A bootstrap test for the expectation of fuzzy random variables

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Abstract

We consider the problem of testing a single hypothesis about the expectation of a fuzzy random variable. For this purpose we take as test statistic a distance between the sample mean and the mean in the null hypothesis. We show that the test rejecting the null hypothesis for large values of the test statistic is consistent. We also prove that the bootstrap can be employed to consistently approximate the null distribution of the test statistic. Finally, we study the finite sample performance of the proposed approximation and compare it with others by means of a simulation study.

Key words: Fuzzy random variable, expectation of a fuzzy random variable, bootstrap, consistency.

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1 Introduction

The concept of fuzzy random variable (frv) was introduced by Puri and Ralescu (1986) as a way to handle the uncertainty due to both: the randomness in the outcomes of a random experiment and the imprecision of these outcomes (to be rigorous, we must say that several concepts of frv were previously given, as the ones in Kwakernaak (1978) and Hirota(1981), but the concept by Puri

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and Ralescu (1986) has been the more successful approach). Since then, many researchers have tried to give the fuzzy analogue of results on traditional (or classical) random variables (we refer this way to random variables taking values in $\mathbb{R}^d$, for some fixed $d \in \mathbb{N}$) as the strong law of large numbers and the central limit theorem (CLT). This is very important because these results are the base of many inferential procedures for classical random variables, which could be extended to frv.

Nevertheless, neither all results on classical random variables can be extended to the fuzzy context, nor when it is possible the extension is unique. For example, while the normal distribution plays a central role in the theory of classical random variables, the concept of normal frv, introduced by Puri and Ralescu (1985), is of very limited application, since in practice the assumption of normality of a frv (either exact or approximate) is unrealistic in many situations because any two variable values are forced to differ by a translation. An example of the nonuniqueness of extensions of classical results is the different versions of the CLT that there exists in the literature on frv, each depending on the considered distance in the space of fuzzy numbers (see for example, Klement, Puri and Ralescu (1986), Shoumei, Ogura, Proske and Puri (2003), Körner (2000)).

The importance of the CLT in the classical context is that it provides a way to easily estimate the distribution of sample means, and hence it gives a base to make approximate inferences on the expectation of a population, where by approximate we mean that they would be exact if the sample size were "infinity". In the fuzzy context, in addition to the existence of several versions of the CLT, there is another practical disadvantage: the limit law is, in most cases, difficult to handle. Therefore, the CLT for frv does not seem to be a very useful tool to make inferences on the expected value of frv.

Nevertheless, in the classic context there are other ways to consistently approximate the distribution of a statistic that could be extended to the fuzzy context. An example is the bootstrap. Since it was introduced by Efron (1979), there have been a huge literature on the topic (for classical random variables), whose main aim is to show that the bootstrap works in many situations, in the sense that it provides consistent estimates of the distribution of many statistics (see for example the book by Shao and Tu (1995)). The objective of this paper is to show that the bootstrap can be employed to consistently estimate the distribution of a distance between the sample mean of a sample of frv and the expectation of the population generating the sample. This will allow us to give a useful test for testing hypothesis on the expectation of frv.

The paper is organized as follows. Section 2 contains some preliminary results and establishes the notation that will be used along the paper. In Section 3 we study the nonnull asymptotic behaviour of the test proposed by Körner (2000) and discuss some problems related to the asymptotic approximation to the null distribution of the test statistic. In Section 4 we show that the bootstrap can be employed to consistently estimate the null distribution of the test statistic. To study and compare the finite sample performance of both approximations, we have carried out a simulation study. The obtained results are displayed in Section 5. Finally, Section 6 concludes and indicates some possible extensions of the results in this paper.

2 Preliminaries

A fuzzy set of $\mathbb{R}^d$ is a function $u : \mathbb{R}^d \to [0,1]$. The study in this paper is restricted to $\mathcal{F}_c(\mathbb{R}^d)$ the class of normal compact fuzzy sets of $\mathbb{R}^d$, that is, the class of fuzzy sets $u : \mathbb{R}^d \to [0,1]$ satisfying

(i) u is normal, i.e., there exists $x_0 \in \mathbb{R}^d$ such that $u(x_0) = 1$,

(ii) for each $0 \leq \epsilon \leq 1$, the ϵ-level set of u,

$$[u]^\epsilon = \begin{cases} \{x \in \mathbb{R}^d \ / \ u(x) \geq \epsilon\}, & \epsilon \in (0,1], \\ \overline{\{x \in \mathbb{R}^d \ / \ u(x) > 0\}}, & \epsilon = 0, \end{cases}$$

is convex and compact.

A linear structure in $\mathcal{F}_c(\mathbb{R}^d)$ is defined via the following operations

$$(u+v)(x) = \sup_{y \in \mathbb{R}^d} \min\{u(y), v(x-y)\}, \ \ (\lambda u)(x) = \begin{cases} u(x\lambda^{-1}) & \text{if } \lambda \neq 0, \\ \chi_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases}$$

where $u, v \in \mathcal{F}_c(\mathbb{R}^d)$, $\lambda \in \mathbb{R}$ and χ_A denotes the characteristic function of $A \subseteq \mathbb{R}^d$. Note that $[u+v]^\epsilon = [u]^\epsilon + [v]^\epsilon$ and $[\lambda u]^\epsilon = \lambda [u]^\epsilon$, $\forall \ u, v \in \mathcal{F}_c(\mathbb{R}^d)$, $\forall \ \epsilon \in [0,1]$, $\forall \ \lambda \in \mathbb{R}$.

For any $u \in \mathcal{F}_c(\mathbb{R}^d)$, the support function of u, $s_u(\cdot,\cdot) : S^{d-1} \times [0,1] \to \mathbb{R}$, is defined by

$$s_u(y,\epsilon) = \sup\{\langle y, a\rangle, \ a \in [u]^\epsilon, \ y \in S^{d-1}\},$$

where $\langle \cdot,\cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$, $S^{d-1} = \{y \in \mathbb{R}^d \ / \ \|y\| = 1\}$ and $\|.\|$ is the Euclidean norm. The mapping $u \mapsto s_u$ is an isomorphism of $\mathcal{F}_c(\mathbb{R}^d)$ onto the cone of continuous functions $C(S^{d-1} \times [0,1])$, satisfying

$$s_{\lambda u + \mu v} = \lambda s_u + \mu s_v, \quad u, v \in \mathcal{F}_c(\mathbb{R}^d), \ \lambda, \mu \geq 0.$$

We can endow $\mathcal{F}_c(\mathbb{R}^d)$ with several metrics. In this work we consider the following

$$D_2(u,v) = \left\{ d \int_0^1 \int_{S^{d-1}} |s_u(y,\epsilon) - s_v(y,\epsilon)|^2 \, \nu(\mathrm{d}y)\mathrm{d}\epsilon \right\}^{1/2},$$

$\forall u, v \in \mathcal{F}_c(\mathbb{R}^d)$, where ν is the normalized Lebesgue measure on S^{d-1}, $\nu(S^{d-1}) = 1$. The corresponding norm is $\|u\|_2 = D_2(u, \chi_{\{0\}})$.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Following Puri and Ralescu (1986), a frv X is a Borel measurable function, $X : \Omega \to (\mathcal{F}_c(\mathbb{R}^d), D_\infty)$, where for $u, v \in \mathcal{F}_c(\mathbb{R}^d)$

$$D_\infty(u,v) = \sup_{\epsilon > 0} \sup_{y \in S^{d-1}} |s_u(y,\epsilon) - s_v(y,\epsilon)| \, .$$

The expectation of X is the fuzzy set $\mathbb{E}X \in \mathcal{F}_c(\mathbb{R}^d)$ whose level sets satisfy

$$[\mathbb{E}X]^\epsilon = \mathbb{E}X_\epsilon$$

where $X_\epsilon : \Omega \to \mathcal{K}_c(\mathbb{R}^d) = \{\text{compact convex subsets of } \mathbb{R}^d\}$ defined by $X_\epsilon(w) = [X(w)]^\epsilon$, is a random compact set and $\mathbb{E}X_\epsilon$ is the Aumann expectation (Aumann (1965)),

$$\mathbb{E}X_\epsilon = \{\mathbb{E}Z \ / \ Z \in L_1(\Omega, \mathcal{A}, P) \text{ and } Z(w) \in X_\epsilon(w) \text{ a.s.}\},$$

where $\mathbb{E}Z$ is the expectation of the random vector Z. Each random vector Z in the definition of $\mathbb{E}X_\epsilon$ is called a selection of X_ϵ.

Klement, Puri and Ralescu (1986) gave the first CLT for frv. Their result has been extended by several authors. An example is the paper by Shoumei, Ogura, Proske and Puri (2003). The limit in these papers is with respect to the metric D_∞. Here we consider the CLT in Körner (2000) that uses the metric D_2. An advantage of this result is that the limit distribution can be characterized by the eigenvalues of an operator.

Theorem 2.1 *Let $X_1, X_2, ..., X_n$ be independent and identically distributed frv with $\mathbb{E}\|X_1\|_2^2 < \infty$. Then*

$$nD_2(\bar{X}_n, \mathbb{E}X_1)^2 \longrightarrow \sum_{k=1}^{\infty} \lambda_k \chi_{1k}^2,$$

weakly, where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i, \chi^2_{11}, \chi^2_{12}, ...$ *are independent chi-square variates with one degree of freedom and the set* $\{\lambda_k\}$ *are the eigenvalues of the covariance operator* C_X *of* X_1 *defined as*

$$fC_Xg = \mathbb{E}f(s_X - \mathbb{E}s_X)g(s_X - \mathbb{E}s_X), \quad f, g \in L_2^*(S^{d-1} \times [0, 1]).$$

3 A test for the expectation of a frv

Let $X_1, X_2, ..., X_n$ be independent and identically distributed (iid) frv with $\mathbb{E}\|X_1\|_2^2 < \infty$ and let $\mu_0 \in \mathcal{F}_c(\mathbb{R}^d)$. Theorem 2.1 characterizes the limit distribution of $nD_2(\bar{X}_n, \mathbb{E}X_1)^2$ and hence it allows us to give an asymptotically α-level test, for some fixed $0 < \alpha < 1$, for testing the null hypothesis

$$H_0 : \mathbb{E}X_1 = \mu_0.$$

Specifically, the test rejects H_0 if

$$T_n = nD_2(\bar{X}_n, \mu_0)^2 > t_{1-\alpha}, \tag{1}$$

where $t_{1-\alpha}$ is the $(1-\alpha)$-quantile of the distribution of the random variable $Y = \sum_{k=1}^\infty \lambda_k \chi^2_{1k}$, with $\{\lambda_k\}$ and $\{\chi^2_{1k}\}$ as defined in Theorem 2.1.

Test (1) was proposed by Körner (2000). Next we study its asymptotic nonnull behaviour.

Theorem 3.1 *Let* $X_1, X_2, ..., X_n$ *be iid frv with* $\mathbb{E}\|X_1\|_2^2 < \infty$ *and mean* $\mathbb{E}X_1 = \mu \in \mathcal{F}_c(\mathbb{R}^d)$. *Then*

$$D_2(\bar{X}_n, \mu_0) \longrightarrow D_2(\mu, \mu_0) \quad a.s. \quad as\ n \to \infty.$$

PROOF. Since $s_{\bar{X}_n}(y, \epsilon) = \frac{1}{n}\sum_{i=1}^n s_{X_i}(y, \epsilon)$, we have that

$$D_2(\bar{X}_n, \mu_0)^2 = \frac{d}{n^2}\sum_{i,j=1}^n h(X_i, X_j)$$

with

$$h(u, v) = \int_0^1 \int_{S^{d-1}} g(u; y, \epsilon)g(v; y, \epsilon)\nu(\mathrm{d}y)\mathrm{d}\epsilon,$$

$$g(u; y, \epsilon) = s_u(y, \epsilon) - s_{\mu_0}(y, \epsilon).$$

Hence

$$D_2(\bar{X}_n, \mu_0)^2 = \frac{d}{n^2} \sum_{i \neq j} h(X_i, X_j) + \frac{d}{n^2} \sum_{i=1}^{n} h(X_i, X_i).$$

Now, since $\mathbb{E}h(X_1, X_2) = D_2(\mu, \mu_0)^2$ and $\mathbb{E}h(X_1, X_1) = \mathbb{E}\|X_1\|_2^2 < \infty$, the result follows from the last equality, Theorem 5.4.A in Serfling (1980) and the SLLN. □

As an immediate consequence of Theorem 3.1 we have the following.

Corollary 3.2 *Let $X_1, X_2, ..., X_n$ be iid frv with $\mathbb{E}\|X_1\|_2^2 < \infty$ and mean $\mathbb{E}X_1 = \mu \neq \mu_0$. Then*

$$T_n \longrightarrow \infty \quad a.s. \quad as\ n \to \infty.$$

From Corollary 3.2 it follows that any test rejecting H_0 for large values of T_n is strongly consistent against any fixed alternative. In particular, $P(T_n > t_{1-\alpha}) \to 1$ when H_0 is not true.

A problem with test (1) is the computation of the critical point $t_{1-\alpha}$, or equivalently, the calculus of the asymptotic p-value $p_{asym} = P(Y \geq t_{obs})$, where t_{obs} is the observed value of the test statistic T_n. Any of these tasks requires: first, to obtain the set of eigenvalues $\{\lambda_k\}$ of the covariance operator C_X; and then to approximate the distribution of $Y = \sum_{k=1}^{\infty} \lambda_k \chi_{1k}^2$, since the distribution of Y is only known for special cases, that is, for especial sequences of eigenvalues. Some authors have proposed several procedures to approximate the distribution of Y (see for example Rao and Scott (1981)).

The big problem with test (1) is the computation of the set of eigenvalues $\{\lambda_k\}$. This is very difficult for two main reasons: one, because to obtain the eigenvalues of an operator is not an easy question; the other reason is that, in most cases, the operator C_X is unknown, since it depends on some population parameters and the only information that we usually have on the population is the random sample $X_1, X_2, ..., X_n$. To appreciate these facts, we next consider frv taking values in a commonly used subset of $\mathcal{F}_c(\mathbb{R})$: the class of LR-fuzzy numbers, that will be denoted by $\mathcal{F}_{LR}(\mathbb{R})$.

Fuzzy sets in $\mathcal{F}_{LR}(\mathbb{R})$, that we denote $u = \{m, l, r\}_{LR}$, have the form

$$\{m, l, r\}_{LR} = m - lu_L + ru_R,$$

where u_L and u_R are fuzzy numbers with ϵ-level sets $[u_L]^\epsilon = [0, L^{-1}(\epsilon)]$ and $[u_R]^\epsilon = [0, R^{-1}(\epsilon)]$, $\epsilon \in (0, 1]$, $L, R : [0, \infty) \to [0, 1]$ being fixed left continuous and non-increasing functions with $L(0) = R(0) = 1$. The functions L and R are called the left and right shape functions, m the modal point and $l, r \geq 0$ are the left and right spreads, respectively, of the LR-fuzzy number. The ϵ-level sets of an LR-fuzzy number u are

$$[u]^\epsilon = [m - lL^{-1}(\epsilon),\, m + rR^{-1}(\epsilon)], \quad \epsilon \in (0, 1].$$

Its support function is

$$s_u(u, \epsilon) = \begin{cases} -m + lL^{-1}(\epsilon) & \text{if } u = -1, \\ m + rR^{-1}(\epsilon) & \text{if } u = 1. \end{cases}$$

The D_2 distance between two LR-fuzzy numbers $u_a = \{m_a, l_a, r_a\}_{LR}$ and $u_b = \{m_b, l_b, r_b\}_{LR}$ is

$$\begin{aligned} D_2(u_a, u_b)^2 = {} & (m_a - m_b)^2 + R_2(r_a - r_b)^2 + L_2(l_a - l_b)^2 \\ & + 2(m_a - m_b)\left[R_1(r_a - r_b) - L_1(l_a - l_b)\right], \end{aligned}$$

where

$$L_1 = \frac{1}{2}\int_0^1 L^{-1}(x)\mathrm{d}x, \quad L_2 = \frac{1}{2}\int_0^1 L^{-1}(x)^2\mathrm{d}x$$

and R_1, R_2 are similarly defined.

Let m, l, r be three random variables with $P(l \geq 0) = P(r \geq 0) = 1$, a random LR-fuzzy number is defined by $X = \{m, l, r\}_{LR}$. To ensure that $\mathbb{E}\|X\|_2^2 < \infty$, the random variables m, l, r must have finite second order moment, that is, $\mathbb{E}m^2$, $\mathbb{E}l^2$, $\mathbb{E}r^2 < \infty$, and L_2, $R_2 < \infty$. Next Theorem, due to Körner (2000), gives a way to calculate the eigenvalues of the covariance operator of a random LR-fuzzy number.

Theorem 3.3 *Let $X = \{m, l, r\}_{LR}$ be a random LR-fuzzy number with $\mathbb{E}\|X\|_2^2 < \infty$. Then the eigenvalues of the covariance operator C_X are equal to the eigenvalues of the matrix*

$$K_X = \begin{pmatrix} C_{mm} - L_1C_{lm} + R_1C_{rm} & L_2C_{lm} - L_1C_{mm} & R_1C_{mm} + R_2C_{rm} \\ C_{lm} - L_1C_{ll} + R_1C_{rl} & L_2C_{ll} - L_1C_{lm} & R_1C_{lm} + R_2C_{rl} \\ C_{rm} - L_1C_{rl} + R_1C_{rr} & L_2C_{rl} - L_1C_{rm} & R_1C_{rm} + R_2C_{rr} \end{pmatrix},$$

where for $z, y \in \{m, l, r\} : C_{zy} = \mathbb{E}(z - \mathbb{E}z)(y - \mathbb{E}y)$.

From Theorems 2.1 and 3.3, to apply test (1) to testing H_0 for random LR-fuzzy numbers we have to know the variances and covariances of the random variables m, l, r, which is not very usual in most practical situations. To calculate the critical point $t_{1-\alpha}$ or the asymptotic p-value $p_{asym} = P(Y \geq t_{obs})$ we may proceed as follows: first, estimate K_X from the data; second, approximate the distribution of Y by that of $\hat{Y} = \sum_{k=1}^{3} \hat{\lambda}_k \chi^2_{1k}$, where $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$ are the eigenvalues of $\hat{K}_X$, the estimator of K_X; and third, approximate the distribution of $\hat{Y}$ by some method as (Rao and Scott (1981)):

(a) $\hat{\lambda}\chi^2_r$, where $r = \text{rank}\{\hat{K}_X\}$ and $\hat{\lambda} = \sum_{k=1}^{3} \hat{\lambda}_k / r$.
(b) $\hat{\lambda}_{(1)}\chi^2_r$, where r is as before and $\hat{\lambda}_{(1)} \geq \hat{\lambda}_{(2)} \geq \hat{\lambda}_{(3)}$.
(c) $\hat{\lambda}(1 + \theta^2)\chi^2_\nu$ where r, $\hat{\lambda}$ are as before, $\nu = r/(1 + \theta^2)$ and $\theta^2 = \sum_{i=1}^{r} (\hat{\lambda}_{(i)} - \hat{\lambda})^2 / r\hat{\lambda}$.

4 A bootstrap test for the expectation of a frv

In Section 3 we have shown that any test rejecting H_0 for large values of T_n has nice properties. To decide when rejecting H_0, we need to know the null distribution of T_n which, in general, is quite difficult and so, in most cases, one have to approximate it. A way to do this is by considering its limiting null distribution, but we have seen that this is not operational.

When H_0 is true, T_n has as weak limit a linear combination of independent chi-square variates (Theorem 2.1), usually called a ω^2-distribution. In the classical context, that is, for random variables, the bootstrap has become a very useful tool for estimating the distribution of statistics converging in law to a ω^2-distribution (see for example Babu (1984), Arcones and Giné (1992), Jiménez-Gamero, Muñoz-García and Pino-Mejías (2003)). In this section we show that the bootstrap also works in the context considered in this paper, that is, it can be employed to consistently approximate the null distribution of T_n.

Let $X_1, X_2, ..., X_n$ be iid frv. Given $X_1, X_2, ..., X_n$, let $\mathbf{X}^* = (X_1^*, X_2^*, ..., X_n^*)$ be a bootstrap sample, that is, $X_1^*, X_2^*, ..., X_n^*$ are iid frv such that

$$P_*(X_1^* = X_j) = \frac{1}{n}, \quad j = 1, 2, ..., n,$$

where P_* denotes the bootstrap probability law, that is, the conditional probability, given the original sample $X_1, X_2, ..., X_n$. Let $T_n^* = nD_2(\bar{X}_n^*, \bar{X}_n)^2$, with $\bar{X}_n^* = \frac{1}{n}\sum_{i=1}^{n} X_i^*$. We call the conditional distribution of T_n^*, given $X_1, X_2, ...,$ X_n, the null bootstrap distribution of T_n. Next theorem shows that the null bootstrap distribution of T_n, $P_*(T_n^* \leq x)$, consistently estimates the null distribution of T_n, $P_{H_0}(T_n \leq x)$, where $P_{H_0}(.)$ denotes the probability when H_0 is true.

Theorem 4.1 *Let $X_1, X_2, ..., X_n$ be iid frv with $\mathbb{E}\|X_1\|_2^4 < \infty$. Then*

$$\sup_{x \in \mathbb{R}} |P_*(T_n^* \leq x) - P_{H_0}(T_n \leq x)| = o(1).$$

PROOF. From the proof of Theorem 3.1,

$$D_2(\bar{X}_n, \mu_0)^2 = \frac{d}{n^2} \sum_{i,j=1}^{n} h(X_i, X_j).$$

For fixed $u \in \mathcal{F}_c(\mathbb{R}^d)$, we have that

$$\mathbb{E}_{H_0} h(X_1, u) = \mathbb{E}_{H_0} h(X_1, X_2) = 0,$$

where $\mathbb{E}_{H_0}$ denotes the expectation when H_0 is true. Therefore, $D_2(\bar{X}_n, \mu_0)^2$ is a degenerate degree 2 V-statistic. The hypothesis $\mathbb{E}\|X_1\|_2^4 < \infty$ implies that $\mathbb{E}h(X_1, X_1)^2 < \infty$ and $\mathbb{E}h(X_1, X_2)^2 < \infty$. Hence, the result follows from Theorem 3.5 in Arcones and Giné (1992) $\square$.

Remark 4.2 *The asymptotic approximation in Theorem 2.1 to the null distribution on T_n assumes that $\mathbb{E}\|X_1\|_2^2 < \infty$, while Theorem 4.1 requires $\mathbb{E}\|X_1\|_2^4 < \infty$ for the consistency of the null bootstrap distribution of T_n. This second stronger assumption cannot be dropped, since if we only assume that $\mathbb{E}\|X_1\|_2^2 < \infty$ the bootstrap can fail. Bickel and Freedman (1981) have given a counterexample showing this fact.*

Remark 4.3 *It is important to note that Theorem 4.1 holds whether or nor H_0 is true. If H_0 is indeed true, Theorem 4.1 implies that the null bootstrap distribution of T_n converges to its null distribution almost surely. If H_0 is not true, then Theorem 4.1 says that the null bootstrap distribution of T_n converges to the distribution of $nD_2(\bar{X}_n, \mathbb{E}X_1)^2$.*

Remark 4.4 *Note that T_n^* is not an "exact bootstrap copy" of T_n, as it is $\tilde{T}_n^* = nD_2(\bar{X}_n^*, \mu_0)^2$. The reason to take T_n^* instead of $\tilde{T}_n^*$ is that the bootstrap distribution of $\tilde{T}_n^*$ does not consistently estimate the null distribution of T_n, since $\tilde{T}_n^*$ is not a is a degenerate degree 2 V-statistic as it is T_n when the null hypothesis is true.*

For testing H_0 we consider the following test: reject H_0 if

$$T_n > t_{1-\alpha}^*, \tag{2}$$

where $t^*_{1-\alpha}$ is the $(1-\alpha)$-quantile of the null bootstrap distribution of T_n, or equivalently, if the bootstrap p-value, $p_{boot} = P_*(T_n^* \geq t_{obs})$, is less or equal than α. From Theorem 4.1, the test (2) has asymptotically level α. Also, as a consequence of Corollary 3.2, the test (2) is consistent against any fixed alternative.

In general, the values $t^*_{1-\alpha}$ and p_{boot} are not known, but they can be approximated by simulation: first, generate B bootstrap samples, $\mathbf{X}^{*1}, \mathbf{X}^{*2}, ..., \mathbf{X}^{*B}$, and then approximate the null bootstrap distribution of T_n by the empirical distribution function of $T_n^{*1}, T_n^{*2}, ..., T_n^{*B}$, where $T_n^{*b} = nD_2(\bar{X}_n^{*b}, \bar{X}_n)^2$ and $\bar{X}_n^{*b}$ is the sample mean of $\mathbf{X}^{*b}$, $1 \leq b \leq B$.

5 Simulations

The results in Theorems 2.1 and 4.1 give two ways to approximate the null distribution of T_n for testing H_0: the asymptotic null distribution and the bootstrap null distribution, respectively. Both approximations are consistent, that is, they work when the sample size is large. To compare the finite sample performance of these approximations we have carried out a simulation experiment. In this experiment we have considered random LR-fuzzy numbers because, as we saw in Section 3, in this particular case it is possible to estimate the eigenvalues of the covariance operator. We have taken $L(x) = R(x) = \max\{0, 1-x\}$, m, l, r independent such that $m \sim N(0,1)$ and r and l are uniformly distributed on the interval $(0,1)$, $r, l \sim U(0,1)$. We have generated 10000 random samples of size $n = 10$. For each sample we have calculated the observed value of the test statistic T_n for testing the null hypothesis

$$H_{01} : \mathbb{E}\{m, l, r\}_{LR} = \{0, 1/2, 1/2\}_{LR}.$$

To approximate the p-value of the observed test statistic we have first estimated K_X (see Theorem 3.3) by estimating the variances and covariances of m, l, r by its sample variances and covariances, respectively; next we have obtained the eigenvalues of the resulting estimator of K_X, $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$; and finally, to approximate the p-value of the observed test statistic we have used the three approximations to the distribution of $\hat{Y} = \sum_{k=1}^{3} \hat{\lambda}_k \chi^2_{1k}$ at the end of Section 3. This way we have obtained three estimates of the asymptotic p-value. We refer to them as asymp (a), asymp (b) and asymp (c), respectively. To see the goodness of these approximations we have calculated the "exact" asymptotic p-value. We refer to it as asymp. To do this, we have first calculated the eigenvalues of K_X, $\lambda_1, \lambda_2, \lambda_3$; next, we have generated 10000 values of three independent χ^2_1 distributions: $\chi^2_{11}(h), \chi^2_{12}(h), \chi^2_{13}(h)$; and finally, we have taken the asymptotic p-value as the relative frequency of $\omega(h) = \sum_{k=1}^{3} \lambda_k \chi^2_{1k}(h)$,

$1 \leq h \leq 10000$, greater or equal than the observed value of the test statistic. We have also estimated the p-value by means of the bootstrap. We refer to it as boot. To approximate the ideal bootstrap estimator we have generated $B = 1000$ bootstrap samples.

		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	mean	var	KS
n=10	asymp	0.0112	0.0480	0.0945	0.502	0.083	0.009
	asymp (a)	0.0801	0.1395	0.1832	0.578	0.137	0.224
	asymp (b)	0.0086	0.0223	0.0365	0.786	0.072	0.448
	asymp (c)	0.0587	0.1143	0.1626	0.535	0.113	0.115
	boot	0.0410	0.0959	0.1550	0.464	0.091	0.063
n=20	asymp	0.0093	0.0496	0.0960	0.497	0.082	0.014
	asymp (a)	0.0678	0.1231	0.1672	0.590	0.130	0.224
	asymp (b)	0.0035	0.0106	0.0231	0.800	0.064	0.468
	asymp (c)	0.0453	0.0970	0.1438	0.545	0.108	0.119
	boot	0.0246	0.0756	0.1279	0.478	0.086	0.032
n=30	asymp	0.0108	0.0462	0.0894	0.502	0.082	0.014
	asymp (a)	0.0571	0.1136	0.1557	0.599	0.129	0.232
	asymp (b)	0.0023	0.0096	0.0190	0.808	0.060	0.474
	asymp (c)	0.0362	0.0861	0.1330	0.555	0.107	0.125
	boot	0.0181	0.0630	0.1175	0.489	0.085	0.018
n=40	asymp	0.0099	0.0484	0.0965	0.509	0.083	0.024
	asymp (a)	0.0568	0.1112	0.1530	0.609	0.129	0.250
	asymp (b)	0.0014	0.0071	0.0161	0.813	0.059	0.486
	asymp (c)	0.0337	0.0839	0.1300	0.564	0.107	0.145
	boot	0.0154	0.0605	0.1112	0.499	0.086	0.014

Table 1: Simulated size, mean, variance and KS statistic for testing H_{01}.

We have considered three nominal sizes: 0.01, 0.05 and 0.10. For each approximation, we have calculated the relative number of p-values less or equal than the nominal size, that is, the simulated size. We have also obtained the mean and the variance of the p-values. Since H_{01} is true, if the considered approximations were exact, then the calculated p-values would be a random sample from a uniform distribution on the interval (0,1). So, as a global measure of the performance of the approximations we have calculated the Kolmogorov-Smirnov test statistic of uniformity (KS) for each set of 10000 p-values obtained in each approximation. We have repeated the above experiment for $n = 20, 30, 40, 50, 100, 200$. The obtained results are displayed in Table 1.

		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	mean	var	KS
n=50	asymp	0.0112	0.0504	0.0912	0.504	0.081	0.013
	asymp (a)	0.0543	0.1094	0.1489	0.604	0.127	0.237
	asymp (b)	0.0014	0.0075	0.0167	0.813	0.058	0.481
	asymp (c)	0.0336	0.0845	0.1265	0.559	0.105	0.130
	boot	0.0156	0.0581	0.1089	0.494	0.083	0.012
n=100	asymp	0.0104	0.0464	0.0901	0.507	0.083	0.015
	asymp (a)	0.0489	0.1022	0.1445	0.609	0.126	0.242
	asymp (b)	0.0014	0.0063	0.0133	0.818	0.055	0.485
	asymp (c)	0.0273	0.0746	0.1209	0.565	0.105	0.139
	boot	0.0125	0.0507	0.1006	0.502	0.084	0.006
n=200	asymp	0.0111	0.0498	0.0943	0.501	0.083	0.008
	asymp (a)	0.0545	0.1055	0.1493	0.602	0.127	0.233
	asymp (b)	0.0010	0.0065	0.0149	0.813	0.057	0.479
	asymp (c)	0.0306	0.0788	0.1234	0.559	0.106	0.130
	boot	0.0133	0.0557	0.1037	0.497	0.084	0.008

Table 1. (continuation).

Looking at Table 1 we see that the bootstrap approximation behaves much better than approximations asymp (a), asymp (b) and asymp (c). The size of the asymptotic approximation (a) is always (for all the considered sample sizes) much larger than the nominal level; for the size of the asymptotic approximation (b) it happens the opposite: it is always much smaller than the nominal level; the mean and the variance of these approximations does not match the theoretical values of a uniform distribution on (0,1), 0.5 and 1/12=0.083, respectively; the KS statistics for testing uniformity of these approximations are quite large, specially for the asymptotic approximation (b). From among the considered asymptotic approximations, the asymptotic approximation (c) is the one having sizes, mean and variance closest to the ideal values. Nevertheless, its behaviour is poorer than the bootstrap, which works very well even for small sample sizes. What it is really happening is that the considered approximations to the true asymptotic estimator are rather little accurate. Looking at Table 1 we see that the true asymptotic approximation fits quite satisfactorily the actual null distribution of the test statistic, even for $n = 10$. For $n \geq 30$ the true asymptotic (which, in most practical cases, can never be calculated) and the bootstrap (which can be always easily calculated) estimates has an almost identical performance. The above assertions can better seen by looking at Figure 1, which displays the histograms of the p-values for testing H_{01} with sample size $n = 50$.

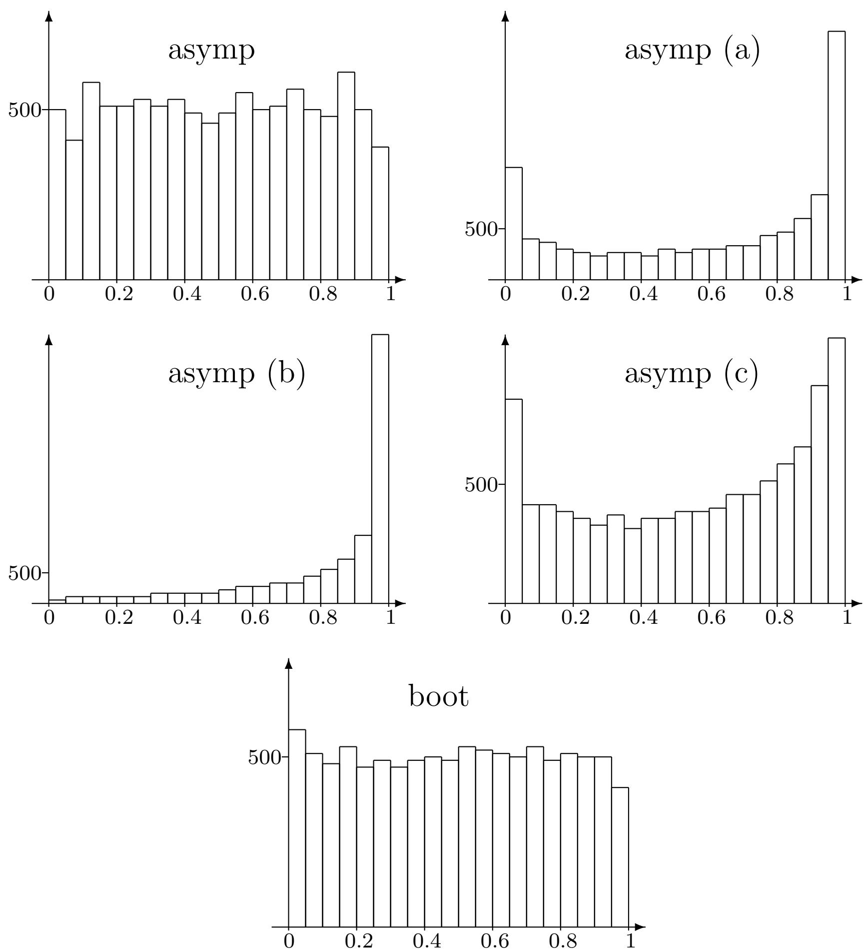

Figure 1: histograms of the p-values for testing H_{01} with $n = 50$.

We have repeated the above experiment with R, L and m, as before and l, r iid with common distribution χ^2_1, χ^2_3 and χ^2_6, for testing the null hypotheses $H_{02}: \mathbb{E}\{m, l, r\}_{LR} = \{0, 1, 1\}_{LR}$, $H_{03}: \mathbb{E}\{m, l, r\}_{LR} = \{0, 3, 3\}_{LR}$ and $H_{04}: \mathbb{E}\{m, l, r\}_{LR} = \{0, 6, 6\}_{LR}$, respectively. The obtained results are quite similar for these three hypotheses, so we only display those for testing H_{04}, which are exhibited in Table 2. Figure 2 displays the histograms of the p-values for testing H_{04} with sample size $n = 50$.

		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	mean	var	KS
n=10	asymp	0.0115	0.0502	0.0964	0.505	0.083	0.015
	asymp (a)	0.0418	0.1015	0.1490	0.512	0.104	0.064
	asymp (b)	0.0037	0.0157	0.0291	0.708	0.069	0.312
	asymp (c)	0.0191	0.0612	0.1134	0.432	0.065	0.118
	boot	0.0267	0.0891	0.1516	0.446	0.087	0.079
n=20	asymp	0.0124	0.0503	0.0959	0.504	0.082	0.012
	asymp (a)	0.0339	0.0891	0.1389	0.521	0.101	0.068
	asymp (b)	0.0028	0.0133	0.0304	0.699	0.069	0.298
	asymp (c)	0.0140	0.0525	0.1030	0.452	0.067	0.086
	boot	0.0206	0.0715	0.1292	0.471	0.085	0.045
n=30	asymp	0.0101	0.0507	0.0958	0.503	0.083	0.012
	asymp (a)	0.0301	0.0825	0.1328	0.523	0.100	0.069
	asymp (b)	0.0022	0.0119	0.0269	0.694	0.069	0.285
	asymp (c)	0.0111	0.0495	0.0976	0.461	0.069	0.078
	boot	0.0155	0.0655	0.1213	0.480	0.086	0.032
n=40	asymp	0.0102	0.0497	0.0958	0.500	0.082	0.010
	asymp (a)	0.0290	0.0773	0.1279	0.523	0.098	0.065
	asymp (b)	0.0015	0.0116	0.0266	0.691	0.068	0.281
	asymp (c)	0.0104	0.0466	0.0911	0.463	0.068	0.077
	boot	0.0147	0.0589	0.1130	0.482	0.084	0.027
n=50	asymp	0.0114	0.0567	0.1002	0.497	0.083	0.010
	asymp (a)	0.0294	0.0801	0.1329	0.521	0.100	0.067
	asymp (b)	0.0021	0.0129	0.0282	0.684	0.070	0.266
	asymp (c)	0.0104	0.0459	0.0975	0.463	0.070	0.075
	boot	0.0148	0.0616	0.1181	0.482	0.085	0.032

Table 2: Simulated size, mean, variance and KS statistic for testing H_{04}.

		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	mean	var	KS
n=100	asymp	0.0109	0.0498	0.0977	0.503	0.083	0.010
	asymp (a)	0.0248	0.0759	0.1238	0.528	0.099	0.070
	asymp (b)	0.0012	0.0110	0.0266	0.683	0.070	0.267
	asymp (c)	0.0078	0.0451	0.0921	0.473	0.071	0.060
	boot	0.0130	0.0579	0.1071	0.493	0.085	0.014
n=200	asymp	0.0101	0.0507	0.0965	0.501	0.082	0.0121
	asymp (a)	0.0230	0.0720	0.1228	0.528	0.097	0.068
	asymp (b)	0.0015	0.0119	0.0260	0.679	0.070	0.261
	asymp (c)	0.0079	0.0417	0.0912	0.474	0.070	0.063
	boot	0.0120	0.0527	0.1037	0.495	0.083	0.009

Table 2: (continuation).

Looking at Table 2 we see that in this case the asymptotic approximations (a), (b) and (c) work better than before. Nevertheless, we again conclude that these approximations are far from the true asymptotic approximation and that the bootstrap behaves much better than the asymptotic approximations (a), (b) and (c).

6 Concluding remarks

In Section 3 we have shown that the test rejecting the null hypothesis for large values of the D_2 distance between the sample mean and mean in the null hypothesis is consistent against any fixed alternative. To obtain the critical region of the test me must approximate the null distribution of the considered test statistic. We have identified some operational problems associated with the asymptotic approximation in Theorem 2.1. In Section 4 we have proved that the bootstrap can be used to consistently estimate the null distribution of the test statistic. In contrast to the asymptotic approximation, the bootstrap approximation can be easily implemented. Moreover, the finite sample results displayed in Section 5, show that the behaviour of the considered estimates of the asymptotic approximation is poorer than that of the bootstrap, which works quite well even for small sample sizes.

Although there is a large number of published papers on the application of the bootstrap methodology for classical random variables, the literature on the application of the bootstrap in the fuzzy context is rather scarce. A work related with ours is that by Montenegro, Colubi, Casals and Gil (2004).

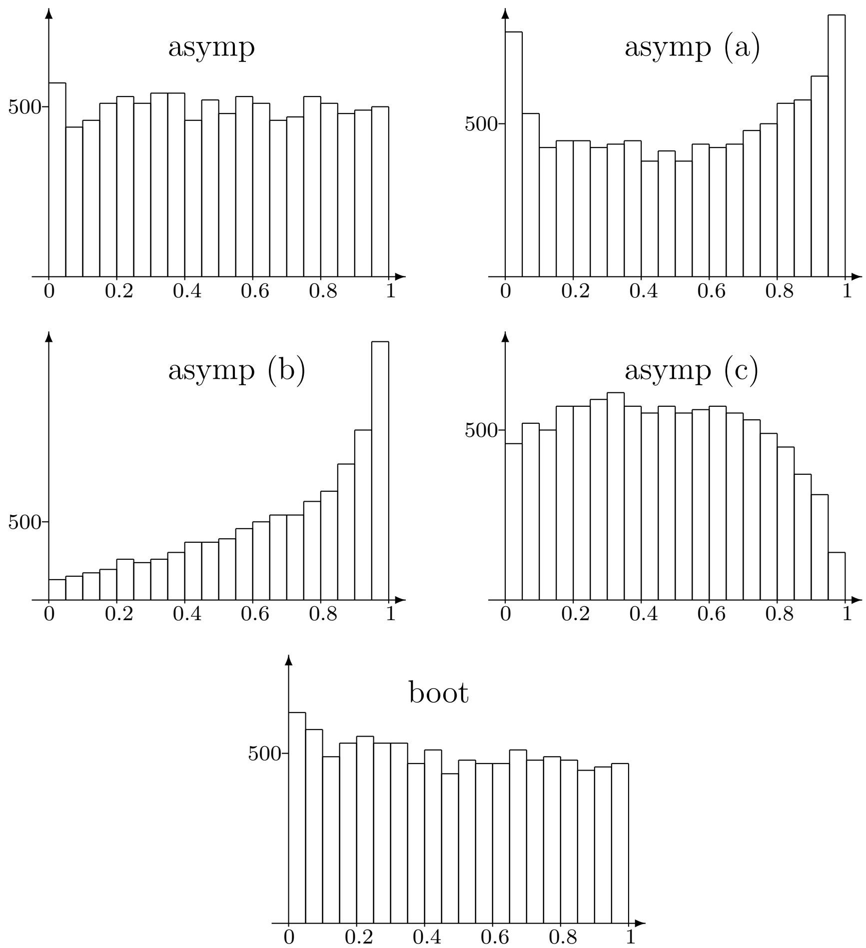

Figure 2: histograms of the p-values for testing H_{04} with $n = 50$.

In the classical context, the bootstrap has become a very powerful tool for estimating the sampling distribution of a statistic and its characteristics. We think that the development of an adequate bootstrap theory in the fuzzy context would be very profitable because, as we noted before, in this context the asymptotic approximations are, in most cases, difficult to handle and hence they are useless to make inferences.

In particular and in relation to the work in this article (a single null hypothesis on the mean of a population), possible extensions are the study of a composite null hypothesis on the mean of a population and the comparison of the mean of two populations. Also, the distance considered in this article and the one in Montenegro, Colubi, Casals and Gil (2004) are particular cases on a more general distance, as it was observed by Näther (2001). It would be very interesting to study if the results in this paper are valid for the general distance in

Näther (2001).

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