A BOUNDARY CUSP SINGULAR POINT AND REVERSIBLE VECTOR FIELDS ON THE PLANE

RONALDO A. GARCIA AND MARCO A. TEIXEIRA

Abstract. In this paper we describe the bifurcation diagram of a boundary cusp of codimension three, i.e, a Bogdanov-Takens singular point in the boundary of the semi plane $\{(x, y) \in \mathbb{R}^2 : \ x \geq 0\}$. This study is applied to the analysis of the behavior of singularity of the germ of vector field $X_0(x, y) = (y, 2x(x^4 + x^2y))$ in the class of reversible vector fields. We classify the generic three parameter families of reversible vector fields $X_{a,b,c}$ with $(a, b, c) \in (\mathbb{R}^3, 0)$ and $X_{a,b,c} = X_0$.

1. Introduction

In qualitative theory of differential equations it is usual to classify phase portraits up to orbital C^0-equivalence, [AL], [P], [PM]; in the theory of singularities of vector fields one studies the classification problem by listing the codimension k (for some finite k) singularities and presenting their normal forms and versal unfolding, [Ar], [CLW].

In this paper we follow such approach to first study a codimension three boundary singularity of a vector field defined in the semi plane $\mathbb{M} = \{(x, y) \in \mathbb{R}^2 : \ x \geq 0\}$ and we obtain the bifurcation diagram. Next, as a straightforward application (see [T3]), we are able to pull back this bifurcation diagram into the context of a special kind of dynamical systems which form our main interest: the class of reversible systems.

We recall that generic bifurcations occurring in one and two parameter families of vector fields in surfaces with or without boundary have been extensively studied, see for example, [Ar], [S3] and [T3] and references there in.

Our starting point is to consider germs of vector fields at $0 \in \partial\mathbb{M} = \{(x, y) \in \mathbb{R}^2 : \ x = 0\}$ with nilpotent $1 - jet\ y\partial/\partial x$ and $2 - jet\ C^\infty$ conjugated to:

$$X_0(x, y) = y\frac{\partial}{\partial x} + (x^2 + \alpha xy)\frac{\partial}{\partial y}, \ \ \alpha = \pm 1 \tag{1}$$

The following 3-parameter family of vector fields defined in $\mathbb{M}$

The authors were partially supported by CNPq and FUNAPE/UFG. This work was done under the project PRONEX/FINEP/MCT - Conv. 76.97.1080.00 - Teoria Qualitativa das Equações Diferenciais Ordinárias.

$$y\frac{\partial}{\partial x}+(x^2+ax+c+\alpha y(x+b))\frac{\partial}{\partial y},\quad \alpha=\pm 1 \tag{2}$$

generically unfolds X_0; it will be called a *quadratic typical family.*

For a given vector field X defined on $\mathbb{R}^2$ one distinguishes two types of symmetry properties. A diffeomorphism φ on $\mathbb{R}^2$ is a symmetry if $\varphi_* \circ X = X\circ\varphi$ and a reversing symmetry if $\varphi_*\circ X = -X\circ\varphi$. The classical equivariant theory comes from the first definition. In the last case when φ is an involution, X is called $\varphi - reversible$.

Let φ be a germ of an involution in $(\mathbb{R}^2,0)$ $\varphi^2=id$, with $DetD\varphi(0)=-1$. We say that a (germ of a) vector field (in $\mathbb{R}^2,0$) is $\varphi - reversible$ if

$$\varphi_*\circ X=-X\circ\varphi.$$

We fix a coordinate system in $(\mathbb{R}^2,0)$ such that the involution takes the form $\varphi_0(x,y)=(-x,y)$. This is not a restriction, since our treatment is local and it is very known that any involution φ with $Det\varphi'(0)=-1$ can be brought to this canonical form by a smooth change of coordinates.

We recall that any orbit of a (φ_0-) reversible X is called symmetric provided that it meets the set $S=Fix\varphi_0$; otherwise it is an asymmetric orbit. Observe that asymmetric orbits appear in pairs and any symmetric, periodic orbit or critical point, cannot be an attractor or a repeller. For more on reversible systems see [L] and [T3].

In this paper we also want to study the following singularity.

$$Y_0(x,y)=(y,2x(x^4+\alpha x^2y)),\quad \alpha=\pm 1.$$

We observe that in the reversible context this non-hyperbolic saddle singularity has codimension 3, whereas in the world of smooth vector fields it has codimension greater than 5.

A 3-parameter family of vector fields defined in $\mathbb{R}^2$ expressed by

$$y\frac{\partial}{\partial x}+2x[(x^4+\alpha x^2y)+ax^2+by+c]\frac{\partial}{\partial y}. \tag{3}$$

generically unfolds Y_0; it will be called a *reversible polynomial typical family.*

In [T3] all the symmetric singularities of codimension 0, 1 and 2 are classified. It is presented a technique which enables to classify in a simple manner those singularities and it consists to make a special change of coordinates around the point and address the analysis to the study of the contact between a general system and $\partial\mathbb{M}$.

Let us fix some notations:

$\Gamma^+(\mathbb{M})$ = smooth vector fields defined on the region $\mathbb{M}=\{(x,y)\in\mathbb{R}^2 : x\geq 0\}$;

$\Gamma^R(\mathbb{R}^2)$ = smooth reversible vector fields defined on $(\mathbb{R}^2,0)$.

All the above sets are endowed with the C^∞−topology, [H].

First of all we recall the C^0-equivalence in $\Gamma^+(\mathbb{M})$.

We say that two vector fields X an Y in $\Gamma^+(\mathbb{M})$ are C^0 -equivalent if there is an $\partial\mathbb{M}-$preserving homeomorphism h on $\mathbb{M}$, which sends orbits of X in orbits of Y.

Two families of vector fields X_λ and Y_ν defined in a domain $\mathbb{M}$ are fiber-C^0- equivalent if there exists a homeomorphism $\nu = \phi(\lambda)$ between the parameter spaces and a family of homeomorphisms $H_\lambda : \mathbb{M} \to \mathbb{M}$ such that H_λ is a C^0- equivalence between X_λ and $Y_{\phi(\lambda)}$.

We state now the main results of this work.

Theorem 1. The bifurcation diagram of the quadratic typical family given by equation 2 with $\alpha = \pm 1$ is as shown in Fig. 1 below. This diagram is a topological cone with vertex at 0 and there are eleven distinct phase portraits which are structurally stable (open regions of the diagram of bifurcation) and thirteen points of codimension two. The lines in the diagram are the bifurcations of codimension one. The phase portrait in each open connected region of the diagram is as shown in Fig. 2, 3, 4, 5, 6, 7 and 9.

Remark 1. *In Fig. 1 is represented the restriction of the bifurcation set to a hemisphere of $a^2 + b^2 + c^2 = 1$. Observe that outside a topological disk the family has no singular point. In fact there exists a unique non transverse contact between the vector field and the boundary which is an internal quadratic tangency.*

Theorem 2. Let X be a vector field in $\Gamma^+(\mathbb{M})$ such that $j^2X(0) = (y, x^2 + \alpha xy)$. Then the universal unfolding of X is fiber-C^0- equivalent to the quadratic typical family given by equation 2.

Theorem 3. The bifurcation diagram of the reversible typical family given by equation 3 in the parameter space (a, b, c) is homeomorphic to that of Fig. 1.

The paper is organized as follows. In Section 2 are described the bifurcation diagram, in a general context, of codimension two singularities. In Section 3 all bifurcations of codimension two that appear in the quadratic typical family are analyzed. In Section 4 an outline of proof of theorem 2 is presented. In Section 5 the proof of theorem 1 is presented. In Section 6 the proof of theorem 3 is given. Section 7 is dedicated to concluding remarks.

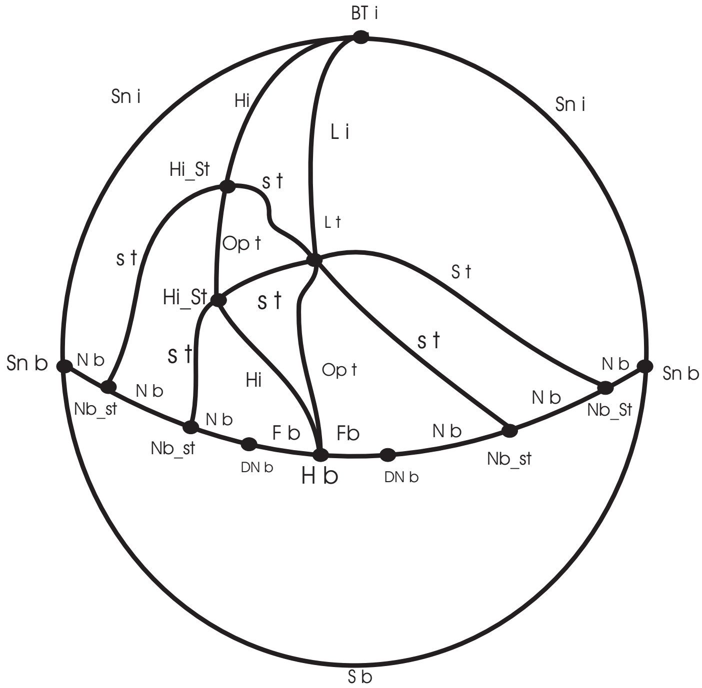
2. Bifurcations of codimension two in generic families

In this section we describe some phenomena of codimension two in generic families of vector fields defined in a boundary region in the plane.

Also the two parameter family of vector fields obtained from a translation followed by a rotation will be analyzed.

2.1. **Rotated and translated vector fields.**

Let $\mathbb{M} = \{(x, y) \in \mathbb{R}^2 : \ x \geq 0\}$.



BT i - Bogdanov-Takens bifurcation in the interior
Hi- interior Hopf point of cod. 1
Sni - interior saddle node of cod. 1
Sb - saddle in the boundary
Fb- focus in the boundary
Nb- node in the boundary
Hb- Hopf in the boundary
DN b- degenerate node in the boundary
Li - interior loop of cod. 1
Lp - loop tangent to the boundary
Op t - hyperbolic periodic orbit tangent to the boundary
St- separatrix of saddle tangent to the boundary
Snb- saddle node in the boundary
Nb_st - node in the boundary and connection of strong separatrix of node and saddle separatrix
Hi_st - interior Hopf and separatrix tangent to the boundary

Figure 1. Bifurcation diagram of the quadratic typical family

Consider a vector field $X = (P, Q) \in \Gamma^+(\mathbb{M})$ and the following two parameter family, [D], [S3],

$$X_{\omega,\nu} = R_\omega(X + \nu\frac{\partial}{\partial x}) \tag{4}$$

where $\nu \in$ and R_ω stands for the rotation by an angle ω in the plane $\mathbb{R}^2$ with the canonical orientation.

Therefore the rotated vector field $R_\omega(X) = P_\omega \partial/\partial x + Q_\omega \partial/\partial y$ is given by

$$\begin{pmatrix} P_\omega \\ Q_\omega \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}$$

2.2. **Bifurcations of codimension 1 and 2 and auxiliary results.**

As usual, we first detect the bifurcations that occur generically for families of vector fields in $\Gamma^+(\mathbb{M})$. We distinguish some of them:

(1) The tangency points between X and $\partial\mathbb{M}$ and their contact order.
(2) The singular points, contained or not in $\partial\mathbb{M}$.
(3) The relative position between the invariant manifolds and $\partial\mathbb{M}$ (the contact order).
(4) The periodic orbits.
(5) Periodic orbits tangent to $\partial\mathbb{M}$.
(6) Saddle separatrices tangent to $\partial\mathbb{M}$.
(7) Homoclinic orbits tangent to $\partial\mathbb{M}$.
(8) Orbits tangent to $\partial\mathbb{M}$ in more than one point.
(9) Heteroclinic orbits tangent to $\partial\mathbb{M}$.

A crucial part of this work is an attempt to locate the above objects in the bifurcation diagram of the *quadratic typical family.*

In this section we will describe various phenomena of codimension two bifurcations necessary to obtain the bifurcation diagram of the *quadratic typical family.*

The next proposition is proved in [T2] and [T3] and we include it here for completeness.

Proposition 1. *Consider a generic two parameter family of vector fields $X_{\epsilon,\delta}$ in $\Gamma^+(\mathbb{M})$ such that X_0 has a codimension two singularity, which is a saddle-node contained in the boundary with transversal invariant manifolds (center and stable), with one hyperbolic sector and two parallel regions in $\mathbb{M}$. Then the bifurcation diagram of the saddle node in the boundary is as shown in Fig. 2.*

Remark 2. *A vector field X having a saddle node in the boundary as above stated is C^0-equivalent to $X_1(x,y) = (-(x-y)^2 + y, y)$. The local central manifold W^c is contained in the line $y = -x$. The two-parameter family $X_{\varepsilon,\delta}(x,y) = X_1(x,y) - (\delta - \varepsilon x)\frac{\partial}{\partial x}$ is an universal unfolding of X_1. A direct analysis shows that the codimension one bifurcations are: hyperbolic node in the boundary (for $\delta = 0$) and a non-degenerate saddle-node in the interior*

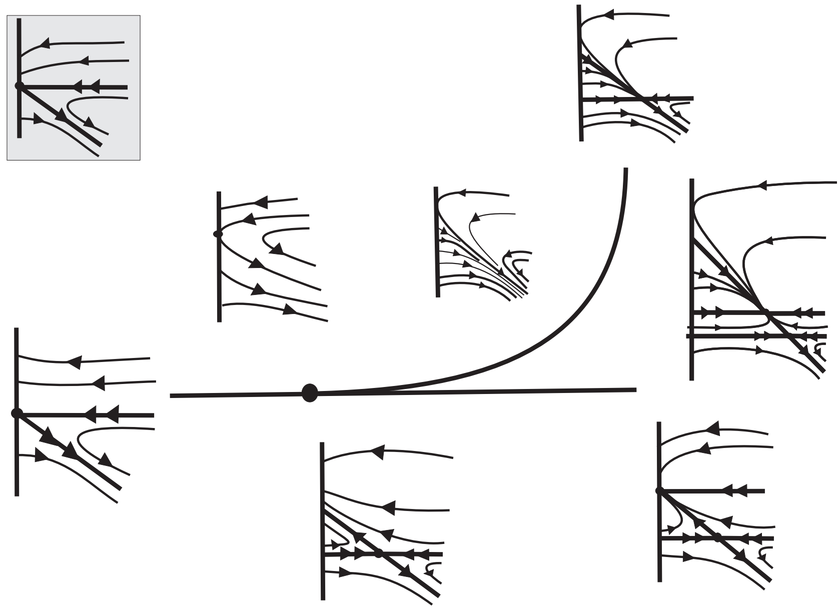


FIGURE 2. Internal Saddle Node in the Boundary

of the region M expressed by the relations $4\delta = \varepsilon^2$ and $\varepsilon > 0$. Observe that all stable and unstable manifolds are transverse to the boundary.

Proposition 2. *Consider a two parameter family of vector fields in $\Gamma^+(\mathbb{M})$ having a (generic) Hopf singular point contained in the boundary. The bifurcation diagram of the Hopf singular point (codimension two) contained in the boundary is as in the Fig. 3 and is diffeomorphic to the bifurcation diagram of the geometric family $X_{\omega,\nu}$ given by equation 4.*

Proof. Consider a two parameter family $X_{\epsilon,\delta}$ of smooth vector field on the plane having a singular point of Hopf type in the boundary of the semi plane $\{(x,y) \in \mathbb{R}^2 : \ x \geq 0\}$. The equations $X_{\epsilon,\delta}(0,y) = 0, \ divX_{\epsilon,\delta}(0,y) \neq 0$ and $X_{\epsilon,\delta}(x,y) = 0, \quad divX_{\epsilon,\delta}(x,y) = 0, \quad x > 0$ define two transversal curves. The first one describes the location of the hyperbolic singularities in the boundary and the other is formed by the Hopf singular points in the interior of $\mathbb{M}$. After a diffeomorphic change of coordinates we can consider these curves given respectively by $\delta = 0$ and $\epsilon = 0, \ \delta > 0$. Recall that the hyperbolic periodic orbit that emerges from the Hopf singular point has radius approximately equal to $\sqrt{\epsilon}$ from the singular point $(\delta, 0)$. So it follows that the curve of periodic orbits tangent to the boundary is well approximated by $\epsilon = \delta^2, \ \delta > 0$. See [T3] where this singularity was also analyzed. $\square$

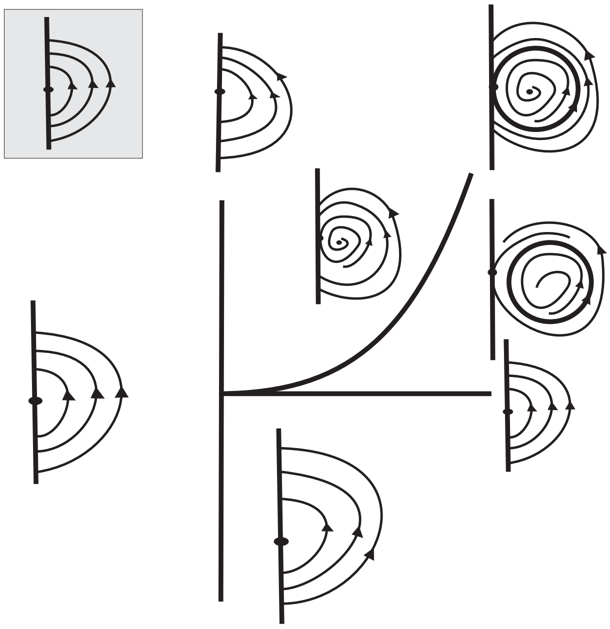


FIGURE 3. Hopf Singular Point in the Boundary

A boundary degenerate node is a hyperbolic node contained in the boundary with first jet conjugated to $x\partial/\partial x+(x+y)\partial/\partial y$ (repeller) or $-x\partial/\partial x+-(x+y)\partial/\partial y$ (attractor).

Proposition 3. *Consider a two parameter family of vector fields $X_{\epsilon,\delta}$ in $\Gamma^+(\mathbb{M})$ such that X_0 has a degenerate attracting node in the boundary. Suppose that this family is generic. Then the bifurcation diagram of the boundary degenerate node is as in the Fig. 4 and it is diffeomorphic to the bifurcation diagram of the geometric family $X_{\omega,\nu}$ given by equation 4.*

Proof. Consider a two parameter family of vector fields $X_{\epsilon,\delta}$ having a hyperbolic degenerate node in the boundary. The equations $X_{\epsilon,\delta}(0,y) = 0, \quad divX_{\epsilon,\delta}(0,y) - 4(detDX_{\epsilon,\delta}(0,y))^2 \neq 0$ define a regular curve in the parameter space having hyperbolic singularities in the boundary of focus and node (two different eigenvalues) types. We observe that the strong separatrix of the node in the boundary is transversal to the boundary and must be considered as a separatrix since it is a topological invariant for C^0-equivalence. Moreover, outside this curve each system has one or zero singular point (focus or node) in the interior of $\mathbb{M}$. See [T2] where this situation was also analyzed. □

Proposition 4. *Consider a two parameter family of vector fields $X_{\epsilon,\delta}$ in $\Gamma^+(\mathbb{M})$ such that X_0 has a nodal singularity in the boundary in such a way that there exists a connection between its strong separatrix and a separatrix of*

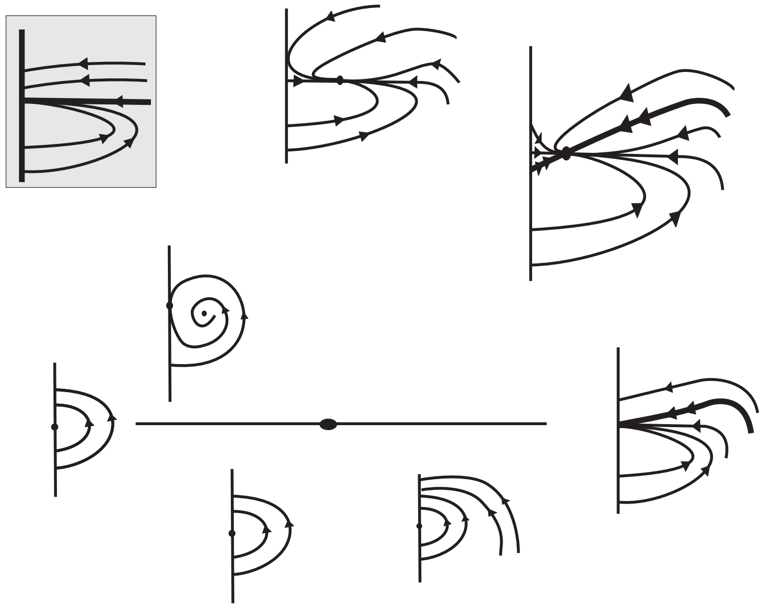


FIGURE 4. Bifurcation diagram of a Degenerate Hyperbolic Node in the Boundary

a hyperbolic saddle contained in the interior of $\mathbb{M}$. *Suppose that this family is generic. Then the bifurcation diagram of this codimension two phenomena is as in the Fig. 5 and is diffeomorphic to the bifurcation diagram of the geometric family* $X_{\omega,\nu}$ *given by equation 4.*

Proof. Let $X_{\epsilon,\delta}$ a two parameter family of vector fields having a hyperbolic node in the boundary and a separatrix connection. We have two phenomena of codimension one bifurcations, a hyperbolic node in the boundary and a quadratic tangency of separatrix of saddle with the boundary. In this case the family $X_{\epsilon,\delta}$ is equivalent to the family $R_\omega(X_0 + \nu\frac{\partial}{\partial x})$. In the parameter space (ω, ν) the lines of codimension one bifurcations are given by $\nu.\omega = 0$. The parameter of translation ν has the effect to move the hyperbolic node from the boundary and ω is appropriated to disconnect transversally the separatrix connections, see [S1] and [S3]. □

Proposition 5. *Consider a two parameter family of vector fields* $X_{\epsilon,\delta}$ *in* $\Gamma^+(\mathbb{M})$ *such that* X_0 *has a Hopf singular point (codimension one) in the interior of* $\mathbb{M}$ *and a quadratic tangency with the boundary of an unstable separatrix of a hyperbolic saddle contained in the interior of* $\mathbb{M}$. *Suppose that this family is generic, i.e, the two codimension one bifurcations unfold independently. Then the bifurcation diagram of this codimension two phenomena is as shown the Fig. 6 and is diffeomorphic to the bifurcation diagram of the geometric family* $X_{\omega,\nu}$ *given by equation 4.*

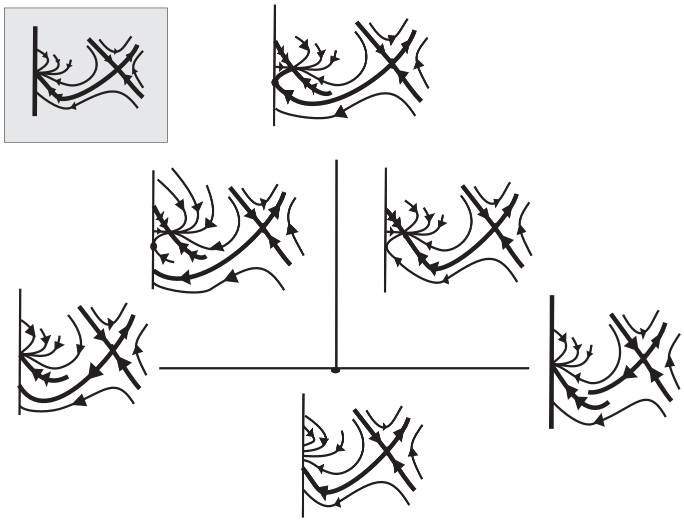


FIGURE 5. Bifurcation diagram of a node in the boundary with a connection between the strong separatrix and a separatrix of saddle

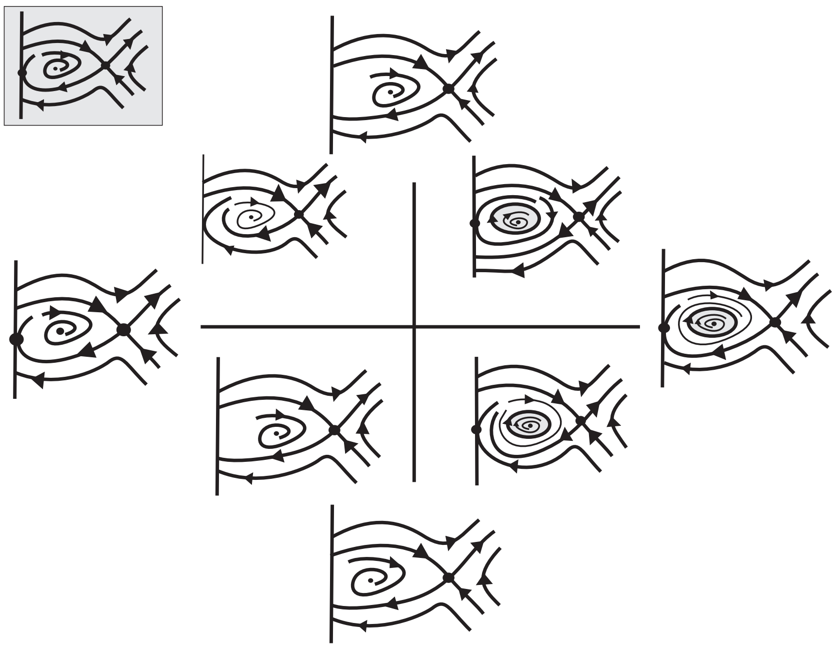


FIGURE 6. Bifurcation diagram of a Hopf Point and Tangent Separatrix of Saddle

Proof. Under the hypothesis of the proposition the bifurcations of codimension one present on a universal unfolding of X_0 are: the Hopf point in the interior, and so by the Implicit Function Theorem is a regular curve and a quadratic tangency of separatrix of saddle which also is a regular curve transversal to the first one.

For the geometric family $R_\omega(X_0 + \nu\frac{\partial}{\partial x})$ in the curve $\omega = 0$ we have the codimension one Hopf bifurcation and the tangency of separatrix of saddle with the boundary happens along a regular curve $\nu = \nu(\omega)$, $\nu'(0) \neq 0$. $\square$

Proposition 6. *Consider a two parameter family of vector fields $X_{\epsilon,\delta}$ in $\Gamma^+(\mathbb{M})$such that X_0 has an attracting loop of a hyperbolic saddle p_0 point which is tangent quadratically to the boundary. Suppose also that the loop unfolds generically. The bifurcation diagram of the tangent loop is as in the Fig. 7.*

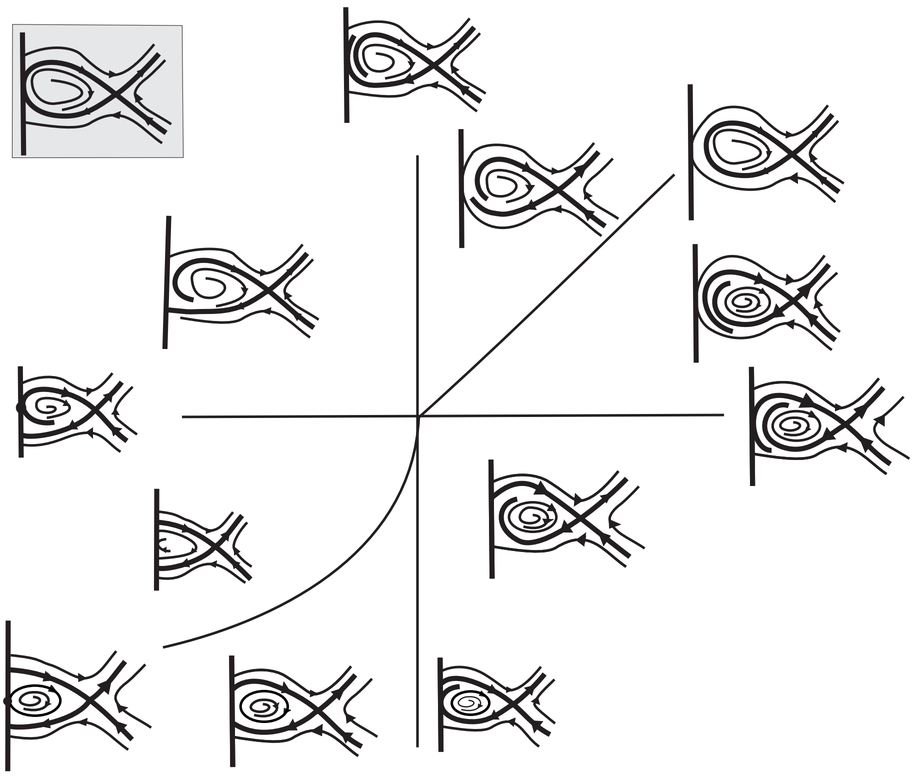


Figure 7. Tangent Loop of a Hyperbolic Saddle Point

Proof. To fix thoughts, we point out that in the Fig. 8 below, the singular point is a hyperbolic saddle contained in the region $\mathbb{M}$ with eigenvalues $\lambda_s < 0$, $\lambda_u > 0$ and $\lambda = \mid \lambda_u/\lambda_s \mid > 1$.

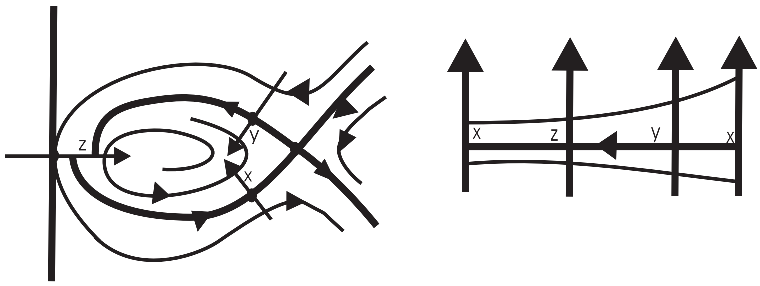


FIGURE 8. Unfold of an attractor tangent loop and the Return map

We begin by considering the transversal sections Σ_1, Σ_2 and Σ_3 with coordinates x, y and z respectively. Here (x, y) are the linearized coordinates of the saddle.

The transition maps $z_1(x) = ax + h.o.t$ and $z_2(y) = y + h.o.t$ associated to a 2-parameter family of vector fields $X_{\epsilon,\delta}$ have, in view of $C^\infty-$Preparation Theorem, [R], the following developments.

$$z_1 = (ax + \epsilon)\varphi(x, \epsilon, \delta)$$
$$z_2 = (y + \delta)\psi(y, \epsilon, \delta), \qquad \varphi(0) = \psi(0) = 1$$

Therefore a periodic orbit in the region $\mathbb{M}$ is defined by the equation

$$(ax + \epsilon)\varphi(x, \epsilon, \delta) = (y + \delta)\psi(y, \epsilon, \delta) > 0$$
$$y = x^\lambda, \quad \lambda > 1$$

A loop in the region $\mathbb{M}$ is defined by the equation

$$\epsilon\varphi = \delta\psi$$

A tangent periodic orbit is defined by

$$(ax + \epsilon)\varphi(x, \epsilon, \delta) = (y + \delta)\psi(y, \epsilon, \delta) = 0$$
$$y = x^\lambda, \quad \lambda > 1$$

A tangency point between an invariant separatrix and the boundary, (say $W^s(p) \cap \partial\mathbb{M}$) is defined by

$$(ax + \epsilon)\varphi(x, \epsilon, \delta) = 0$$
$$x = 0$$

Analogously, a tangency point between $W^u(p)$ and the boundary $\partial\mathbb{M}$, is defined by

$$(y + \delta)\psi(y, \epsilon, \delta) = 0$$
$$y = 0$$

Solving the equations above we get the desired proof and the bifurcation diagram undergoes as sketched in Fig. 7. □

The following result is classical, see for example [Ar], [B1], [B2], [CLW], [DRS], [DRSZ], [RW], [R] and [Ta].

Proposition 7. *Consider a two parameter family of vector fields in* $\Gamma^+(\mathbb{M})$ *having a Takens-Bogdanov singular point (codimension two) in the interior of* $\mathbb{M}$*. Suppose that this family is generic. Then the bifurcation diagram of this codimension two singular point is as shown the Fig. 9.*

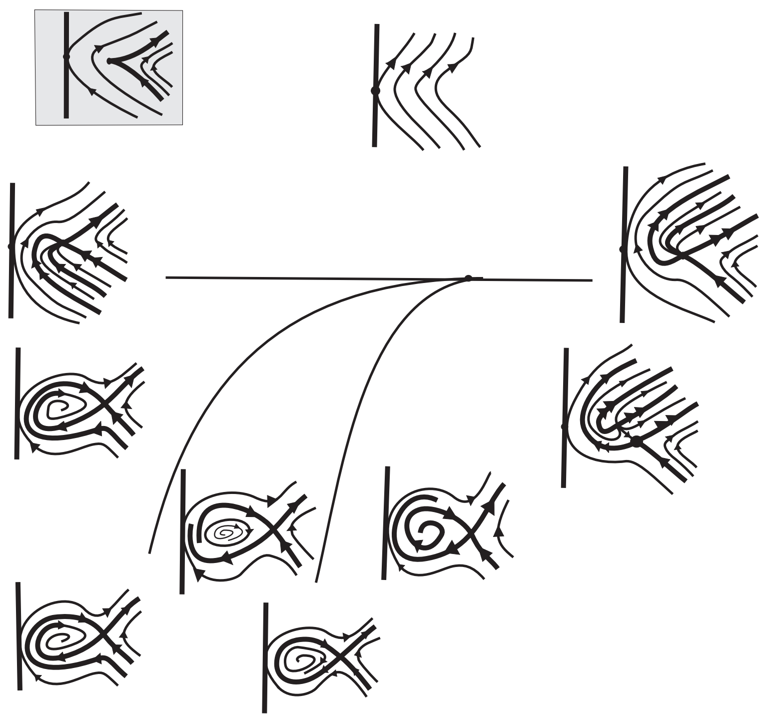


FIGURE 9. Bogdanov-Takens bifurcation in the interior: $\alpha = -1$

Remark 3. *In the Fig. 9 we recall that when* $\alpha = 1$ *the loop is a repeller, while for* $\alpha = -1$ *the loop is an attractor. But the bifurcation diagrams are diffeomorphic.*

3. SINGULARITIES OF CODIMENSION ONE AND TWO IN THE QUADRATIC TYPICAL FAMILY

In the *quadratic typical family* the following bifurcations occur:

(i) **Codimension one bifurcations**:

(1) tangency point (internal and external) between a periodic hyperbolic orbit and the boundary;
(2) tangency point between a separatrix of a hyperbolic saddle and the boundary;

(3) tangency point between a strong separatrix of a hyperbolic node and the boundary;
(4) a hyperbolic saddle or node contained in the boundary with transverse separatrices (recall that the nodal strong invariant manifold is also a separatrix and it must be distinguished);
(5) a saddle-node disjoint from the boundary;
(6) a Hopf singular point disjoint from the boundary;
(7) a simple loop of a hyperbolic singular point disjoint from the boundary;

These kind of bifurcations are well known and we refer to the following basic references [AL], [Ar], [DRS], [DRSZ], [R], [S1], [S2], [S3], [T1], [T3].

(ii) **Codimension two bifurcations**:
(1) Bogdanov-Takens cusp point disjoint from the boundary, [Ta], [B1], [B2], [RW];
(2) saddle-node contained in the boundary with transverse invariant manifolds (center and stable or unstable), [T2];
(3) Hopf singular point contained in the boundary, [T3];
(4) quadratic tangency between a loop of a hyperbolic saddle point and the boundary;
(5) Hopf singular point in the interior and a separatrix of saddle tangent to the boundary;
(6) hyperbolic node in the boundary with connection of separatrices (strong separatrix of the node with a separatrix of saddle);
(7) degenerate node in the boundary, [T2].

For sake of completeness and also to show that the *quadratic typical family* satisfies the required properties in the sequel we state several propositions. Their proofs follow from the propositions of section 2. In fact, we intend to verify that in each case the *quadratic typical family* is generic as required in section 2. They will be omitted.

Proposition 8. *In the bifurcation diagram of the quadratic typical family there are two points of codimension two (saddle-node contained in the boundary with transversal invariant manifolds (center and stable or unstable)). The bifurcation diagram of these points are homeomorphic to that of Fig. 2 of section 2.*

Proposition 9. *In the bifurcation diagram of the quadratic typical family there is a point of codimension two (Hopf singularity in the boundary). The bifurcation diagram of this point is homeomorphic to that of Fig. 3 of section 2.*

Proposition 10. *In the bifurcation diagram of the quadratic typical family there are two points of codimension two (boundary degenerate node). The bifurcation diagram of both points are homeomorphic to that of Fig. 4 of section 2.*

Proposition 11. *In the bifurcation diagram of the quadratic typical family there are four points of codimension two (node in the boundary with a connection between the strong separatrix and a separatrix of saddle). The bifurcation diagram of all points are homeomorphic to that of Fig. 5 of section 2.*

Proposition 12. *In the bifurcation diagram of the quadratic typical family there are two points of codimension two (Hopf singularity in the interior and separatrix of saddle tangent to the boundary). The bifurcation diagram of both points are homeomorphic to that of Fig. 6 of section 2.*

Proposition 13. *Consider the quadratic typical family $X_\lambda = X_{a,b,c}$. Then there exists a parameter $\lambda_0 = (a_0, b_0, c_0)$ such that X_{λ_0} has a loop of a hyperbolic saddle p_0 point tangent quadratically to the boundary which unfolds generically. Therefore bifurcation diagram of the tangent loop of the quadratic typical family is homeomorphic to that of Fig. 7 of section 2.*

Proposition 14. *In the bifurcation diagram of the quadratic typical family given by equation 2 there is a codimension two cusp singularity(Takens-Bogdanov type) disjoint from the boundary. The bifurcation diagram of this point is homeomorphic to that of Fig. 9 of section 2.*

Proof. The quadratic typical family has, for $a^2 - 4c = 0$, and $a < 0$, a cusp in the interior of $\mathbb{M}$ which satisfy the hypothesis of Takens-Bogdanov Theorem for the two parameter family $X_{a,b,a^2/4}$. □

4. Proof of Theorem 2

It follows from [RW] that the versal unfolding of the cusp (Bogdanov-Takens) singularity of codimension two can be expressed by the following polynomial normal form, up to $C^0 - fiber$ equivalence.

$$X_{b,c} = y\frac{\partial}{\partial x} + (x^2 + c + \alpha y(x+b))\frac{\partial}{\partial y}. \tag{5}$$

Next we consider the deformation $X_{a,b,c} = X_{b,c} + ax\frac{\partial}{\partial x}$; its main effect consists to provide an universal unfolding of the singularities in the region $\mathbb{M}$.

Observing that $X_{a,b,c}$ considered as a family of smooth vector fields on the plane, $\Xi^\infty(\mathbb{R}^2)$, is induced from $X_{b,c}$ the result follows. It is enough to observe that all new bifurcations arise when one fixes the vector field $X_{b,c}$ and one moves slightly the boundary.

5. Proof of Theorem 1

The bifurcation diagram given in Fig. 1 is the simplest diagram which contains the thirteen codimension two phenomena of bifurcation that occur in the *quadratic typical family* as formulated in propositions 8, 9, 10, 11, 12, 13 and 14.

The main new ingredient here is the presence of the boundary. The tangent loop, the two Hopf point in the interior with separatrix of saddle tangent and the four node in the boundary with separatrix connections can be determined by the Implicit Function Theorem. All the other points of codimension two are determined by algebraic equations.

6. Proof of Theorem 3

A smooth reversible vector field $X \in \Gamma^R(\mathbb{R}^2)$ has the following form:

$$X(x, y) = (p(x^2, y), xq(x^2, y)),$$

with p, q being smooth real functions.

For a given reversible system, in [T3], the following technique was employed: to perform a special change of coordinates around the singularity such that the analysis of the original system can be transferred to study the generic contact between a general system and a smooth curve in $(\mathbb{R}^2, 0)$. Here we follow these ideas.

Consider now the planar fold mapping $h(x, y) = (u, v)$ expressed by $u = x^2$ and $v = y$. We have then $\mathbb{M} = Im(h)$.

A simple calculation yields

$$Y(u, v) = (2\sqrt{u}p(u, v), \sqrt{u}q(u, v))$$

where

$$Y(u, v) = Dh(h^{-1}(u, v).X(h^{-1}(u, v)$$

for $u \geq 0$.

It follows that on the open semi plane $x > 0$ the vector field X is C^0-*equivalent* to $Z = Z(X)$, defined on $u > 0$ where $Z(u, v) = (p(u, v), q(u, v))$. Observe that Z can be smoothly extended to a full neighborhood of 0 in $\mathbb{R}^2$. Due to the symmetric properties of X with respect the line $x = 0$ (the symmetric axis of X) we deduce that the knowledge of the phase portrait of X is directly achieved by the knowledge of the phase portrait of $Z(X)$.

The tangency set T_Z between Z and $\partial\mathbb{M}$ plays a important role in the analysis of phase portrait of X and is represented by $T_Z = \{(x, y); x = 0 \text{ and } Zf = 0\}$ where $f(x, y) = x$.

We say that $p \in \partial\mathbb{M}$ is a generic tangency (or a fold tangency) of Z if $Z(p) \neq 0$, $Zf(p) = 0$ and $Z^2 f(p) \neq 0$. Any tangency point between $Z(X)$ and $u = 0$ corresponds to a singular point of X and vice-versa.

Applying this fold map to the equation below

$$y\frac{\partial}{\partial x} + (x^2 + ax + c + \alpha y(x + b))\frac{\partial}{\partial y}, \quad \alpha = \pm 1 \tag{6}$$

we obtain the three parameter family of reversible systems given by equation 3, called *reversible polynomial typical family.*

In the Fig. 10 below we show a part of the bifurcation diagram of the three family given by equation 3. All the other phase portrait are obtained in the same way, i.e., by symmetry and time-reversibility.

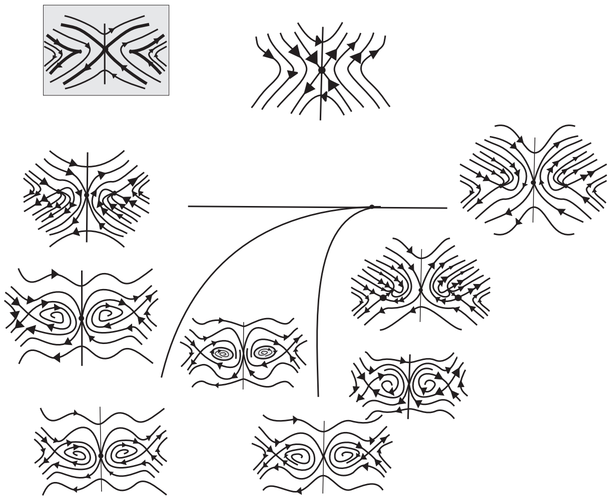


FIGURE 10. Bifurcation diagram of a reversible vector field with cusps of Bogdanov-Takens type

7. Final Comments

As said earlier our main goal was to present an analysis of the versal unfolding of a degenerate saddle in the reversible context. This means that one always wants to respect symmetries and reversibilities that are present in the original system, which brings us, as shown in [T3], a strong link between the theory of vector fields defined in manifolds with boundary and singularity theory. Our main guide in conducting the work was [DRS].

In a forthcoming paper we intend to get similar results for the system $X(x,y) = (y, -2x(x^4 + \alpha x^2 y))$ (degenerate reversible center).

It is worthwhile to mention that the results obtained here can have applications in the theory of stability of discontinuous differential equations, [MS], [ST] and in divergent-free systems in 3D as shown in [BTY].

There is evidence that the transference from the results obtained here in systems defined in manifolds with boundary to reversible systems is straightforward. This analogy is in fact the fundamental point in our approach.

References

[AL] A.A. Andronov, E. A. Leontovich et al., Theory of Bifurcations of Dynamic Systems in the Plane, IPST, Jerusalem, 1966.

[Ar] V. Arnold, Dynamical Systems V, Bifurcation Theory and Catastrophe Theory, Encyclopaedia of Math. Sciences, **5**, Springer Verlag, 1994.

[B1] V. Bogdanov, *Bifurcation of a limit cycle for a family of vector fields on the plane,* Selecta Math. **01**, (1981), pp. 373-388.

[B2] V. Bogdanov, *Versal deformations of a singular point of a vector field on the plane in the case of zero eigenvalues,* Selecta Math. **01**, (1981), pp. 389-421.

[BTY] C. Buzzi, M. A. Teixeira and J. Yang; *Hopf-zero bifurcations of reversible vector fields*, Nonlinearity, **14**, (2001), pp. 623-638

[CLW] S. Chow, C. Li and D. Wang, Normal Forms and Bifurcations of Planar Vector Fields, Cambridge Univ. Press, (1994),

[DRS] F. Dumortier, R. Roussarie, and J. Sotomayor ; *Generic 3-parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part. The cusp case,* Ergodic Theory and Dynamical Systems , (1987), pp. 375-413.

[DRSZ] F. Dumortier, R. Roussarie, J. Sotomayor and H. Zoladek, Bifurcations of planar vector fields: Nilpotent Singularities and Abelian Integrals, Springer Lecture Notes in Math. **1480**, (1991).

[D] G. Duff, *Limit Cycles of Rotated Vector Fields*, Annals of Math. **57** (1953).

[H] M. Hirsh, Differential Topology, Springer Verlag, (1982).

[L] J. Lamb, Reversing Symmetries in Dynamical Systems, Thesis U. of Amsterdam, 1994.

[MS] A. Machado, J. Sotomayor, *Structurally Stable Discontinuous Vector Fields in the Plane,* Qualitative Theory of Dynamical Systems, **03**, (2002), pp. 227-250.

[P] M. Peixoto, *Structural stability on two dimensional manifolds,* Topology, **1**, (1962), pp. 101-120.

[PM] J. Palis, W. de Melo, Geometric Theory of Dynamical Systems, Springer Verlag, (1982).

[R] R. Roussarie, Bifurcation of Planar Vector Fields and Hilbert's Sixteenth Problem, Birkhauser, Berlin, (1998).

[RW] R. Roussarie and F. Wagener, *A study of Takens- Bogdanov Bifurcations,* Resenhas do IME-USP, **02**, (1995), pp. 1–25.

[S1] J. Sotomayor, *La mesure de l'ensemble de bifurcation des champs de vecteurs dans le plan,* C. R. Acad. Sci. Paris, **39**, (1981).

[S2] J. Sotomayor, *Generic one-parameter families of vector fields on 2-manifolds,* Publ. Math. IHES, **43**, (1974), pp. 5–46.

[S3] J. Sotomayor, Curvas definidas por equações diferenciais no plano, IMPA, Rio de Janeiro, (1981).

[ST] J. Sotomayor and M.A. Teixeira, *Regularization of Discontinuous Vector Fields,* International Conference on Differential Equations (Equadiff 95), Word Scientific, (1998).

[Ta] F. Takens, *Applications of Global Analysis, part II,* Comm. of Math, Inst. Rijksuniv. Utrecht, **3**, (1974).

[T1] M. A. Teixeira, *Generic bifurcations in manifolds with boundary*, Journal of Differential Equations **25** (1977) pp. 65–89.

[T2] M.A. Teixeira, *Generic bifurcation of certain singularities,* Bolletino della Unione Matematica Italiana, **16-B**, (1979), 238–254.

[T3] M.A. Teixeira, *Singularities of Reversible Vector Fields,* Physica D, **100**, (1997), pp. 101–118.

Ronaldo Alves Garcia
Instituto de Matemática e Estatística

Universidade Federal de Goiás
CEP 74001-970, Caixa Postal 131
Goiânia, GO, Brazil
E.mail: ragarcia@mat.ufg.br

Marco Antonio Teixeira
Departamento de Matemática -IMEEC
Universidade de Campinas
CEP 13081-970, Caixa Postal 6065
Campinas, S.P., Brazil
E.mail: teixeira@ime.unicamp.br