_X), \end{aligned} \tag{32}$$

provided $c_3 \leq \widetilde{c}k$, (This condition and (30) implies a small condition of field g in the norms $\|\cdot\|_{(a,\infty)}$ and $\|\cdot\|_{(b,\infty)}$). Moreover,

$$\begin{aligned} \|\Theta_{m+1}\|_X &\leq 2c_2\|w_m\|_Y\|\theta_m\|_X + 2c_2\|u_{m-1}\|_Y\|\Theta_m\|_X \\ &\leq 2c_2k(\|w_m\|_Y + \|\Theta_m\|_X). \end{aligned} \quad (33)$$

From (32), (33), we obtain

$$\begin{aligned} \text{Max}\{\|w_{m+1}\|_Y, \|\Theta_{m+1}\|_X\} &\leq \widetilde{c}k\text{Max}\{\|w_m\|_Y, \|\theta\|_X\} \leq \dots \quad (34) \\ &\leq (\widetilde{c}k)^{m+1}a_0, \ \forall m = 0, 1, \dots \end{aligned}$$

Note that $u_m(t) = \sum_{j=0}^m w_j(t), \ \theta(t) = \sum_{j=0}^m \Theta_j(t)$; since $\tilde{c}k < 1$ (by (31)), from (34) we conclude that there exists functions $u \in Y, \ \theta \in X$ such that when $m \longrightarrow \infty$,

$$u_m \longrightarrow u \text{ in } Y, \quad \theta_m \longrightarrow \theta \text{ in } X.$$

Note that

$$\begin{aligned} \Big\| -\int_{-\infty}^t e^{-(t-s)A}P(u_m\cdot\nabla u_m)(s)ds &+ \int_{-\infty}^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds\Big\|_Y \leq \\ &\leq \Big\|\int_{-\infty}^t e^{-(t-s)A}P((u_m-u)\cdot\nabla u_m)(s)ds\Big\|_Y \\ &+ \Big\|\int_{-\infty}^t e^{-(t-s)A}P(u\cdot\nabla(u_m-u))(s)ds\Big\|_Y \\ &\leq 2c_1\|u_m-u\|_Y\|u_m\|_Y + 2c_1\|u\|_Y\|u_m-u\|_Y \\ &< \|u_m-u\|_Y, \ \forall m. \end{aligned}$$

Then in Y

$$-\int_{-\infty}^t e^{-(t-s)A}P(u_m\cdot\nabla u_m)(s)ds \longrightarrow \int_{-\infty}^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds. \quad (35)$$

Analogously,

$$\begin{aligned} \Big\| -\int_{-\infty}^t e^{-(t-s)B}(u_m\cdot\nabla\theta_m)(s)ds &+ \int_{-\infty}^t e^{-(t-s)B}(u\cdot\nabla\theta)(s)ds\Big\|_X \\ &\leq \Big\|\int_{-\infty}^t e^{-(t-s)B}((u_m-u)\cdot\nabla\theta_m)\Big\|_X \\ &+ \Big\|\int_{-\infty}^t e^{-(t-s)B}(u\cdot\nabla(\theta_m-\theta))\Big\|_X \\ &\leq 2c_2\|u_m-u\|_Y\|\theta_m\|_X + 2c_2\|u\|_Y\|\theta_m-\theta\|_X \\ &< \|u_m-u\|_Y + \|\theta_m-\theta\|_X, \ \forall m. \end{aligned}$$

Then in X

$$-\int_{-\infty}^t e^{-(t-s)B}(u_m\cdot\nabla u_m)(s)ds \longrightarrow \int_{-\infty}^t e^{-(t-s)B}(u\cdot\nabla u)(s)ds. \quad (36)$$

Finally, when $m \longrightarrow \infty$,

$$\Big\| - \int_{-\infty}^{t} e^{-(t-s)A} P(g(\theta_m - \theta))\Big\|_Y \leq c_3 \|\theta_m - \theta\|_X \longrightarrow 0. \tag{37}$$

From (35), (36) and (37) we conclude that (u, θ) is a solution of system of integral equations (12), (13).

Periodicity.

Being f a periodic function with period $\tau > 0$, the functions u_m and θ_m are also periodic with the same period τ. Consequently, the limit (u, θ) is periodic with period τ.

Uniqueness.

Supposed that (u_1, θ_1) is another solution of (12)-(13), such that $\|u_1\|_Y \leq k,\ \|\theta_1\|_X \leq k$, being k the constant of (31). Working as before, we encounter that

$$\|\theta - \theta_1\|_X \leq 2c_2 k \|u - u_1\|_Y + 2c_2 k \|\theta - \theta_1\|_X,$$
$$\|u - u_1\|_Y \leq c_3 \|\theta - \theta_1\|_X + 2c_1 k \|u - u_1\|_Y.$$

Hence, if $M \equiv \mathrm{Max}\{\|u - u_1\|_Y, \|\theta - \theta_1\|_X\}$ we have

$$M \leq \widetilde{c} k M,$$

because $c_3 \leq \widetilde{c} k$, implying that $\theta = \theta_1,\ u = u_1$.

3.2 Strong Solution. Proof of Theorem 2.

In this subsection we shall that the periodic solution (u, θ) constructed in the Theorem 4, is actually a solution of the differential system (1)-(5), assuming in addition that f and g satisfy adequate regularity conditions. To demonstrate the theorem 5, we need the following result of local existence of strong solutions to the initial boundary value problem for (1)-(5). This result follows the arguments of Kato [20] and Giga [13]. Let us first give the definition of Strong solution of the initial value problem (1)-(5).

Definition 15 *Let $a \in L_\sigma^{(n,\infty)}, b \in L^{(n,\infty)}$. A duple (v, w) defined on $(t_0, t_1) \times \Omega$ is called a strong solution of (1)-(5) with initial value (a, b) if*

1. $v \in BC_w([t_0, t_1); L_\sigma^{(n,\infty)})^n \cap C^1((t_0, t_1); L_\sigma^{(n,\infty)})^n,\ w \in BC_w([t_0, t_1); L^{(n,\infty)}) \cap C^1((t_0, t_1); L^{(n,\infty)})$,
2. $Au \in C((t_0, t_1); L_\sigma^{(n,\infty)})^n,\ Bw \in C((t_0, t_1); L^{(n,\infty)}),\ t_0 < t < t_1$,
3. $v_t + Au + P(v.\nabla v) = P(wg),$ *in* $L_\sigma^{(n,\infty)},\ x \in \Omega,\ t_0 < t < t_1$,
4. $w_t + Bw + (u \cdot \nabla w) = f$ *in* $L^{(n,\infty)},\ x \in \Omega,\ t_0 < t < t_1$,

where BC_w denotes the class of bounded and weakly-$*$ continuous functions, together with

$$\lim_{t\to t_0}(v(t),\phi)=(a,\phi),\ \lim_{t\to t_0}(v(t),\varphi)=(b,\varphi),$$

for all $\phi\in L_\sigma^{(n/(n-1),1)}(\Omega)^n,\varphi\in L_\sigma^{(n/(n-1),1)}(\Omega)$.

Our result on the local existence of strong solutions now reads:

Theorem 16 *Let $n/2<q<n$ and $1<l<\infty$ such that $1/q<1/l<1/q+1/n$. Suposed that $a\in(L_\sigma^{(n,\infty)})^n\cap(L_\sigma^{(q^*,\infty)})^n$, $b\in L^{(n,\infty)}\cap L^{(q^*,\infty)}$, where $q^*=nq/(n-q)$, $f\in BC(\mathbb{R};L^{(l,\infty)})$ being Holder continuous with value in $L^{(n,\infty)}$, $g\in L^{(b,\infty)}\cap L^{(n,\infty)}$ with $b>n/2$. Then there exist $T\in(0,1]$ such that for all $t_0\in\mathbb{R}$ exist an unique strong solution of the problem (1)-(5) at (t_0,t_0+T) with initial value $v(t_0)=a,w(t_0)=b$. Moreover, the solution satisfies $v\in BC((t_0,t_0+T);L_\sigma^{(q^*,\infty)})^n,\ w\in BC((t_0,t_0+T);L^{(q^*,\infty)})$, with*

$$\sup_{t_0<t<t_0+T}\|v(t)\|_{(q^*,\infty)}\le C_1,\qquad \sup_{t_0<t<t_0+T}\|w(t)\|_{(q^*,\infty)}\le C_2, \tag{38}$$

where C_1,C_2 are independents of t_0. Here T is estimated as

$$T\equiv\Big[\frac{\tilde{k}}{c_1 Max\{\|a\|_{(n/\alpha,\infty)},\|b\|_{(n/\alpha,\infty)}+\|f\|_{BC(\mathbb{R};L^{(l,\infty)})}\}}\Big]^{\frac{\alpha-1}{2}} \tag{39}$$

with C_3

Proposition 17 *Let $n/2<q<n$ and $1<l<\infty$ such that $1/q<1/l<1/q+1/n$. Suposed that $a\in(L_\sigma^{(n,\infty)})^n\cap(L_\sigma^{(q^*,\infty)})^n$, $b\in L^{(n,\infty)}\cap L^{(q^*,\infty)}$, where $q^*=nq/(n-q)$, $f\in BC(\mathbb{R};L^{(l,\infty)})$ with value in $L^{(n,\infty)}$, $g\in L^{(b,\infty)}\cap L^{(n,\infty)}$ with $b>n/2$. Then there exist $T\in(0,1]$ and functions v,w in the class $v\in BC_w([t_0,t_0+T);L_\sigma^{(n,\infty)})^n$, $w\in BC_w([t_0,t_0+T);L^{(n,\infty)})$ with $v\in BC((t_0,t_0+T);L^{(q^*,\infty)})^n$, $w\in BC((t_0,t_0+T);L^{(q^*,\infty)})$ and $(t-t_0)^{1/2}\nabla v\in BC_w((t_0,t_0+T);L_\sigma^{(n,\infty)})^{n\times n}$, $(t-t_0)^{1/2}\nabla w\in BC_w((t_0,t_0+T);L_\sigma^{(n,\infty)})^n$, such that for all $t_0\in\mathbb{R}$,*

$$\begin{aligned} v(t) &= e^{-(t-t_0)A}a+\int_{t_0}^t e^{-(t-s)A}P(wg)ds-\int_{t_0}^t e^{-(t-s)A}P(v\cdot\nabla v)ds, && (40)\\ w(t) &= e^{-(t-t_0)B}b+\int_{t_0}^t e^{-(t-s)B}fds-\int_{t_0}^t e^{-(t-s)B}(v\cdot\nabla w)ds. && (41)\end{aligned}$$

Moreover, the functions v,w satisfy that $t^{1/4}v\in BC((t_0,t_0+T);L_\sigma^{2n})^n$, $t^{1/4}w\in BC((t_0,t_0+T);L^{2n})$. Here T is estimated as

$$T\equiv\Big[\frac{\tilde{k}}{c_1 Max\{\|a\|_{(n/\alpha,\infty)},\|b\|_{(n/\alpha,\infty)}+\|f\|_{BC(\mathbb{R};L^{(l,\infty)})}\}}\Big]^{\frac{\alpha-1}{2}} \tag{42}$$

Proof.

Let us construct the solutions of integral equations (40)-(41) according to the following scheme:

$$v_{m+1}(t) = v_0(t) + \int_{t_0}^{t} e^{-(t-s)A} P(w_m g) ds - \int_{t_0}^{t} e^{-(t-s)A} P(v_m \cdot \nabla v_m) ds, \quad (43)$$

$$w_{m+1}(t) = w_0(t) + \int_{t_0}^{t} e^{-(t-s)B} f ds - \int_{t_0}^{t} e^{-(t-s)B} (v_m \cdot \nabla w_m) ds, \quad (44)$$

where $v_0(t) = e^{-(t-t_0)A}a,\ w_0(t) = e^{-(t-t_0)B}b$.

Since this Lemma deals with only local existence of solutions, we may assume that $0 < T \leq 1$. Let $\alpha = n/q^*,\ q^* = nq/(n-q)$. Then $0 < \alpha < 1$. We shall need the following lemmas

Lemma 18 *The sequences (43), (44) satisfy the following estimates*

$$\sup_{t_0<t<t_0+T} (t-t_0)^{(1-\alpha)/2} \|v_m(t)\|_{(n/\alpha,\infty)} \leq K_{m,1}, \quad (45)$$

$$\sup_{t_0<t<t_0+T} (t-t_0)^{(1-\alpha)/2} \|w_m(t)\|_{(n/\alpha,\infty)} \leq K_{m,2},\ m = 0, 1, ... \quad (46)$$

for some positive constants $K_{m,1}, K_{m,2}$ which are independents of t_0. Moreover, there exist (v, w) with

$$(t-t_0)^{(1-\alpha)/2} v(.) \in BC((t_0, t_0+T); L_\sigma^{(n/\alpha,\infty)})^n,$$
$$(t-t_0)^{(1-\alpha)/2} w(.) \in BC((t_0, t_0+T); L^{(n/\alpha,\infty)}),$$

such that

$$\lim_{m\to\infty} \sup_{t_0<t<t_0+T} (t-t_0)^{(1-\alpha)/2} \|v_m(t) - v(t)\|_{(n/\alpha,\infty)} = 0,$$

$$\lim_{m\to\infty} \sup_{t_0<t<t_0+T} (t-t_0)^{(1-\alpha)/2} \|w_m(t) - w(t)\|_{(n/\alpha,\infty)} = 0.$$

Proof. The proof is done by induction, in fact,

$$\|v_0\|_{(n/\alpha,\infty)} \leq c\|a\|_{(n/\alpha,\infty)},\ \|w_0\|_{(n/\alpha,\infty)} \leq c\|b\|_{(n/\alpha,\infty)},$$

to $t_0 < t < t_0 + T$, where c is independent of t_0. Consequently,

$$\sup_{t_0<t<t_0+T} (t-t_0)^{(1-\alpha)/2} \|v_0(t)\|_{(n/\alpha,\infty)} \leq cT^{(1-\alpha)/2} \|a\|_{(n/\alpha,\infty)} \equiv K_{0,1} \quad (47)$$

$$\sup_{t_0<t<t_0+T} (t-t_0)^{(1-\alpha)/2} \|w_0(t)\|_{(n/\alpha,\infty)} \leq cT^{(1-\alpha)/2} \|b\|_{(n/\alpha,\infty)} \equiv K_{0,2}. \quad (48)$$

Assume true (45),(46). We will prove (45),(46) for the case $m+1$. Note that for all $\phi \in C_{0,\sigma}^\infty$ and all $t_0 < t < t_0 + T$, the lemma 7 implies

$$\left|\left(- \int_{t_0}^{t} e^{-(t-s)A} P(v_m \cdot \nabla v_m)(s) ds, \phi \right)\right| = \left| \int_{t_0}^{t} (v_m \otimes v_m(s), \nabla e^{-(t-s)A} \phi) ds \right|$$

$$\leq \int_{t_0}^{t} \|v_m(s)\|^2_{(n/\alpha,\infty)} \|\nabla e^{-(t-s)A}\phi\|_{(n/(n-2\alpha),1)} ds$$
$$\leq c\int_{t_0}^{t} (t-s)^{-\alpha/2-1/2} \|v_m(s)\|^2_{(n/\alpha,\infty)} ds \cdot \|\phi\|_{(n/(n-\alpha),1)}$$
$$\leq cB((1-\alpha)/2,\alpha) K^2_{m,1} (t-t_0)^{-(1-\alpha)/2} \|\phi\|_{(n/(n-\alpha),1)}$$

where $B(.,.)$ denote the function beta and $c = c(n,q)$ is independent of t_0. By duality we have

$$\Big\| \int_{t_0}^{t} e^{-(t-s)A} P(v_m \cdot \nabla v_m)(s) ds \Big\|_{(n/\alpha,\infty)} \leq C_{1,1} K^2_{m,1} (t-t_0)^{-(1-\alpha)/2},\ t_0 < t < t_0 + T.$$

with $C_{1,1} = C_{1,1}(n,q)$. Now,

$$\begin{aligned}
\Big\| \int_{t_0}^{t} e^{-(t-s)A} P(g w_m)(s) ds \Big\|_{(n/\alpha,\infty)} &\leq c\int_{t_0}^{t} (t-s)^{-n/2b} \|g\|_{(b,\infty)} \|w_m(s)\|_{(n/\alpha,\infty)} ds \\
&\leq c\|g\|_{(b,\infty)} (t-t_0)^{(1-\alpha)/2} (t-t_0)^{-(1-\alpha)/2} \\
&\times \int_{t_0}^{t} \|w_m(s)\|_{(n/\alpha,\infty)} (t-s)^{-n/2b} \\
&\leq c(t-t_0)^{-(1-\alpha)/2} K_{m,2} \|g\|_{(b,\infty)} \\
&\leq (t-t_0)^{-(1-\alpha)/2} C_{2,1} K_{m,2}.
\end{aligned}$$

Hence,

$$\sup_{t_0<t<t_0+T} (t-t_0)^{(1-\alpha)/2} \|v_{m+1}\|_{(n/\alpha,\infty)} \leq K_{0,1} + C_{1,1}K^2_{m,1} + C_{2,1}K_{m,2}. \quad (49)$$

Now, for all $\phi \in C_0^\infty$ and all $t_0 < t < t_0 + T$,

$$\Big|\Big(-\int_{t_0}^{t} e^{-(t-s)B} (v_m \cdot \nabla w_m)(s) ds, \phi\Big)\Big| = \Big|\int_{t_0}^{t} (v_m \otimes v_m(s), \nabla e^{-(t-s)B}\phi) ds\Big|$$

$$\leq \int_{t_0}^{t} \|v_m(s)\|_{(n/\alpha,\infty)} \|w_m(s)\|_{(n/\alpha,\infty)} \|\nabla e^{-(t-s)B}\phi\|_{(n/(n-2\alpha),1)} ds$$
$$\leq c\int_{t_0}^{t} (t-s)^{-\alpha/2-1/2} \|v_m(s)\|_{(n/\alpha,\infty)} \|w_m(s)\|_{(n/\alpha,\infty)} \cdot \|\phi\|_{(n/(n-\alpha),1)} ds$$
$$\leq cB((1-\alpha)/2,\alpha) K_{m,1} K_{m,2} (t-t_0)^{-(1-\alpha)/2} \|\phi\|_{(n/(n-\alpha),1)}.$$

By duality,

$$\Big\| \int_{t_0}^{t} e^{-(t-s)B} (v_m \cdot \nabla w_m)(s) ds \Big\|_{(n/\alpha,\infty)} \leq C_{1,2} K_{m,2} K_{m,1} (t-t_0)^{-(1-\alpha)/2},\ t_0 < t < t_0 + T.$$

Now, using the Lemma 10 we have

$$\begin{aligned}\int_{t_0}^{t}\|e^{-(t-s)B}f(s)\|_{(n/\alpha,\infty)} &\leq c\int_{t_0}^{t}\|\nabla e^{-(t-s)B}f(s)\|_{(q,\infty)} \\ &\leq c\int_{t_0}^{t}(t-s)^{-n/2(1/l-1/q)-1/2}\|f(s)\|_{(l,\infty)}ds \\ &\leq c\|f\|_{BC(\mathbb{R};L^{(l,\infty)})}(t-t_0)^{-(1-\alpha)/2+3/2-n/2l},\end{aligned}$$

for all $t_0 < t < t_0+T$ with $c = c(n,q,l)$. Since $1/l < 1/q+1/n$, we have $(1-\alpha)/2 < 3/2-n/2l$ hence the above estimate yields

$$(t-t_0)^{(1-\alpha)/2}\Big\|\int_{t_0}^{t}e^{-(t-s)B}f(s)\Big\|_{(n/\alpha,\infty)} \leq c\|f\|_{BC(\mathbb{R},L^{(b,\infty)})}T^{(1-\alpha)/2}. \quad (50)$$

Consequently,

$$\sup_{t_0<t<t_0+T}(t-t_0)^{(1-\alpha)/2}\|w_{m+1}(t)\|_{(n/\alpha,\infty)} \leq K_{0,2}+c\|f\|_{BC(\mathbb{R};L^{(l,\infty)})}T^{(1-\alpha)/2}+C_{1,2}K_{m,1}K_{m,2}. \quad (51)$$

Then, we can take $K_{m+1,1}, K_{m+1,2}$ being respectively, $K_{0,1}+C_{1,1}K_{m,1}^2+C_{2,1}K_{m,2},\ K_{0,2}+c\|f\|_{BC(\mathbb{R};L^{(l,\infty)})}T^{(1-\alpha)/2}+C_{1,2}K_{m,1}K_{m,2}$.
Letting $K_m = \mathrm{Max}(K_{m,1},K_{m,2}),\ m=1,2,...$, from (51) and (**??**) we have

$$K_{m+1} \leq K_0+\widetilde{C}K_m^2+C_{1,1}K_m, \quad (52)$$

where $K_0 = c_1T^{(1-\alpha)/2}\mathrm{Max}\{\|a\|_{(n/\alpha,\infty)}, \|b\|_{(n/\alpha,\infty)}+\|f\|_{BC(\mathbb{R};L^{(l,\infty)})}\}$ and $\widetilde{C} = \mathrm{Max}\{C_{1,1},C_{1,2}\}$.
If we consider

$$C_{2,1}<1,\ K_0 < \frac{(1-C_{2,1})^2}{4\widetilde{C}} \quad (53)$$

we have that

$$K_m < \frac{(1-C_{2,1})-\sqrt{(1-C_{2,1})^2-4\widetilde{C}K_0}}{2\widetilde{C}} \equiv k < \frac{1}{2\widetilde{C}},\ \forall m=0,1,2,... \quad (54)$$

Assuming (53) and working as section 2, we can conclude, due the uniform estimate, with respect to m, (54) the existence of a duple (v,w) such that

$$(t-t_0)^{(1-\alpha)/2}v(.) \in BC((t_0,t_0+T);L_\sigma^{(n/\alpha,\infty)})^n, \quad (55)$$

$$(t-t_0)^{(1-\alpha)/2}w(.) \in BC((t_0,t_0+T);L^{(n/\alpha,\infty)}), \quad (56)$$

satisfying

$$\lim_{m\to\infty}\sup_{t_0<t<t_0+T}(t-t_0)^{(1-\alpha)/2}\|v_m(t)-v(t)\|_{(n/\alpha,\infty)}=0, \quad (57)$$

$$\lim_{m\to\infty}\sup_{t_0<t<t_0+T}(t-t_0)^{(1-\alpha)/2}\|w_m(t)-w(t)\|_{(n/\alpha,\infty)}=0. \quad (58)$$

Lemma 19 *If K_0 defined by (53) is small sufficiently, then the limit (v, w) given by Lemma 18 satisfies the following estimate*

$$(t-t_0)^{1/2}\nabla v(.) \in BC((t_0, t_0+T); L_\sigma^{(n,\infty)})^{n\times n}, \quad (59)$$

$$(t-t_0)^{1/2}\nabla w(.) \in BC((t_0, t_0+T); L^{(n,\infty)})^{n}, \quad (60)$$

with

$$\lim_{m\to\infty} \sup_{t_0<t<t_0+T} (t-t_0)^{1/2}\|\nabla v_m(t) - \nabla v(t)\|_{(n,\infty)} = 0, \quad (61)$$

$$\lim_{m\to\infty} \sup_{t_0<t<t_0+T} (t-t_0)^{1/2}\|\nabla w_m(t) - \nabla w(t)\|_{(n,\infty)} = 0. \quad (62)$$

Proof. The proof is done by induction, in fact, we will prove that

$$\sup_{t_0<t<t_0+T} (t-t_0)^{1/2}\|\nabla v_m(t)\|_{(n,\infty)} \le J_{m,1}, \quad (63)$$

$$\sup_{t_0<t<t_0+T} (t-t_0)^{1/2}\|\nabla w_m(t)\|_{(n,\infty)} \le J_{m,2}, \quad (64)$$

for some constants $J_{m,1}, J_{m,2}$ which are independents of t_0, $m = 0, 1, ...$
Note that by lemma 7

$$\|\nabla v_0\|_{(n,\infty)} \le C(t-t_0)^{-1/2}\|a\|_{(n,\infty)},$$
$$\|\nabla w_0\|_{(n,\infty)} \le C(t-t_0)^{-1/2}\|b\|_{(n,\infty)},$$

where $C = C(n)$ is independent of t_0. Hence we can take $J_{0,1}$ and $J_{0,2}$ being respectively, $C\|a\|_{(n,\infty)}$, $C\|b\|_{(n,\infty)}$.
Supposed true the inequalities (63),(64). Then

$$\begin{aligned}\left\|\nabla \int_{t_0}^t e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)\right\|_{(n,\infty)} &\le \int_{t_0}^t (t-s)^{-n/2(\alpha/n)-1/2}\|v_m(s)\|_{(n/\alpha,\infty)}\|\nabla v_m(s)\|_{(n,\infty)} \\ &\le cK_{m,1}J_{m,1}\int_{t_0}^t (t-s)^{-\alpha/2-1/2}(s-t_0)^{\alpha/2-1}ds \\ &\le C_{3,1}kJ_{m,1}(t-t_0)^{-1/2},\end{aligned}$$

for all $t_0 < t < t_0+T$, where $C_{3,1} = C_{3,1}(n, q)$ is independent of t_0.
Now

$$\begin{aligned}\left\|\nabla \int_{t_0}^t e^{-(t-s)A}P(gw_m)(s)ds\right\|_{(n,\infty)} &\le \int_{t_0}^t (t-s)^{-n/2(\alpha/n)-1/2}\|g\|_{(n,\infty)}\|w_m(s)\|_{(n/\alpha,\infty)}ds \\ &\le c\|g\|_{(n,\infty)} \sup_{t_0<t<t_0+T}\|w_m(t)\|_{(n/\alpha,\infty)}\int_{t_0}^t (t-s)^{-(\alpha+1)/2}ds \\ &\le cB((1-\alpha)/2, (1+\alpha)/2)k\|g\|_{(n,\infty)}.\end{aligned}$$

Therefore,

$$\sup_{t_0<t<t_0+T}(t-t_0)^{1/2}\|\nabla v_{m+1}\|_{(n,\infty)} \le J_{0,1} + C_{3,1}kJ_{m,1} + C_{4,1}k\|g\|_{(n,\infty)}. \quad (65)$$

Now, for all $t_0 < t < t_0 + T$,

$$\begin{aligned}\Big\|\nabla\int_{t_0}^{t} e^{-(t-s)B}(v_m\cdot\nabla w_m)(s)\Big\|_{(n,\infty)} &\le \int_{t_0}^{t}(t-s)^{-n/2(\alpha/n)-1/2}\|v_m(s)\|_{(n/\alpha,\infty)}\|\nabla w_m(s)\|_{(n,\infty)}\\ &\le cK_{m,1}J_{m,2}\int_{t_0}^{t}(t-s)^{-\alpha/2-1/2}(s-t_0)^{\alpha/2-1}ds\\ &\le C_{2,2}kJ_{m,2}(t-t_0)^{-1/2},\end{aligned}$$

where $C_{2,2}$ is independent of t_0. As

$$\Big\|\nabla\int_{t_0}^{t} e^{-(t-s)B}f(s)ds\Big\|_{(n,\infty)} \le c(t-t_0)^{-1/2}\|f\|_{BC(\mathbb{R};L^{(n,\infty)})},$$

we conclude that

$$\sup_{t_0<t<t_0+T}(t-t_0)^{1/2}\|w_{m+1}(t)\|_{(n,\infty)} \le J_{0,2}+C_{2,2}kJ_{m,2}+c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})}. \quad (66)$$

Then we can to take $J_{m+1,1}$ and $J_{m+1,2}$ being respectively,

$$J_{0,1}+C_{3,1}kJ_{m,1}+C_{4,1}k\|g\|_{(n,\infty)},\quad J_{0,2}+C_{2,2}kJ_{m,2}+c\|f\|_{BC(\mathbb{R};L(n,\infty))}.$$

Let $J_m = \mathrm{Max}\{J_{m,1}, J_{m,2}\}$, $m = 1, 2, ...$ and $J_0 = \mathrm{Max}\{J_{0,1}+C_{4,1}k\|g\|_{(n,\infty)},\ J_{0,2}+c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})}\}$. Then

$$J_{m+1} \le J_0 + k\widetilde{C}J_m, \quad (67)$$

where $\widetilde{C} = \mathrm{Max}\{C_{3,1}, C_{2,2}\}$.
Consequently, if

$$k < 1/\widetilde{C} \quad (68)$$

we have a uniform estimate for the sequence $\{J_m\}$ give by

$$J_m \le \frac{J_0}{1-\widetilde{C}k} \equiv J,\ m = 0, 1, ...$$

Assuming (68) for a moment, we can see that the limits v, w satisfy (59)-(60) and the proof of lemma is finished.

Lemma 20 *The limit (v, w) given by Lemma 18, Lemma* **??** *satisfies the following estimate*

$$(t-t_0)^{1/4}v(.) \in BC((t_0, t_0+T); L^{2n}_\sigma)^n, \quad (69)$$
$$(t-t_0)^{1/4}w(.) \in BC((t_0, t_0+T); L^{2n}), \quad (70)$$

with

$$\lim_{m\to\infty}\sup_{t_0<t<t_0+T}(t-t_0)^{1/4}\|v_m(t)-v(t)\|_{2n} = 0, \quad (71)$$

$$\lim_{m\to\infty}\sup_{t_0<t<t_0+T}(t-t_0)^{1/4}\|w_m(t)-w(t)\|_{2n} = 0. \quad (72)$$

Proof. As the Lemmas before, the proof is done by induction. In fact, we will prove that there exist constants $N_{m,1}, N_{m,2}$, which are independents of t_0, such that

$$\begin{aligned} \|v_m(t)\|_{2n} &\leq N_{m,1}(t-t_0)^{-1/4} \qquad (73) \\ \|w_m(t)\|_{2n} &\leq N_{m,2}(t-t_0)^{-1/4}. \qquad (74) \end{aligned}$$

Since $L^{(p_0,\infty)} \cap L^{(p_1,\infty)} \subset L^p$ and $\|f\|_p \leq C(p_o, p_1, \lambda)\|f\|^{1-\lambda}_{(p_0,\infty)}\|f\|^{\lambda}_{(p_1,\infty)}$ provided that $p_0 \neq p_1, 0 < \lambda < 1$ and $1/p = (1-\lambda)/p_0 + \lambda/p_1$, we have

$$\begin{aligned} \|v_0(t)\|_{2n} &\leq C(t-t_0)^{-1/4}\|a\|_{(n,\infty)}, \\ \|w_0(t)\|_{2n} &\leq C(t-t_0)^{-1/4}\|b\|_{(n,\infty)}, \end{aligned}$$

where $C = C(n)$ is independent of t_0. Hence, we define $N_{0,1}$ and $N_{0,2}$ as $C\|a\|_{(n,\infty)}$ and $C\|b\|_{(n,\infty)}$, respectively.
Assuming true (73)-(74), we can prove that (73)-(74) hold for the case $m+1$. In fact, note that for all $\phi \in C^\infty_{0,\sigma},\ \varphi \in C^\infty_0$, we have

$$\begin{aligned} \Big|\Big(-\int_{t_0}^t e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)ds, \phi\Big)\Big| &\leq \int_{t_0}^t \|v_m\otimes v_m\|_n\|\nabla e^{-(t-s)A}\phi\|_{n'} \\ &\leq C\int_{t_0}^t \|v_m\|^2_{2n}(t-s)^{-3/4}\|\phi\|_{(2n)'}, \\ \Big|\Big(-\int_{t_0}^t e^{-(t-s)B}(v_m\cdot\nabla w_m)(s)ds, \varphi\Big)\Big| &\leq \int_{t_0}^t \|w_m\cdot v_m\|_n\|\nabla e^{-(t-s)B}\varphi\|_{n'} \\ &\leq C\int_{t_0}^t \|v_m\|_{2n}\|w_m\|_{2n}(t-s)^{-3/4}\|\varphi\|_{(2n)'}. \end{aligned}$$

Hence by duality

$$\begin{aligned} \Big\|\int_{t_0}^t e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)ds\Big\|_{2n} &\leq C_{1,1}N^2_{m,1}, \\ \Big\|\int_{t_0}^t e^{-(t-s)B}(v_m\cdot\nabla w_m)(s)ds\Big\|_{2n} &\leq C_{1,2}N_{m,1}N_{m,2}. \end{aligned}$$

We also note that

$$\begin{aligned} \Big\|\int_{t_0}^t e^{-(t-s)A}P(gw_m)\Big\|_{2n} &\leq \int_{t_0}^t \|e^{-(t-s)A}P(gw_m)\|_{2n} \leq \int_{t_0}^t (t-s)^{-1/2}\|g\|_{(n,\infty)}\|w_m\|_{(2n)} \\ &\leq c\|g\|_{(n,\infty)}N_{m,2}. \end{aligned}$$

and

$$\begin{aligned} \Big\|\int_{t_0}^t e^{-(t-s)B}f(s)ds\Big\|_{2n} &\leq \int_{t_0}^t \|e^{-(t-s)B}f(s)\|_{2n} \leq \int_{t_0}^t (t-s)^{-1/4}\|f(s)\|_{(n,\infty)}ds \\ &\leq c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})}. \end{aligned}$$

The inequalities above imply that

$$\sup_{t_0<t<t_0+T}(t-t_0)^{1/4}\|v_m\|_{2n} \leq N_{0,1}+C_{1,1}N_{m,1}^2+C_{2,1}N_{m,2}$$
$$\sup_{t_0<t<t_0+T}(t-t_0)^{1/4}\|w_m\|_{2n} \leq N_{0,2}+C_{1,2}N_{m,1}N_{m,2}+c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})},$$

with $C_{2,1}=c\|g\|_{(n,\infty)}$. As before, letting $N_m=\mathrm{Max}(N_{m,1},N_{m,2}), m=1,2,...$ and $N_0=\mathrm{Max}(N_{0,2}+c\|f\|_{BC(\mathbb{R};L^{(n,\infty)})},N_{0,1})$, we obtain

$$N_{m+1}\leq N_0+\tilde{C}N_m^2+C_{2,1}N_m,,$$

where $\tilde{C}=\mathrm{Max}(C_{1,1},C_{1,2})$. If we consider

$$C_{2,1}<1,\quad N_0<\frac{(1-C_{2,1})^2}{4\tilde{C}}, \tag{75}$$

we have that the sequence $\{N_m\}_{m=0}^{m=\infty}$ is bounded with

$$N_m\leq\frac{(1-C_{2,1})-\sqrt{(1-C_{2,1})^2-4N_0\tilde{C}}}{2\tilde{C}},\ m=0,1,...$$

Assuming (75) and working as the lemmas 18, 19, we conclude the proof of Lemma.
Using the Lemma 18 and Lemma 19, we will prove that $v\in BC((t_0,t_0+T);L_\sigma^{(n,\infty)}\cap L_\sigma^{(q^*,\infty)})^n$ and $w\in BC((t_0,t_0+T);L^{(n,\infty)}\cap L^{(q^*,\infty)})$; for this, we need prove that

$$\sup_{t_0<t<t_0+T}\|v_m(t)\|_{(n/s,\infty)}\leq M_{1,s,m},\ s=\alpha,\quad s=1, \tag{76}$$
$$\sup_{t_0<t<t_0+T}\|w_m(t)\|_{(n/s,\infty)}\leq M_{2,s,m},\ s=\alpha,\quad s=1, \tag{77}$$

with $M_{1,s,m},M_{2,s,m}$ independent of t_0. Calculations similar to Lemma 18, Lemma 19, yields

$$M_{1,\alpha,0}=C\|a\|_{(n/\alpha,\infty)},\ M_{1,1,0}=C\|a\|_{(n,\infty)},$$
$$M_{2,\alpha,0}=C\|b\|_{(n/\alpha,\infty)},\ M_{2,1,0}=C\|b\|_{(n,\infty)},$$

where $C=C(n,s)$ is independent of t_0. Suppose by induction that (76), (77) are true.
Note that

$$\Big|\Big(-\int_{t_0}^t e^{-(t-s)A}P(v_m\cdot\nabla v_m)(s)ds,\phi\Big)\Big|\leq$$

$$
\begin{aligned}
&\leq \Big| \int_{t_0}^{t} (v_m \otimes v_m(s), \nabla e^{-(t-s)A}\phi) ds \Big| \\
&\leq \int_{t_0}^{t} \|v_m\|_{(n/\alpha,\infty)} \|v_m\|_{(n/s,\infty)} \|\nabla e^{-(t-s)A}\phi\|_{(n/(n-\alpha-s),1)} ds \\
&\leq CK_{m,1} M_{1,s,m} \int_{t_0}^{t} (t-s)^{-\alpha/2-1/2} (s-t_0)^{-(1-\alpha)/2} \cdot \|\phi\|_{(n/(n-s),1)} ds \\
&\leq CkM_{1,s,m} B((1-\alpha)/2, (1+\alpha)/2) \|\phi\|_{(n/(n-s),\infty)},
\end{aligned}
$$

for all $\phi \in C_{0,\sigma}^{\infty}$ and all $t_0 < t < t_0 + T, C = C(n,q,s)$ is independent of t_0; By duality we have consequently,

$$
\sup_{t_0 < t < t_0+T} \Big\| \int_{t_0}^{t} e^{-(t-s)A} P(v_m \cdot \nabla v_m)(s) ds \Big\|_{(n/s,\infty)} \leq C_{5,1} k M_{1,s,m}, \ s = 1, \alpha, \quad (78)
$$

where $C_{5,1}$ independent of t_0.
Note that

$$
\Big\| \int_{t_0}^{t} e^{-(t-s)A} P(g w_m) \Big\|_{(n/s,\infty)} \leq c \|g\|_{(b,\infty)} \int_{t_0}^{t} (t-s)^{-n/2b} \|w_m(s)\|_{(n/s,\infty)} ds \leq C_{6,1} M_{2,s,m}, \quad (79)
$$

with $C_{6,1} = c\|g\|_{(n,\infty)}$ and

$$
\begin{aligned}
\| \int_{0}^{t} e^{-(t-s)B} f(s) ds \|_{(n/\alpha,\infty)} &\leq c \|f\|_{BC(\mathbb{R};\mathbb{L}^{(<,\infty)})} \\
\| \int_{0}^{t} e^{-(t-s)B} f(s) ds \|_{(n,\infty)} &\leq cT.
\end{aligned}
$$

Now, for all $\phi \in C_0^{\infty}$ and all $t_0 < t < t_0 + T$,

$$
\Big| \Big(- \int_{t_0}^{t} e^{-(t-s)B} (v_m \cdot \nabla w_m)(s) ds, \phi \Big) \Big| \leq
$$

$$
\begin{aligned}
&\leq \Big| \int_{t_0}^{t} (w_m \cdot v_m(s), \nabla e^{-(t-s)B}) \phi) ds \Big| \\
&\leq \int_{t_0}^{t} \|v_m\|_{(n/\alpha,\infty)} \|w_m\|_{(n/s,\infty)} \|\nabla e^{-(t-s)B}\phi\|_{(n/(n-\alpha-s),1)} ds \\
&\leq CK_{m,1} M_{2,s,m} \int_{t_0}^{t} (t-s)^{-\alpha/2-1/2} (s-t_0)^{-(1-\alpha)/2} ds \cdot \|\phi\|_{(n/(n-s),1)} \\
&\leq CkM_{2,s,m} B((1-\alpha)/2, (1+\alpha)/2) \|\phi\|_{(n/(n-s),1)}.
\end{aligned}
$$

Consequently,

$$
\sup_{t_0 < t < t_0+T} \Big\| \int_{t_0}^{t} e^{-(t-s)B} (v_m \cdot \nabla w_m)(s) ds \Big\|_{(n/s,\infty)} \leq C_{4,2} k M_{2,s,m}, \ s = 1, \alpha, \quad (80)
$$

where $C_{4,2}$ is independent of t_0.
Hence, from (78)-(80) we can take

$$\begin{aligned} M_{1,s,m+1} &= M_{1,s,0} + C_{5,1}kM_{1,s,m} + C_{6,1}M_{2,s,m}, && (81)\\ M_{2,s,m+1} &= M_{2,s,0} + c\|f\|_{BC(\mathbb{R};L^{(l,\infty)})} + C_{4,2}kM_{2,s,m}. && (82) \end{aligned}$$

Letting

$$\begin{aligned} M_{s,m} &= \text{Max}\{M_{1,s,m}, M_{2,s,m}\} \\ M_{s,0} &= \text{Max}\{M_{1,s,0}, M_{2,s,0} + c\|f\|_{BC(\mathbb{R};L^{(l,\infty)})}\} \\ \widetilde{C} &= \text{Max}\{C_{5,1}, C_{4,2}\}, \end{aligned}$$

from (81),(82) we obtain

$$M_{s,m+1} \leq M_{s,0} + k\widetilde{C}M_{s,m} + C_{6,1}M_{s,m}, \quad m = 0, 1, ... \; s = 1, \alpha.$$

Then, if

$$C_{6,1} < 1, \quad k\widetilde{C} < 1, \tag{83}$$

we have that

$$M_{s,m} \leq \frac{M_{s,0}}{1 - k\widetilde{C} - C_{6,1}}, \quad m = 0, 1, ..., s = 1, \alpha,$$

which yields $v \in BC((t_0, t_0 + T); L_\sigma^{(n,\infty)} \cap L_\sigma^{(q^*,\infty)})^n$ and $w \in BC((t_0, t_0 + T); L^{(n,\infty)} \cap L^{(q^*,\infty)})$, with

$$\begin{aligned} \lim_{m\to\infty} \sup_{t_0<t<t_0+T} \|v_m(t) - v(t)\|_{(n/s,\infty)} &= 0, \; s = 1, \alpha, \\ \lim_{m\to\infty} \sup_{t_0<t<t_0+T} \|w_m(t) - w(t)\|_{(n/s,\infty)} &= 0, \; s = 1, \alpha. \end{aligned}$$

Now we see that under the conditions (53),(68), (83), the limit (v, w) belongs to the class required in the Proposition. Moreover, there holds

$$\int_{t_0}^t e^{-(t-s)A} P(v_m \cdot \nabla v_m)(s) ds \longrightarrow \int_0^t e^{-(t-s)A} P(v \cdot \nabla v)(s) ds, \text{ in } L_\sigma^{(n,\infty)}, \tag{84}$$

$$\int_{t_0}^t e^{-(t-s)A} P(w_m g)(s) ds \longrightarrow \int_{t_0}^t e^{-(t-s)A} P(wg)(s) ds, \text{ in } L_\sigma^{(n,\infty)}, \tag{85}$$

$$\int_0^t e^{-(t-s)B} P(v_m \cdot \nabla w_m)(s) ds \longrightarrow \int_0^t e^{-(t-s)A} P(v \cdot \nabla w)(s) ds, \text{ in } L^{(n,\infty)}, \tag{86}$$

uniformly in $t \in (t_0, t_0+t)$ as $m \to \infty$. In fact, note that by Lemma 7, Lemma 18 and Lemma 19, we have

$$
\begin{aligned}
\Big\| \int_{t_0}^t e^{-(t-s)A} P(v_m \cdot \nabla v_m)(s) ds - \int_{t_0}^t e^{-(t-s)A} P(v \cdot \nabla w)(s) ds \Big\|_{n,\infty} &\leq \\
\int_{t_0}^t \| e^{-(t-s)A} P((v_m - v) \cdot \nabla w_m)(s) \|_{(n,\infty)} ds &+ \\
\int_{t_0}^t \| e^{-(t-s)A} P(v \cdot \nabla (v_m - v))(s) \|_{(n,\infty)} ds &+ \\
\int_{t_0}^t (t-s)^{-\alpha/2} \| v_m(s) - v(s) \|_{(n/\alpha,\infty)} \| \nabla v_m(s) \|_{(n,\infty)} ds &+ \\
\int_{t_0}^t (t-s)^{-\alpha/2} \| v(s) \|_{(n/\alpha,\infty)} \| \nabla (v_m(s) - v(s)) \|_{(n,\infty)} ds &\times \\
cB(1-\alpha/2, \alpha/2)(J \sup_{t_0 < s < t_0 + T} (s-t_0)^{(1-\alpha)/2} \| v_m(s) - v(s) \|_{(n/\alpha,\infty)}) &+ \\
k \sup_{t_0 < s < t_0+T} (s-t_0)^{1/2} \| \nabla v_m(s) - \nabla v(s) \|_{(n,\infty)}) \longrightarrow 0.
\end{aligned}
$$

By other hand,

$$
\begin{aligned}
&\Big\| \int_{t_0}^t e^{-(t-s)A} P(w_m g)(s) ds - \int_{t_0}^t e^{-(t-s)A} P(wg)(s) ds \Big\|_{(n,\infty)} \leq \int_{t_0}^t \| e^{-(t-s)A} P((w_m - w)g)(s) \|_{(n,\infty)} ds \\
&\leq \int_{t_0}^t (t-s)^{-n/2b} \| g \|_{(b,\infty)} \| w_m(s) - w(s) \|_{(n/\alpha,\infty)} \leq c(s-t_0)^{(1-\alpha)/2} \| w_m(s) - w(s) \|_{(n/\alpha,\infty)} \longrightarrow 0. \quad (87)
\end{aligned}
$$

Analogously, we obtain (86).
Now we will prove the weak continuity on the initial dada. We first noting that for any $\varphi \in L_\sigma^{(n',1)}$ and $\phi \in L^{(n',1)}$ we have

$$
\begin{aligned}
|(e^{-(t-t_o)A} a - a, \varphi)| &= |(a, e^{-(t-t_0)A} \varphi - \varphi)| \\
&\leq \| a \|_{(n,\infty)} \| e^{-(t-t_0)A} \varphi - \varphi \|_{(n',1)} \to 0, \ t \to t_0^+. \\
|(e^{-(t-t_o)B} b - b, \phi)| &= |(b, e^{-(t-t_0)A} \varphi - \varphi)| \\
&\leq \| b \|_{(n,\infty)} \| e^{-(t-t_0)B} \phi - \phi \|_{(n',1)} \to 0, \ t \to t_0^+.
\end{aligned}
$$

As $\|v\|_{(q^*,\infty)}, \|w\|_{(q^*,\infty)} \leq c$ and $t^{1/4}\|v\|_{2n}, t^{1/4}\|w\|_{2n} \leq c$, we can obtain

$$
\lim_{t \to t_0} \Big(\int_{t_0}^t e^{-(t-s)A} P(wg) - \int_{t_0}^t e^{-(t-s)A} P(v \cdot \nabla v), \phi \Big) = 0,
$$
$$
\lim_{t \to t_0} \Big(\int_{t_0}^t e^{-(t-s)B} f - \int_{t_0}^t e^{-(t-s)B} (v \cdot \nabla w), \varphi \Big) = 0.
$$

Letting $m \to \infty$ in (43)-(44), we see by estimates above that (v, w) is a solution of (40)-(41).

The proof for uniqueness is standard, so we ma omit it.
It remains to estimate the time-interval T of existence in terms of the prescribed data. As k is determined by (54), there exists a constant $\tilde{k}$ independent of t_0 such that if $K_0 \leq \tilde{k}$, then the conditions (53),(68), (83) are satisfied. Now, from (52) we see that T may be chosen as

$$T \equiv \Big[\frac{\tilde{k}}{c_1 \mathrm{Max}\{\|a\|_{(n/\alpha,\infty)}, \|b\|_{(n/\alpha,\infty)} + \|f\|_{BC(\mathbb{R};L^{(l,\infty)})}\}}\Big]^{\frac{\alpha-1}{2}} \tag{88}$$

This complete the proof of Proposition.

Remark 21 *The solution (v,w) of integral equations (37)-(40) satisfies that*

$$v \in BC(t_0, t_0+T, L^p_\sigma)^n, \ \ w \in BC(t_0, t_0+T, L^p),$$

for all $p \in (n, q^*)$, with

$$\|v\|_p \leq C\|v\|^{1-\lambda}_{(n,\infty)} \|u\|^{\lambda}_{(q^*,\infty)}, \ \ \|w\|_p \leq C\|w\|^{1-\lambda}_{(n,\infty)} \|w\|^{\lambda}_{(q^*,\infty)}, \tag{89}$$

where λ is such that $1/p = (1-\lambda)/n + \lambda/q^*$.

Proof of Theorem 16. Being (v,w) the integral solution of (40)-(41), we can prove the time Holder continuity of

$$F(v,w) \equiv -P(v\cdot\nabla v) + P(wg), \ \ G(v,w) \equiv -(v\cdot\nabla w) + f,$$

in the $L^{(n,\infty)}$ space. Indeed, we follow the ideas of [22], and use the Theorem 3.3.4 of [29].

Proof of Theorem 5. Let (u,θ) the periodic solution of the integral equations (12),(13) given by Theorem 4. As $u \in Y, \theta \in X$, we have by the Lemma (10) that $u \in BC(\mathbb{R}; L^{(n,\infty)}_\sigma \cap L^{q^{(*,\infty)}})^n, \ \theta \in BC(\mathbb{R}; L^{(n,\infty)} \cap L^{(q^*,\infty)})$. Let T defined by (39). By Theorem 16, for every $t_0 \in \mathbb{R}$, there exists a unique strong solution (v,w) of (1)-(5) on (t_0, t_0+T) with the initial data $(u(t_0), \theta(t_0))$. From (31), (38) we have

$$\sup_{t_0<t<t_0+T} \|v(t)\|_{(q^*,\infty)} + \sup_{t_0<t<t_0+T} \|\nabla u(t)\|_{(q,\infty)} \leq C_{7,1} \tag{90}$$

$$\sup_{t_0<t<t_0+T} \|w(t)\|_{(q^*,\infty)} + \sup_{t_0<t<t_0+T} \|\nabla \theta(t)\|_{(q,\infty)} \leq C_{7,2} \tag{91}$$

where $C_{7,1}, C_{7,2}$ are independents of t_0. Replacing (a,b) by $(u(t_0), \theta(t_0))$ in (40),(41), by (12), (13), we can see

$$\begin{aligned} u(t) - v(t) &= -\int_{t_0}^{t} e^{-(t-s)A} P(u\cdot\nabla u)(s) + \int_{t_0}^{t} e^{-(t-s)A} P(v\cdot\nabla v) \\ &+ \int_{t_0}^{t} e^{-(t-s)A} P(\theta f) - \int_{t_0}^{t} e^{-(t-s)A} P(wg) \\ &= -\int_{t_0}^{t} e^{-(t-s)A} P((u-v)\cdot\nabla u) - \int_{t_0}^{t} e^{-(t-s)A} P(v\cdot\nabla(u-v)) \\ &+ \int_{t_0}^{t} e^{-(t-s)A} P((\theta - w)g) \\ &\equiv I_1(t) + I_2(t) + I_3(t), \quad t_0 < t < t_0+T, \end{aligned} \tag{92}$$

$$\begin{aligned}\theta(t) - w(t) &= -\int_{t_0}^{t} e^{-(t-s)B}(u\cdot\nabla\theta)(s) + \int_{t_0}^{t} e^{-(t-s)B}(v\cdot\nabla w)(s)\\ &= -\int_{t_0}^{t} e^{-(t-s)B}((u-v)\cdot\nabla\theta) - \int_{t_0}^{t} e^{-(t-s)B}(v\cdot\nabla(\theta-w))(s)\\ &\equiv I_4(t) + I_5(t),\ t_0 < t < t_0+T, \end{aligned} \tag{93}$$

Note que

$$\begin{aligned}\|I_1(t)\|_{(n,\infty)} &\le C\int_{t_0}^{t}(t-s)^{-n/2(1/q)}\|(u-v)(s)\|_{(n,\infty)}\|\nabla u(s)\|_{(q,\infty)}ds\\ &\le C\sup_{s\in\mathbb{R}}\|\nabla u(s)\|_{(q,\infty)}\sup_{t_0<s<t_0+t}\|(u-v)(s)\|_{(n,\infty)}(t-t_0)^{(1-n/2q)}, \end{aligned} \tag{94}$$

for all $t_0 < t < t_0 + T$, where $C = C(n,q)$ is independent of t_0.

$$\begin{aligned}|(I_2(t),\phi)| &= \Big|\int_{t_0}^{t}(v\otimes(u-v)(s)\nabla e^{-(t-s)A}\phi)ds\Big|\\ &\le C\int_{t_0}^{t}\|v(s)\|_{(q^*,\infty)}\|\nabla e^{-(t-s)A}\phi\|_{(q',1)}\|u(s)-v(s)\|_{(n,\infty)}\\ &\le C\sup_{t_0<s<t_0+t}\|v(s)\|_{(q^*,\infty)}\sup_{t_0<s<t_0+t}\|u(s)-v(s)\|_{(n,\infty)}(t-t_0)^{1-n/2q}\|\phi\|_{(n',1)}, \end{aligned}$$

for all $\phi \in C^\infty_{0,\sigma}$ and for all $t_0 < t < t_0 + T$, where C is independent of t_0. By duality, we have that

$$\|I_2(t)\|_{(n,\infty)} \le C\sup_{t_0<s<t_0+t}\|v(s)\|_{(q^*,\infty)}\sup_{t_0<s<t_0+t}\|u(s)-v(s)\|_{(n,\infty)}(t-t_0)^{1-n/2q}, \tag{95}$$

for all $t_0 < t < t_0 + T$.

$$\begin{aligned}\|I_3(t)\|_{(n,\infty)} &\le C\int_{t_0}^{t}(t-s)^{-n/2(1/b)}\|\theta(s)-w(s)\|_{(n,\infty)}\|g\|_{(b,\infty)}\\ &\le C\sup_{t_0<s<t_0+t}\|\theta(s)-w(s)\|_{(n,\infty)}(t-t_0)^{1-n/2b}, \end{aligned} \tag{96}$$

for all $t_0 < t < t_0 + T$, with C independent of t_0.
An similar analysis to I_1, I_2 implies

$$\|I_4(t)\|_{(n,\infty)} \le C\sup_{t_0<s<t_0+t}\|\nabla\theta(s)\|_{(\tilde{q},\infty)}\sup_{t_0<s<t_0+t}\|(u-v)(s)\|_{(n,\infty)}(t-t_0)^{(1-n/2\tilde{q})}, \tag{97}$$

$$\|I_5(t)\|_{(n,\infty)} \le C\sup_{t_0<s<t_0+t}\|v(s)\|_{(q^*,\infty)}\sup_{t_0<s<t_0+t}\|\theta(s)-w(s)\|_{(n,\infty)}(t-t_0)^{1-n/2q}, \tag{98}$$

for all $t_0 < t < t_0 + T$, with C independent of t_0.
From (90)-(98) follow that for all $t_0 < t < t_0 + T$,

$$\begin{aligned}\|u(t)-v(t)\|_{(n,\infty)} &\le C_1\Big(\sup_{t_0<s<t_0+t}\|u(s)-v(s)\|_{(n,\infty)}(t-t_0)^{1-n/2\tilde{q}} +\\ &+ \sup_{t_0<s<t_0+t}\|\theta(s)-w(s)\|_{(n,\infty)})(t-t_0)^{1-n/2b}\Big) \end{aligned} \tag{99}$$

$$\|\theta(t)-w(t)\|_{(n,\infty)} \le C_2\sup_{t_0<s<t_0+t}\|\theta(s)-w(s)\|_{(n,\infty)}(t-t_0)^{1-n/2q}, \tag{100}$$

where C_1, C_2 are independents of t_0.
For all $t_0 < t < t_0 + T$ let $E(t) = \text{Max}\{\|u(t) - v(t)\|_{(n,\infty)}, \|\theta(t) - w(t)\|_{(n,\infty)}\}$.
Hence, from (99)-(72) follow that for all $t_0 < t < t_0 + T$

$$E(t) \leq C_3 \sup_{t_0 < s < t_0 + t} E(s)(t - t_0)^{1 - n/2p},$$

where $p = Max(b, q)$. Therefore, $E(t) \leq C_3 \sup_{t_0 < s < t_0 + t} E(s) T^{1 - n/2p}$. Taking $\varsigma \equiv \min\{(1/2C_3)^{2p/(2p-n)}, T\}$ we conclude that

$$E(t) \leq 1/2 \sup_{t_0 < t < t_0 + t} E(s), \tag{101}$$

for all $t_0 < t < t_0 + T$, and hence we encounter

$$E(t) \equiv 0 \text{ on } [t_0, t_0 + \varsigma).$$

Since ς can be taken independently of t_0, we have $E(t) \equiv 0$ on $[t_0, t_0 + T)$. This imply that $u = v$ on $[t_0, t_0 + T), \theta = w$ on $[t_0, t_0 + T)$.Finally, as t_0 is arbitrary by Theorem 16, we conclude that (u, θ) is the required solution in Theorem 5.

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References

[1] Benilan, P., Brezis, H., Crandall, M., A semilinear Equation in $L^1(R^n)$, Ann. Scuola Norm. Sup. Pisa, Serie V 2 (1975) 523-555.

[2] Bergh, J., Löfström, J., *Interpolation Spaces.* Springer Verlag, Berlin, 1976.

[3] Borchers, W., Miyakawa, T., *Algebraic L^2 decay for Navier-Stokes flows in exterior domains.* Acta Math. 165, (1990), 189-227.

[4] Borchers, W., Miyakawa, T., *L^2 decay for Navier-Stokes flows in halfspaces.* Math. Ann. 282, (1988), 139-155.

[5] Borchers, W., Miyakawa, T., *On stability of exterior stationary Navier-Stokes flows.* Acta Math. 174, (1995), 311-382.

[6] Cannon, J. R. and DiBenedetto, E., The initial value problem for the Boussinesq equations with data in L^p, *Approximation Methods for Navier-Stokes Problems,* Edited by Rautmann, R., Lect. Notes in Math., Vol. 771, Springer-Verlag, Berlin, 1980.

[7] Chandrasekhar, S., "Hidrodinamic and Hydromagnetic Stability," Dover, New york, 1981.

[8] Chen, Z., Solutions of the stationary and nonstationary Navier-Stokes equations in exterior damains. Pacific J. Math. 159 (1993), 227-240.

[9] Fife, P. C. and Joseph, D. D., Existence of convective solutions of the generalized Bérnard problem which are analytic in their norm, Arch. Rational Mech. Anal., 33 (1969) 116-138.

[10] Foias, C., Manley, O. and Temam, R., Attractors for the Bérnard problem: existence and physical bounds on their fractal dimension, Nonlinear Anal. T. M.A., 11 (1987), 939-967.

[11] Fujiwara, D., Morimoto, H., *An L^r-Theorem of the Helmholtz decomposition of vectos fields.*, Fac.Sci.Univ.Tokio, sec. IA 24 (1977), 658-700.

[12] Giga, Y. *Analiticy of the semigroup generated by the Stokes operator in L_r spaces.* Math. Z., 187 (1981), 297-329.

[13] Giga, Y. *Solutions for the semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system.* J. Diff. Eq. 62 (1986), 186-212.

[14] Giga, Y., Sorh, H., *On the Stokes operator in exterior domains.* J. Fac. Sci. Univ. Tokyo. Sec IA, 36 (1989), 103-130.

[15] Hishida, T., Existence and Regularizing Properties of Solutions for the Nonstationary Convection Problem, Funkcialaj Ekvacioj, 34(1991) 449-474.

[16] Hishida, T., Global Existence and Exponential Stability of Convection, J. Math. Anal. Appl. 196(1995) 699-721.

[17] Hishida, T., On a class of Stable Steady Flows to the Exterior Convection Problem, J. Diff. Eq. 141(1997) 54-85.

[18] Hunt, R., On $L(p,q)$ spaces. L 'Enseignement Mathématique, t. (12) (4) (1966) 249-276.

[19] Iwashita, H., $L_q - L_r$ *estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in L_q spaces.* Math. Ann. 285 (1989), 265-288.

[20] Kato, T.,*Strong L^p−solutions of the Navier-Stokes equations in $\boldsymbol{R}^m$, with applications to weak solutions,* Math. Z., 187 (1984), 471-480.

[21] Kozono, H., Nakao, M.,*Periodic solutions of the Navier-Stokes equations in unbounded domains.* J. Tohoku Math, 48 (1996), 33-50.

[22] Kozono, H., Ogawa, T., *Some L^p estimate for the exterior Stokes flow and an application to the nonstationary Navier-Stokes equations.* Indiana Univ. Math. J., 41 (1992), 789-808.

[23] Kozono, H., Ogawa, T., *On stability of the Navier-Stokes flows in exterior domains.* Arch. Rational Mech. Anal., 128 (1994), 1-31.

[24] Giga, Y., Miyakawa, T., *Solutions in L_r of the Navier-Stokes initial value problem.* Arch. Rational Mech. Anal., 89 (1985), 276-281.

[25] Maremonti, P., *Existence and stability of time periodic solutions of the Navier-Stokes equations in whole space.* Nonlinearity 4 (1991), 503-529.

[26] Maremonti, P., *Some theorems of existence for solutions of the Navier-Stokes equations with slip boundary conditions in half-space.* Rich. Mat. 40 (1991), 81-135.

[27] Ōeda, K., On the initial value problem for the heat convection equation of Boussinesq approximation in a time-dependent domain, Proc. Japan Acad., Ser. A, 64(1988) 143-146.

[28] Stein, E. M., Weiss, G., Introduction to Fourier analysis on Euclidean spaces, Princeton University Press, Princeton, N.J., 1971.

[29] Tanabe, H., *Equations of evolution.* Pitman, London. 1979.

[30] Taniuchi, Y., *On stability of periodic solutions of the Navier-Stokes equations in unbounded domains* Hokkaido Math. J, 28 (1) (1999), 147-173.

[31] Yamasaki, M., *Solutions in Morrey Spaces of the Navier-Stokes Equations with Time-Dependent External Force* Funk. Ekva., 43 (2000), 419-460.