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Multiple Sign Changing Solutions to Semilinear Elliptic Problems

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Multiple sign changing solutions to semilinear elliptic problems *†

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Abstract

In this paper we establish the existence of multiple sign changing solutions for the semilinear elliptic problem

$$\begin{aligned} -\Delta u = g(u) &\quad \text{in} \quad \Omega \\ u = 0 &\quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $g : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $g(0) = 0$ and which is superlinear or asymptotically linear at infinity.

1 Introduction

Let us consider the problem

$$\begin{aligned} -\Delta u = g(u) &\quad \text{in} \quad \Omega \\ u = 0 &\quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$ and $g : \mathbb{R} \to \mathbb{R}$ is a function that satisfies

(g_1) $g(0) = 0$, $g'(0) < \lambda_1$ and $g \in C^1(\mathbb{R}, \mathbb{R})$.

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(g_2) There exist constants $R > 0$ and $\theta > 2$ such that

$$0 < \theta G(t) \leq tg(t) \quad \text{for all} \quad |t| \geq R,$$

where $G(t) = \int_0^t g(s)ds$.

Supposing that g has a subcritical growth. Ambrosetti and Rabinowitz in [1] proved that (1) has two nontrivial solutions, one positive and another negative. In [10]. Wang have proved that (1) has three nontrivial solutions. More recently, Bartsch, Chang and Wang in [3] have showed that the third solution changes sign and they also proved some additional information about it Morse index. See also Castro, Cossio and Neuberger in [5, 6] and Bartsch and Wang in [2].

We denote by $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \ldots$ the eigenvalues of $(-\Delta, H_0^1)$, where each λ_j occurs in the sequence as often as its multiplicity.

In order to state our results we need some additional conditions that was used by Bartsch and Wang in [2].

(g_3) There exist constants $c > 0$ and $2 < p < 2N/(N-2)$ such that

$$|g'(t)| \leq c(1 + |t|^{p-2}), \quad \forall\, t \in \mathbb{R}.$$

(g_4) There exists $\lambda \in \mathbb{R}$ such that $g'(t) > \lambda$ for all $t \in \mathbb{R}$.

Theorem 1.1 *Suppose that* (g_1)-(g_4) *hold, then problem* (1) *has at least two sign changing solutions.*

In our next theorems we study the asymptotically linear case. We assume that

(g_5) $g'(t) \to \omega \in \mathbb{R}$ as $|t| \to \infty$, and there exists k such that $\lambda_k < \omega < \lambda_{k+1}$.

We note that (g_5) implies (g_3) and (g_4).

Theorem 1.2 *Suppose that* (g_1), (g_2) *and* (g_5) *hold. If* $k > 2$ *then problem* (1) *has two sign changing solutions.*

Since, in the previous theorems, we have a positive and a negative solution we can conclude the following corollary.

Corollary 1.1 *Problem* (1) *has at least four nontrivial solutions, in any of the above situations.*

Theorem 1.3 *Suppose that* (g_1), (g_2) *and* (g_5) *holds. If* $g'(t) \leq \lambda_{k+1}$ *and* $k > 3$, *then problem* (1) *has three sign changing solutions.*

Corollary 1.2 *Problem* (1) *has at least five nontrivial solutions, in the above situation.*

Theorem 1.4 *Suppose that* (g_1), (g_2) *and* (g_5) *hold. If* $g'(t) > g(t)/t$, $k = 4$ *and* $g'(t) \leq \lambda_5$, *then problem* (1) *has at least six nontrivial solutions.*

Now, to study the coercive case, assume that

(g_1') $g(0) = 0$ and $g \in C^1(\mathbb{R}, \mathbb{R})$.

(g_5') $\limsup_{|t|\to\infty} \frac{g(t)}{t} < \lambda_1$.

Theorem 1.5 *Suppose that* (g_1') *and* (g_5') *holds. If* $g'(0) > \lambda_3$ *then* (1) *has at least two sign changing solutions.*

Bartsch and Wang in [2] have proved that problem (1) has one sign changing solution. Assuming that $g'(t) > g(t)/t$, Bartsch, Chang and Wang in [2] have proved that problem (1) has two sign changing solution, in the asymptotically linear case.

2 Critical point theory on ordered Hilbert spaces

In this section we proof an abstract theorem that improves the Theorem 3.6 in [3].

Let H be a Hilbet space and $P_H \subset H$ a closed cone (i.e., $P_E = \overline{P}_E$, $\mathbb{R}^+ P_E \subset P_E$ and $P_E \cap (-P_E) = \{0\}$). Let $X \subset E$ be a Banach space which is densely embedded into E. We set $P := P_E \cap X$ and assume that P has nonempty interior $int(P)$ in X. The elements of $int(P)$ are called positive, those of $-P$ negative and those of $X \setminus P \cup (-P)$ sign changing. We assume that there exists an element $e \in int(P)$ with $\langle u, e\rangle_H > 0$ for all $u \in P \setminus \{0\}$.

Let $\Phi : E \to \mathbb{R}$ be a functional satisfying the hypotheses

(Φ_1) $\Phi \in C^2$, $\Phi(0) = 0$ and Φ satisfies the Palais-Smale condition. Any critical point of Φ lies in X.

(Φ_2) The gradient of Φ is of the form $\nabla\Phi = Id - K$ with $K : H \to H$ a compact operator. In addition, $K(X) \subset X$ K restricted to X is of class C^1 and strong order preserving (i.e., $u - v \in P \Rightarrow K(u) - K(v) \in int(P)$).

(Φ_3) For a critical point u_0 of Φ any eigenvalue of the derivative $DK(u_0)$ lies in X, the largest eigenvalue of $DK(u_0)$ is simple and its eigenspace is spanned by a positive eigenvector.

(Φ_4) One of the following holds:

(i) Φ is bounded below.

(ii) For every $u \in E \setminus \{0\}$ we have $\Phi(tu) \to -\infty$ as $t \to \infty$. There exists $a < 0$ such that $\Phi(u) \leq a$ implies $\Phi'(u)u < 0$.

(iii) There exists a compact self-adjoint linear operator A_H, defined in H, such that $\nabla\Phi(u) = u - A_H u + o(||u||_H)$ as $||u||_H \to \infty$. All eigenvectors of A_H lies in X, the largest eigenvalue is simple and its eigenspace is spanned by a positive eigenvector $v_\infty \in int(P)$ such that $\langle u, v_\infty \rangle > 0$ for every $u \in P \setminus \{0\}$. Moreover, the restriction $A := A_E|_X$ is a bounded linear operator in X. And $\dim\ker(Id - A_H) = 0$ (i.e., the infinity is nondegenerated).

Setting $D := P \cup (-P) \setminus \{0\}$ and $\Phi^a = \{u \in X \ ; \ \Phi(u) \leq a\}$. As in [2, 3] we restrict the negative gradient flow of Φ on H to X and obtain a continuous flow φ^t on X. We have, by [2, Lemma 3.1], that

$$\varphi^t(v) \in int(D) = int(P) \cup int(-P), \quad \forall\, v \in D, \ t > 0.$$

This is the main property of φ^t that we need.

If U is a neighborhood of an isolated critical point u_0 with $\Phi(u_0) = c$, then by the excision we have for the critical groups of u_0

$$C_p(\Phi, u_0) = H_p(\Phi^c \cap U, \Phi^c \cap U \setminus \{u_0\}).$$

The next lemma follows as in the proof of the Theorem 3.4 in [3].

Lemma 2.1 *If c is a critical value and $\Phi^{-1}(c) \cap D = \{u_i\}_{i=1}^r$, then, for $\epsilon > 0$ small enough,*

$$H_j(\Phi^{c+\epsilon} \cup D, \Phi^{c-\epsilon} \cup D) = \bigoplus_{i=1}^{r} C_j(\Phi, u_i).$$

Now we state our abstract critical point theorems. Let $m(0)$ the Morse index of 0 and $n(0) = \dim\ker\Phi''(0)$. We define the Morse index $m(\infty)$ of infinity depending on which case in (Φ_4) holds. In the case (i) we set $m(\infty) = 0$, in the case (ii) $m(\infty) = \infty$, and in the case (iii) we set $m(\infty)$ is the number of negative eigenvalues of $Id - A_H$.

Under the above hypotheses, assuming that $m(\infty) \geq 2$ and $m(0) \leq 1$, the Theorem 3.6 in [3] says that Φ has a critical point which changes sign. The next theorem improves this result and implies the Theorems 1.1 and 1.2. The idea of the proof is standard, similar ideas can be found in Corollary 1.1 in [4] and Theorem 2.1 in [9]. Our approach is similar to the proof of Theorem 3.8 in [3].

Theorem 2.1 *Suppose $m(\infty) \geq 3$, $m(0) = 0 = n(0)$ and that all sign changing critical point are isolated. Then Φ has two critical point which changes sign.*

Proof: By the proof of Theorem 3.6 in [3] we have a sign changing critical point u such that $\Phi(u) = c > 0$ and $C_2(\Phi, u) \neq 0$. Indeed, we have that

$$H_2(\Phi^{c+\epsilon} \cup D, \Phi^{c-\epsilon} \cup D) \neq 0, \tag{2}$$

where $\epsilon > 0$ is small enough. Moreover, for $\alpha > 0$ large enough,

$$H_p(X, \Phi^{-\alpha} \cup D) = H_p(X, \Phi^{-\alpha}) = \delta_{p,m_\infty} G, \quad \forall\, p \in \mathbb{N}, \tag{3}$$

(cf. [2] and [7]).

Now consider the following diagram

$$\begin{array}{ccccc} H_3(X, \Phi^{c+\epsilon} \cup D) & \rightarrow & H_2(\Phi^{c+\epsilon} \cup D, \Phi^{-\alpha} \cup D) & \rightarrow & H_2(X, \Phi^{-\alpha} \cup D) \\ & & \downarrow & & \\ & & H_2(\Phi^{c+\epsilon} \cup D, \Phi^{c-\epsilon} \cup D) & & \\ & & \downarrow & & \\ & & H_1(\Phi^{c-\epsilon} \cup D, \Phi^{-\alpha} \cup D) & & \end{array}$$

Since the vertical maps are part of the exact sequence of $(\Phi^{c+\epsilon} \cup D, \Phi^{c-\epsilon} \cup D, \Phi^{-\alpha} \cup D)$ it follows from (2) that $H_2(\Phi^{c+\epsilon} \cup D, \Phi^{-\alpha} \cup D) \neq 0$ or $H_1(\Phi^{c-\epsilon} \cup D, \Phi^{-\alpha} \cup D) \neq 0$. In the first case we deduce that $H_3(X, \Phi^{c+\epsilon} \cup D) \neq 0$ by (3) and because the top row is part of exact sequence of $(X, \Phi^{c+\epsilon} \cup D, \Phi^{-\alpha} \cup D)$.

Case 1: $H_3(X, \Phi^{c+\epsilon} \cup D) \neq 0$.

As in [3] we can show that there exists a critical value $d > c$ with $H_3(\Phi^{d-\epsilon} \cup D, \Phi^{d-\epsilon} \cup D) \neq 0$. By the Lemma 2.1 we have a critical point w in $X \setminus D$ with $C_3(\Phi, w) \neq 0$ and $\Phi(w) = d$. Since $d > c > 0$ we have that $w \neq u$ and w is nontrivial.

Case 2: $H_1(\Phi^{c-\epsilon} \cup D, \Phi^{-\alpha} \cup D) \neq 0$.

Again we can show that there exists a critical value $d < c$ with $H_1(\Phi^{d-\epsilon} \cup D, \Phi^{d-\epsilon} \cup D) \neq 0$. By the Lemma 2.1 we have a critical point w in $X \setminus D$ with $C_1(\Phi, w) \neq 0$ and $\Phi(w) = d$. Since $d < c$ we have that $w \neq u$. As 0 is a nondegenerated local minimum we get $C_p(\Phi, 0) = \delta_{p0}G$, and thus w is nontrivial.

$\square$

The next theorem implies the Theorem 1.5 and improves the existence result in the Theorem 3.4 in [3]. The proof is similar the proof of the previous theorem.

Theorem 2.2 *Suppose $m(\infty) = 0$, $m(0) \geq 3$ and that all sign changing critical point are isolated. Then Φ has two critical point which changes sign.*

Proof: By the proof of Theorem 3.4 in [3] we have a sign changing critical point u such that $\Phi(u) = c < 0$, $C_1(\Phi, u) \neq 0$. Thus

$$H_1(\Phi^{c+\epsilon} \cup D, \Phi^{c-\epsilon} \cup D) \neq 0, \tag{4}$$

where $\epsilon > 0$ is small enough. We have that, for $\alpha > 0$ large enough,

$$H_p(X, \Phi^{-\alpha} \cup D) = H_p(X, \Phi^{-\alpha}) = H_p(X) = \delta_{p,0}G, \quad \forall\, p \in \mathbb{N}. \tag{5}$$

Now consider the following diagram

$$\begin{array}{ccccc} H_2(X, \Phi^{c+\epsilon} \cup D) & \to & H_1(\Phi^{c+\epsilon} \cup D, \Phi^{-\alpha} \cup D) & \to & H_1(X, \Phi^{-\alpha} \cup D) \\ & & \downarrow & & \\ & & H_1(\Phi^{c+\epsilon} \cup D, \Phi^{c-\epsilon} \cup D) & & \\ & & \downarrow & & \\ & & H_0(\Phi^{c-\epsilon} \cup D, \Phi^{-\alpha} \cup D) & & \end{array}$$

Again, we can conclude that $H_0(\Phi^{c-\epsilon} \cup D, \Phi^{-\alpha} \cup D) \neq 0$ or $H_2(X, \Phi^{c+\epsilon} \cup D) \neq 0$. It follows that there exists a critical point $w \in X \setminus D$ that satisfies $w \neq u$ and either $C_0(\Phi, w) \neq 0$ or $C_2(\Phi, w) \neq 0$. Since $m(0) \geq 3$ the Shifting Theorem (see [7, Corollary 5.1, Chapter 1]) implies that $C_p(\Phi, 0) = 0$ for $p = 0, 1, 2$. Thus w is nontrivial.

$\square$

3 Proof of main Theorems

The natural variational setting is to consider

$$\Phi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega G(u) dx, \quad u \in H_0^1(\Omega). \tag{6}$$

To apply the Theorem 2.1 and the theory from [2, 3], we need to modify the functional.

Let be H the Hilbert space $H_0^1(\Omega)$ equipped with the inner product

$$\langle u, v\rangle := \int_\Omega \nabla u \nabla v dx + \lambda \int_\Omega uv dx,$$

where λ is given by (g_4). Setting $f(t) = g(t) + \lambda t$ and $F(t) = \int_0^t f(s)ds$ we can write Φ as

$$\Phi(u) = \frac{1}{2}||u||_\lambda^2 dx - \int_\Omega F(u)dx, \quad u \in H,$$

where

$$||u||_\lambda^2 = \int_\Omega (|\nabla u|^2 + \lambda u^2)dx.$$

The partial ordering on H is given by the closed cone $P_H = \{u \in H\ ;\ u \geq 0 \text{ almost everywhere}\}$. Let X the Banach space $C_0^1(\Omega)$ with the L^∞ norm. We know that X is dense in H and that $P = X \cap P_E$ has nonempty interior $int(P)$. The normalized eigenfunction φ_1, associated to λ_1, satisfies $\varphi_1 \in int(P)$ and $\langle u, \varphi_1\rangle > 0$ for all $u \in P \setminus \{0\}$.

Proof of Theorem 1.1 and 1.2

We just check the hypotheses of Theorem 2.1. By (g_1) we have that $m(0) = 0 = n(0)$. As in [3] we can show that Φ satisfy $(\Phi_1) - (\Phi_3)$ and $(\Phi_4)(ii)$, $(\Phi_4)(iii)$ in the superlinear and asymptotically linear cases, respectively. We observe that the nonresonance assumptions $(\Phi_4)(iii)$ follows from $w \notin \sigma(-\Delta)$. We have that $m(\infty) = \infty$ in the superlinear case, and $m(\infty) \geq 3$ in the asymptotically linear case, since $w > \lambda_3$.

□

Proof of Theorem 1.3

By (3) we can conclude that there exists a sign changing critical point z such that

$$C_k(\Phi, z) \neq 0, \tag{7}$$

see the Theorem 2.3 in [3]. The theorem follows from $k > 3$, the next *Claim* and the previous theorem.

Claim: $C_p(\Phi, z) = \delta_{pk} G$.

Indeed, let φ_j the eigenfunction associated to λ_j. For $j \geq k+1$ we have

$$\begin{aligned}\Phi''(z)(\varphi_j, \varphi_j) &= \int_\Omega |\nabla \varphi_j|^2 - \int_\Omega g'(z)\varphi_j^2 \\ &\geq \lambda_{k+1} \int_\Omega \varphi_j^2 - \int_\Omega g'(z)\varphi_j^2 \\ &= \int_\Omega (\lambda_{k+1} - g'(z))\varphi_j^2 > 0.\end{aligned}$$

Thus we can conclude that $m(z) + \dim\ker \Phi''(z) \leq k$, where $m(z)$ denotes the Morse index of z. By (7) and a corollary of the Shifting Theorem (see [7, Corollary 5.1, Chapter 1]) we can conclude that $m(z) + \dim\ker \Phi''(z) = k$ and $C_p(F, z) = \delta_{pk} G$.

□

Proof of Theorem 1.4

First we observe that $g'(t) > g(t)/t$ implies that the sign changing critical points have Morse index at least 2. Then, by Shifting Theorem, we can conclude that u and w are such that

$$C_p(\Phi, u) = \delta_{p2} G, \quad \text{and} \quad C_p(\Phi, w) = 0 \quad \text{for} \quad p = 1,\ 2.$$

As in the previous theorem we have that $m(w) + \dim\ker \Phi''(w) \leq 4$. Then $\dim\ker \Phi''(w) = k \leq 2$, since $m(w) \geq 2$. It follows from [7, Corollary 5.1, Chapter 1] that

$$C_p(\Phi, w) = 0 \quad \text{for all} \quad p \neq 3.$$

Let $u^\pm$ be the negative and positive critical points of Φ. We know that

$$C_p(\Phi, u^\pm) = \delta_{p1} G.$$

Suppose that Φ has not another critical points than 0, $u^\pm$, u, w, z. Then the Morse inequality read as

$$(-1)^4 = (-1)^4 + \text{rank} C_3(\Phi, w)(-1)^3 + (-1)^2 + (-1)^1 + (-1)^1 + (-1)^0.$$

This is a contradiction, since $C_3(\Phi, w) \neq 0$.

□

Proof of Theorem 1.5

We can suppose that $|g'(t)| < \lambda$ for all $t \in \mathbb{R}$ and some $\lambda > 0$ (see [8, Theorem 8] and [3, Theorem 2.1]). Thus (g_4) is satisfied. In this case Φ satisfy $(\Phi_1) - (\Phi_3)$. And $(\Phi_4)(i)$ follows from (g_5'). Since $g'(0) > \lambda_3$, we have that $m(0) \geq 3$. Applying the Theorem 2.2 we obtain the Theorem 2.1.

□

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