Continuous-Time Optimization Problems via KT-Invexity

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Abstract

We prove that the notion of KT-invexity for continuous-time nonlinear optimization problem is a necessary and sufficient condition for global optimality of a Karush-Kuhn-Tucker point.

Key words: Continuous-time nonlinear programming, Invexity, KKT-Conditions, Global optimality.

1 Introduction

Consider the continuous-time nonlinear programming problem below.

$$\left.\begin{array}{l} \text{Minimize } \ \phi(x) = \int_0^T f(x(t),t)dt, \\ \\ \text{subject to } g(x(t),t) \leq 0 \text{ a.e. in } [0,T], \\ \qquad\qquad x \in X. \end{array}\right\} \qquad \text{(CNP)}$$

Here X is a nonempty open convex subset of the Banach space $L^n_\infty[0,T]$, $\phi : X \to \mathbb{R}$, $g(x(t),t) = \gamma(x)(t)$ and $f(x(t),t) = \xi(x)(t)$, where $\gamma : X \to \Lambda^m_1[0,T]$ and $\xi : X \to \Lambda^1_1[0,T]$.

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Where $L_\infty^n[0,T]$ denotes the space of all n-dimensional vector valued Lebesgue measurable functions, which are essentially bounded, defined on the compact interval $[0,T] \subset \mathbb{R}$, with norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \max_{1 \le j \le n} \operatorname{ess\,sup}\{|x_j(t)|,\ 0 \le t \le T\},$$

where for each $t \in [0,T]$, $x_j(t)$ is j-th component of $x(t) \in \mathbb{R}^n$; and $\Lambda_1^m[0,T]$ denotes the space of all m-dimensional vector functions which are essentially bounded and Lebesgue measurable, defined on $[0,T]$, with the norm $\|\cdot\|_1$ defined by

$$\|y\|_1 = \max_{1 \le j \le m} \int_0^T |y_j(t)| dt.$$

This class of problems was introduced in 1953 by Bellman [4] in connection with production-inventory "botleneck processes". He considered a type of optimization problems, which is now known as continuous-time linear programming, formulated its dual and provided duality relations. He also suggested some computational procedure. Since then, a lot of authors have extended his theory to wider classes of continuous-time linear problems (e.g. [23], [16], [24], [11,12], [2], [19], [3] and [20]). On the other hand, optimality conditions in the spirit of Karush-Kuhn-Tucker type for continuous nonlinear problems were first investigated by Hanson and Mond [13]. They considered a class of linear constrained nonlinear programming problems. Assuming a nonlinear integrand in the cost function twice differentiable, they linearized the cost function and applied Levinson's duality theory [16] to obtain the Karush-Kuhn-Tucker optimality conditions. Also applying linearization, Farr and Hanson [9] obtained necessary and sufficient optimality conditions for a more general class of continuous-time nonlinear problems (both cost function and constraints were nonlinear). Assuming some kind of constraint qualifications and using direct methods, further generalizations of the theory of optimality conditions for continuous-time nonlinear problems are to be found in Scott and Jefferson [22], Abraham and Buie [1], Reiland and Hanson [21] and Zalmai [25], [26], [27], [28], [29]. The development of nonsmooth necessary optimality conditions for Problem (CNP) was given in [5]. The Sufficient conditions for nonsmooth case was given in [18]. Related results can be found in Craven [7]. However, his arguments are via approximation of smooth functions rather than alternative theorems. None the above works established necessary and sufficient conditions for a KKT point be a global solution of (CNP). We observe that in the case of mathematical programming this results was given by Martin [17]. If we observe the Martin's proof, the main argument involves the use of the Motzkin alternative theorem. In this work, we give a generalization of the Motzkin alternative theorem for Problem (CNP). In fact, this results is an adaptation of a similar result given by Zalmai in [26]. With this result in our possession, we prove, by a similar way as in [17], our main result.

This work is organized as follows. In Section 1, we give the preliminaries and established the Motzkin type theorem of the alternative for the continuous-time case. In Section 2, we recall the notion of invexity for (CNP), give the generalization for (CNP) of the notion of KT-invexity introduced by Martin [17] for the mathematical programming case and give a generalization of the Mangasarian-Fromovitz constraint qualification for the continuous-time case. Also we prove our main result and give an example of a problem that is KT-invex, but it is not invex. Moreover in this example holds the fact that all KKT point is a global solution. Finally, we discuss the relation between the constraint qualification and the feasible cones to the constraints.

2 Preliminaries

Let $\mathbb{F}$ be the set of all feasible solutions to Problem (CNP) (which we suppose nonempty), i.e.,

$$\mathbb{F} = \{x \in X : g(x(t), t) \leq 0 \text{ a.e. in } [0, T]\}.$$

Let V be an open subset of $\mathbb{R}^n$ containing the set $\{x(t) \in \mathbb{R}^n : x \in X,\ t \in [0, T]\}$. We assume that f and g_i (the i-*th* component of g), $i = 1, 2, \ldots, m$, are real functions defined in $V \times [0, T]$. The functions $t \mapsto f(x(t), t)$ and $t \mapsto g(x(t), t)$ are assumed to be Lebesgue measurable and integrable for all $x \in X$. In this paper we assume also that the functions f and g are continuously differentiable (in the Fréchet sense) with respect to their first arguments. We denote by $\nabla f(x(t), t)$ and $\nabla g(x(t), t)$ these derivatives, respectively.

Let

$$I = \{1, 2, \ldots, m\}.$$

For any $x \in \mathbb{F}$, we denote by $I(x)$ the index set of all the binding constraints at x:

$$I(x) = \{i \in I :\ g_i(x(t), t) = 0 \text{ a.e. in } [0, T]\}.$$

About vectors, in this paper they are all collum vectors. We use a prime to denote transposition. Besides, $w \leq 0$ means that $w_i \leq 0$ for all i, and $w < 0$ means that $w_i < 0$ for all i.

In the follows we state a Motzkin type theorem of the alternative useful for the proof of our result.

Theorem 2.1 *Let X be a nonempty open convex subset of $L_\infty^n[0, T]$. Let $p : V \times [0, T] \to \mathbb{R}^k$ and $q : V \times [0, T] \to \mathbb{R}^m$ mappings given by $p(x(t), t) = \pi(x)(t)$ and $q(x(t), t) = B(t)x(t) - b(t)$, respectively, where π is a function of X in*

$\Lambda_1^k[0,T],\ B(t) \in \mathbb{R}^{m\times n}$ *and* $b(t) \in \mathbb{R}^m,\ t \in [0,T]$. *We assume that* p *is convex with respect to its first argument on* $[0,T]$ *and that there does not exist a* $v \in L_\infty^m[0,T] \setminus \{0\},\ v(t) \geq 0$ *a.e. in* $[0,T]$, *such that*

$$B'(t)v(t) = 0 \text{ a.e. in } [0,T]. \tag{1}$$

Then exactly one of the following systems is consistent:

(I) $p(x(t),t) < 0,\ B(t)x(t) \leq b(t)$ *a.e. in* $[0,T]$ *has solution* $x \in L_\infty^n[0,T]$*;*

(II) $\int_0^T \{u'(t)p(x(t),t) + v'(t)[B(t)x(t) - b(t)]\}dt \geq 0$ *for all* $x \in L_\infty^n[0,T]$, *for some* $u \in L_\infty^k[0,T],\ u(t) \geq 0,\ u(t) \neq 0$ *a.e. in* $[0,T]$ *and for some* $v \in L_\infty^m[0,T],\ v(t) \geq 0$ *a.e. in* $[0,T]$.

PROOF. Let $\bar{x} \in L_\infty^n[0,T]$ be a solution for (I). Then for all $u \in L_\infty^k[0,T],\ u(t) \geq 0,\ u(t) \neq 0$ a.e. in $[0,T]$ and $v \in L_\infty^m[0,T],\ v(t) \geq 0$ a.e. in $[0,T]$, we have

$$\int_0^T \{u'(t)p(x(t),t) + v'(t)[B(t)x(t) - b(t)]\}dt < 0.$$

Consequently, (II) has no solution.

Now, we assume that (I) has no solution $x \in L_\infty^n[0,T]$. From Corollary 3.1 in [26], we have that there exist $u \in L_\infty^k[0,T]$ and $v \in L_\infty^m[0,T]$ such that

$$\int_0^T \{u'(t)p(x(t),t) + v'(t)[B(t)x(t) - b(t)]\}dt \geq 0, \tag{2}$$

for all $x \in X,\ u(t) \geq 0,\ v(t) \geq 0$ a.e. in $[0,T]$ and $(u(t),v(t)) \neq 0$ a.e. in $[0,T]$. If $u(t) \neq 0$ a.e. in $[0,T]$ the Theorem is proved. So, we assume that $u(t) = 0$ a.e. in $[0,T]$. From (2) we obtain

$$\int_0^T v'(t)[B(t)x(t) - b(t)]dt \geq 0 \tag{3}$$

for all $x \in X$ and some $v \in L_\infty^m[0,T],\ v(t) \geq 0$ and $v(t) \neq 0$ a.e. in $[0,T]$. If

$B'(t)v(t) \neq 0$ a.e. in $[0,T]$ then, we consider

$$x'(t) = \begin{cases} -v'(t)B(t) & \text{if } \int_0^T v'(t)b(t)dt \geq 0 \\ \dfrac{2[v'(t)b(t)][v'(t)B(t)]}{v'(t)B(t)B'(t)v(t)} & \text{if } \int_0^T v'(t)b(t)dt < 0. \end{cases}$$

In this way we have

$$\int_0^T v'(t)[B(t)x(t) - b(t)]dt < 0.$$

This, it is contradictory with (3). Therefore, $B'(t)v(t) = 0$ a.e. in $[0,T]$. But, this is contradictory with our hypothesis. Thus, $u(t) \neq 0$ a.e. in $[0,T]$.

3 KT-invexity and optimality conditions

In the continuous-time nonlinear programming problem where the functions are differentiable or nonsmooth, the Karush-Kuhn-Tucker conditions provide necessary conditions for an optimum, given certain qualifications on the constraints. See [27] for the differentiable case and [5] for the nonsmooth case.

A problem that continues to evoke very interest is that of finding sufficient conditions for an optimum. The work [28] gives some results in this direction via the pseudoconvexity in the differentiable case and the work [18] studied the nonsmooth case via the notion of invexity (observe that the results obtained in [18] are also valid for the smooth case).

We recall the notion of invexity for the Problem (CNP) in the case that the functions are Fréchet differentiable with respect to their first arguments. We say that the Problem (CNP) is invex if there exists a function $\eta : V \times V \to \mathbb{R}^n$ such that $t \mapsto \eta(x(t), y(t)) \in L^n_\infty[0,T]$ and

$$\begin{aligned} &\phi(x) - \phi(y) \geq \int_0^T \nabla f'(y(t),t)\eta(x(t),y(t))dt \\ &g_i(x(t),t) - g_i(y(t),t) \geq \nabla g_i'(y(t),t)\eta(x(t),y(t)) \text{ a.e in } [0,T],\ i \in I, \end{aligned} \tag{4}$$

for all $x, y \in X$.

Now we repeat the argument used in [18].

We say that a feasible solution y, i.e. $y \in \mathbb{F}$, for (CNP) satisfy the Karush-Kuhn-Tucker condition (we write KKT condition) if there exists a $\lambda \in L^m_\infty[0,T]$ such that

$$\int_0^T \left[\nabla f'(y(t),t) + \sum_{i\in I} \lambda_i(t) \nabla g_i'(y(t),t)\right] z(t)dt = 0,\ \forall z \in L^n_\infty[0,T], \tag{5}$$

$$\lambda_i(t) g_i(y(t),t) = 0,\ \text{a.e. in } [0,T],\ i \in I, \tag{6}$$

$$\lambda(t) \geq 0 \text{ a.e. in } [0,T]. \tag{7}$$

In such case, we say that y is a Karush-Kuhn-Tucker point (we write KKT point).

Let y be a feasible solution for (CNP) that satisfies the KKT condition and suppose that the Problem (CNP) is invex. From (4) and (7), we have

$$\int_0^T [f(x(t),t) - f(y(t),t)]dt - \int_0^T \nabla f'(y(t),t)\eta(x(t),y(t))dt$$
$$+ \int_0^T \sum_{i\in I} \lambda_i(t)[g_i(x(t),t) - g_i(y(t),t) - \nabla g_i'(y(t),t)\eta(x(t),y(t))]dt \geq 0,$$

i.e.,

$$\int_0^T [f(x(t),t) - f(y(t),t)]dt \geq \int_0^T \left[\nabla f'(y(t),t) + \sum_{i\in I}^{m} \lambda_i(t)\nabla g_i'(y(t),t)\right] \eta(x(t),y(t))dt$$
$$- \int_0^T \left[\sum_{i\in I} \lambda_i(t)[g_i(x(t),t) - g_i(y(t),t)]\right] dt.$$

Hence, by using (5) and (6) we obtain

$$\int_0^T [f(x(t),t) - f(y(t),t)]dt \geq - \int_0^T \left[\sum_{i\in I} \lambda_i(t) g_i(x(t),t)\right] dt.$$

Finally, it follows from (7) that

$$\int_0^T [f(x(t),t) - f(y(t),t)]dt \geq 0,\ \forall x \in \mathbb{F}. \tag{8}$$

Therefore $\phi(x) \geq \phi(y),\ \forall x \in \mathbb{F}$, that is, y is a global minimizer of (CNP).

If we observe carefully this proof, we can see first that the inequalities in (4) should only hold for feasible solutions for Problem (CNP), i.e., only for $x, y \in \mathbb{F}$. Second that it is not necessary that

$$g_i(x(t), t) - g_i(y(t), t) \geq \nabla g_i'(y(t), t)\eta(x(t), y(t)) \text{ a.e. in } [0, T]$$

for $i \notin I(y)$, because of the complementary slackness property of a KKT point (6). Also, it is easy to see that the omission of the terms $g_i(x(t), t),\ i \in I$, in (4) does not affect the conclusion (8). With this in mind we introduce a relaxation of invexity, which will be called KT-invexity (see [17] for the mathematical programming case).

Definition 3.1 *The Problem (CNP) is called Karush-Kuhn-Tucker invex (or KT-invex) if there exists a function* $\eta : V \times V \to \mathbb{R}^n$ *such that* $t \mapsto \eta(x(t), y(t)) \in L_\infty^n[0, T]$ *and*

$$\begin{aligned} &\phi(x) - \phi(y) \geq \int_0^T \nabla f'(y(t), t)\eta(x(t), y(t))dt \\ &-\nabla g_i'(y(t), t)\eta(x(t), y(t)) \geq 0 \text{ a.e. in } [0, T],\ i \in I(y), \end{aligned} \tag{9}$$

for all $x, y \in \mathbb{F}$.

Below we state a generalized version of the Mangasarian-Fromovitz constraint qualification for the continuous-time case.

Definition 3.2 (Mangasarian-Fromovitz Constraint Qualification) *We say that the constraint* g *satisfies the Mangasarian-Fromovitz constraint qualification at* $x \in \mathbb{F}$ *if there exists a* $h \in L_\infty^n[0, T]$ *such that*

$$\nabla g_i'(x(t), t)h(t) < 0 \text{ a.e. in } [0, T],\ i \in I(x).$$

Proposition 3.3 g *satisfies the Mangasarian-Fromovitz constraint qualification at* $x \in \mathbb{F}$ *if and only if there do not exist* $v_i \in L_\infty[0, T],\ v_i(t) \geq 0$ *a.e. in* $[0, T],\ i \in I(x)$*, not all null, such that*

$$\sum_{i \in I(x)} v_i(t)\nabla g_i(x(t), t) = 0 \text{ a.e. in } [0, T].$$

PROOF. Necessity: By hypothesis the system

$$\nabla g_i'(x(t), t)h(t) < 0 \text{ a.e. in } [0, T],\ \in I(x),$$

has a solution $h \in L^n_\infty[0,T]$. So the result follows from the Generalized Gordan's Theorem (see [26]).

Sufficiency: The hypothesis implies that the system below is not consistent:

$$\int_0^T \left[\sum_{i\in I(x)} v_i(t)\nabla g_i'(x(t),t) \right] h(t)dt = 0,$$

for all $h \in L_\infty[0,T]$ and for some $v \in L^k_\infty[0,T],\ v(t) \geq 0,\ v(t) \neq 0$ a.e. in $[0,T]$.

So again the result follows from the Generalized Gordan's Theorem.

Theorem 3.4 *We assume that g satisfies the Mangasarian-Fromovitz constraint qualification at each $y \in \mathbb{F}$. Then, every KKT point of (CNP) is a global minimizer if and only if (CNP) is KT-invex.*

PROOF. Sufficiency: We assume that (CNP) is KT-invex. Let $y \in \mathbb{F}$ be a KKT point for (CNP). Then, there exist $\lambda_i \in L_\infty[0,T],\ i \in I(y)$, such that

$$\int_0^T \left[\nabla f'(y(t),t) + \sum_{i\in I(y)} \lambda_i(t)\nabla g_i'(y(t),t) \right] z(t)dt = 0,\ \forall z \in L^n_\infty[0,T], \quad (10)$$

$$\lambda_i(t) \geq 0 \text{ a.e. in } [0,T],\ i \in I(y). \quad (11)$$

From (9) and (11) we obtain

$$\int_0^T [f(x(t),t) - f(y(t),t)]dt - \int_0^T \nabla f'(y(t),t)\eta(x(t),y(t))dt - \int_0^T \sum_{i\in I(y)} \lambda_i(t)\nabla g_i'(y(t),t)\eta(x(t),y(t))dt \geq 0,$$

for all $x \in \mathbb{F}$. Thence,

$$\int_0^T [f(x(t),t) - f(y(t),t)]dt \geq \int_0^T \left[\nabla f'(y(t),t) + \sum_{i\in I(y)} \lambda_i(t)\nabla g_i'(y(t),t) \right] \eta(x(t),y(t))dt,$$

for all $x \in \mathbb{F}$. From (10) we have

$$\int_0^T [f(x(t),t) - f(y(t),t)]dt \geq 0,\ \forall x \in \mathbb{F},$$

and hence $\phi(x) \geq \phi(y),\ \forall x \in \mathbb{F}$, that is, y is a global minimizer for Problem (CNP).

Necessity: Suppose that every KKT point for (CNP) is a global minimizer, and consider any pair of feasible points $x, y \in \mathbb{F}$.

If $\phi(x) < \phi(y)$ then y is not a global minimizer, and hence, by hypothesis, y is not a KKT point, this means, there exist no set of multipliers $\lambda_i \in L_\infty[0,T],\ \lambda_i(t) \geq 0$ a.e. in $[0,T],\ i \in I(y)$, and $\mu > 0$ such that

$$\int_0^T \left[\mu \nabla f'(y(t),t) + \sum_{i\in I(y)} \lambda_i(t) \nabla g_i'(y(t),t)\right] z(t)dt = 0,\ \forall z \in L_\infty^n[0,T].$$

By hypothesis g satisfies the Mangasarian-Fromovitz constraint qualification at y. So by Proposition 3.3 the condition (1) in Theorem 2.1 is satisfied. From this theorem it follows that there exists $z \in L_\infty^n[0,T]$ such that

$$\int_0^T \nabla f'(y(t),t) z(t) dt > 0 \tag{12}$$

$$\nabla g_i'(y(t),t) z(t) \geq 0 \text{ a.e. in } [0,T],\ i \in I(y). \tag{13}$$

Hence, setting

$$\eta(x(t),y(t)) = [\phi(x) - \phi(y)] \left[\int_0^T \nabla f'(y(t),t) z(t) dt\right]^{-1} z(t),$$

we have

$$\phi(x) - \phi(y) - \int_0^T \nabla f'(y(t),t) \eta(x(t),y(t)) dt = 0, \tag{14}$$

and since $\phi(x) < \phi(y)$, it follows from (12) that

$$[\phi(x) - \phi(y)] \left[\int_0^T \nabla f'(y(t),t) z(t) dt\right]^{-1} < 0.$$

Then, from (13) we have

$$\nabla g_i'(y(t),t) \eta(x(t),y(t)) < 0 \text{ a.e. in } [0,T],\ i \in I(y). \tag{15}$$

From (14) and (15) it follows that (CNP) is KT-invex.

If $\phi(x) \geq \phi(y)$ we consider $\eta(x(t), y(t)) = 0$, so that

$$\phi(x) - \phi(y) - \int_0^T \nabla f'(x(t), t)\eta(x(t), y(t))dt \geq 0 \tag{16}$$

and

$$\nabla g_i'(y(t), t)\eta(x(t), y(t)) = 0, \ i \in I(y). \tag{17}$$

From (16) and (17) we obtain that (CNP) is KT-invex.

In the cases above we do not define η for $x, y \in \mathbb{F}$. But we can take $\eta(x(t), y(t)) = 0$ when x or y is not feasible.

The next example show a problem that is not invex, but it is KT-invex.

Example 3.5 *We consider the following problem of continuous-time nonlinear programming:*

$$\textit{Minimize } \phi(x) = \int_0^T [1 - \exp(-x(t))]dt$$
$$\textit{subject to } x(t) \geq 0 \textit{ a.e. in } [0, T],$$

where $x \in L_\infty[0, T]$. Taking $f(x(t), t) = 1 - \exp(-x(t))$ and $g(x(t), t) = -x(t)$. We have that $\bar{x} \equiv 0$ is the unique point that satisfy the KKT conditions. In fact, taking $\lambda(t) = \nabla f(\bar{x}(t), t) = 1$ we have

$$\int_0^T [\nabla f(\bar{x}(t), t) + \lambda(t)\nabla g(\bar{x}(t), t)]z(t)dt = 0,$$

for all $z \in L_\infty[0, T]$. Then $\bar{x} \equiv 0$ satisfy the KKT conditions. Now, we show that it is unique. Let x be such that $x(t) > 0$ a.e. in $[0, T]$. Assume that there exists $\lambda \in L_\infty[0, T]$ such that x satisfy the KKT conditions. So,

$$\int_0^T [\nabla f(x(t), t) + \lambda(t)\nabla g(x(t), t)]z(t)dt = 0, \ \forall z \in L_\infty[0, T],$$
$$\lambda(t)g(x(t), t) = 0 \textit{ a.e. in } [0, T],$$
$$\lambda(t) \geq 0 \textit{ a.e. in } [0, T].$$

Consequently

$$\int_0^T [\nabla f(x(t),t) - \lambda(t)]z(t)dt = 0,\ \forall z \in L_\infty[0,T],$$

$$\lambda(t) = 0\ a.e.\ in\ [0,T],$$

i.e.,

$$\int_0^T \nabla f(x(t),t)z(t)dt = 0,\ \forall z \in L_\infty[0,T].$$

Therefore, $\nabla f(x(t),t) = \exp(-x(t)) = 0\ a.e.\ in\ [0,T]$*. This is absurd.*

Thus, every point that satisfy the KKT conditions is a global minimizer. But, shall this problem be invex? If it is, there exists a function $\eta : V \times V \to \mathbb{R}$ *such that for* $x, y \in X$

$$\int_0^T [f(x(t),t) - f(y(t),t)]dt \geq \int_0^T \nabla f(y(t),t)\eta(x(t),y(t))dt$$
$$-x(t) + y(t) \geq -\eta(x(t),y(t))\ a.e.\ in\ [0,T].$$

But, this implies

$$\int_0^T [f(x(t),t) - f(y(t),t)]dt - \int_0^T \nabla f(y(t),t)[x(t) - y(t)]dt \geq$$
$$\geq \int_0^T [f(x(t),t) - f(y(t),t)]dt - \int_0^T \nabla f(y(t),t)\eta(x(t),y(t))dt \geq 0,$$

since $\nabla f(y(t),t) = \exp(-y(t)) > 0\ a.e.\ in\ [0,T]$*. Hence*

$$\phi(x) - \phi(y) \geq \nabla\phi(y)(x - y),$$

where $\nabla\phi(y)$ *denote the Fréchet derivative of* ϕ *at* y*. Then the problem is invex if and only if it is convex, but this problem is not convex, so that it is not invex. However, it is KT-invex. Indeed, define* $\eta : V \times V \to \mathbb{R}$ *by*

$$\eta(x(t),y(t)) = \frac{\phi(x) - \phi(y)}{\nabla\phi(y)}.$$

So

$$\phi(x) - \phi(y) - \nabla\phi(y)\eta(x(t),y(t)) = \phi(x) - \phi(y) - \nabla\phi(y)\frac{\phi(x) - \phi(y)}{\nabla\phi(y)} = 0.$$

Moreover if x, y are such that $x(t) \geq 0$ and $y(t) = 0$ a.e. in $[0, T]$ then

$$-\nabla g(y(t), t)\eta(x(t), y(t)) = \eta(x(t), 0) = \frac{\phi(x) - \phi(0)}{\nabla\phi(0)} \geq 0,$$

since $y \equiv 0$ is the global minimizer and $\nabla\phi(0) = T > 0$.

Now we discuss briefly the Mangasarian-Fromovitz constraint qualification. It is equivalent to the fact that the intersection of the feasible cones to the binding constraints be nonempty. In the follows we place the definition of a feasible cone to a given set Q at a point in the closure of Q, and we prove that equivalence.

Let Q be a nonempty subset of a Banach space E, and let x be a point in the closure of Q. We say that $h \in E$ is a feasible direction for Q at x if there exist a neighborhood U of h and a real number $\varepsilon_0 > 0$ such that

$$x + \varepsilon\bar{h} \in Q, \text{ for all } 0 < \varepsilon < \varepsilon_0 \text{ and } \bar{h} \in U.$$

The set of all feasible directions generate an open cone with apex at zero. We will denote this cone by $\mathcal{F}(Q, x)$.

If we take

$$Q_i = \{x(t) \in \mathbb{R}^n : g_i(x(t), t) \leq 0 \text{ a.e. in } [0, T],\ t \in [0, T]\}, \quad i \in I,$$

then

$$\mathcal{F}(Q_i, x(t)) = \{h(t) \in \mathbb{R}^n : \nabla g_i'(x(t), t)h(t) < 0 \text{ a.e. in } [0, T]\}, \quad i \in I. \tag{18}$$

For a proof of this and more detail about feasible cones and related concepts we refer the reader to [10].

Proposition 3.6 *g satisfies the Mangasarian-Fromovitz constraint qualification at $x \in \mathbb{F}$ if and only if*

$$\bigcap_{i \in I(x)} \mathcal{F}(Q_i, x) \neq \emptyset.$$

PROOF. It follows directly from (18) and the Definition 3.2.

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