Examples of Irreducible Automorphisms of Handlebodies

Leonardo Navarro Carvalho*
leonardo@ime.unicamp.br

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Abstract

Automorphisms of handlebodies arise naturally in the a classification of automorphisms of three-manifolds. Among automorphisms of handlebodies, there are certain automorphisms called irreducible (or generic), which are analogues of pseudo-Anosov automorphisms of surfaces. We develop a method for constructing a certain range of examples of such automorphisms.

1 Introduction.

1.1 Some history and background

The classification of *automorphisms* (i.e. self-homeomorphisms) of a manifold, up to isotopy, is a natural and important problem. Nielsen addressed the case where the manifold is a compact and connected surface and his results were later substantially improved by Thurston (see [Nie86a, Nie86b, Nie86c, Thu88, HT85]). We briefly state their main result: An automorphism of a surface is, up to isotopy, either *periodic* (i.e., has finite order), *reducible* (i.e., preserves an essential codimension-1 submanifold) or *pseudo-Anosov.* We refer the reader to any of [FLP79, HT85, Thu88, CB88] for details — including the definition of a pseudo-Anosov automorphism. The

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Nielsen-Thurston theory also shows that the reducible case may — as expected — be reduced to the other two. Since periodic automorphisms are relatively easy to understand, the remaining *irreducible* case — pseudo-Anosov — is the most interesting and rich one.

Indeed, pseudo-Anosov automorphisms of surfaces are the subject of intense and wide research (see [Thu88]). We mention two works on the natural problem of building examples of such automorphisms: Penner provides a generating method [Pen88] and a testing algorithm is developed in [BH95].

In [Oer02], Oertel undertakes a similar classification project for a certain class of three-dimensional manifolds. Suppose that a three-dimensional manifold M is compact, connected, orientable and *irreducible* (i.e., every embedded sphere bounds a ball). Assume further that $\partial M \neq \emptyset$. By use of canonical decompositions of M due to Bonahon (determined by his *characteristic compression body* [Bon83a]) and Jaco, Shalen and Johanson (the *JSJ-decomposition* [JS79, Joh79]), the study of automorphisms of M is reduced to the study of automorphisms of *compression bodies* and *handlebodies* (see [Oer02]). We define these types of manifolds:

Definition 1.1. A *handlebody* H is an orientable and connected three-manifold obtained from a three-dimensional ball by attaching a certain finite number g of 1-handles. The integer g is the *genus* of H. It should be clear that $\pi_1(H)$ is isomorphic to the free group F_g on g generators.

A *compression body* is a pair (Q, F), where Q is a three-manifold obtained from a compact surface F (not necessarily connected) in the following way: consider the disjoint union of $F \times I$ with the disjoint union of finitely many balls B and add 1-handles to $(F \times \{1\}) \cup \partial B$, obtaining Q. We allow empty or non-empty ∂F, but F cannot have sphere components. We identify F with $F \times \{0\} \subseteq Q$, which is called the *interior boundary* of (Q, F), denoted by $\partial_i Q$. The *exterior boundary* $\partial_e Q$ of Q is the closure $\overline{\partial Q - \partial_i Q}$. If Q is homeomorphic to the disjoint union of $F \times I$ with balls then (Q, F) is said to be *trivial*.

We may abuse notation and refer to Q as a compression body.

The role of the disjoint union of balls B in the definition of compression body above is a two fold one. It makes some operations of attaching 1-handles to be trivial. For instance, a 1-handle may connect $F \times I$ with a ball or connect two distinct balls. On the other hand, under this definition, a handlebody may be regarded as a connected compression body whose interior boundary $\partial_i Q = F$ is empty. Indeed, these two types of manifolds are quite similar [Bon83a], as is the study of their automorphisms [Oer02]. This research will focus on the case of handlebodies.

The following definition is due to Oertel:

Definition 1.2. An automorphism $f \colon H \to H$ of a handlebody H is said *reducible* if any of the following holds:

1. there exists an f-invariant (up to isotopy) non-trivial compression body (Q, F) with $Q \subseteq H$, $\partial_e Q \subseteq \partial H$ and $F = \partial_i Q \neq \emptyset$ not containing ∂-parallel disc components,

2. there exists an f-invariant (up to isotopy) collection of pairwise disjoint, incompressible, non-∂-parallel and properly embedded annuli, or

3. H admits an f-invariant (up to isotopy) I-bundle structure.

The automorphism f is said *irreducible* (or *generic*, as in [Oer02]) if both of the following conditions hold:

1. $\partial f = f|_{\partial H}$ is pseudo-Anosov, and

2. there exists no *closed reducing surface* F: a closed reducing surface is a surface $F \neq \emptyset$ which is the interior boundary $\partial_i Q$ of a non-trivial compression body (Q, F) such that $Q \subseteq H$, (Q, F) is f-invariant (up to isotopy) and $\partial_e Q = \partial H$.

An obvious remark is that this definition of irreducible automorphism excludes the periodic case.

Theorem 1.3 (Oertel, [Oer02]). *An automorphism of a handlebody is either:*

1. *periodic,*

2. *reducible, or*

3. *irreducible.*

We note that the theorem above is not entirely obvious. For example, one must show that if an automorphism $f : H \to H$ of a handlebody does not restrict to a pseudo-Anosov ∂f on ∂H, then f is actually reducible according to Definition 1.2, or periodic.

Our interest is precisely in the irreducible case, which is in many ways analogous to the pseudo-Anosov case for surfaces (an important similarity is related to the existence of certain invariant projective measured laminations [Oer02]). We shall address the problem of constructing examples of irreducible automorphisms of handlebodies.

This is already settled for genus two handlebodies: a previous work of Bonahon [Bon83b] imply that any automorphism of a genus two handlebody whose restriction to the boundary is pseudo-Anosov automorphism is then irreducible (see also [Lon88]). That is not true for higher genus (see Appendix A).

We also note that in [FL80] the authors build an automorphism of a genus two handlebody which 1) restricts to the boundary as a pseudo-Anosov automorphism thus, as mentioned before, is irreducible, and 2) induces the identity on the fundamental group. Such an example is very interesting. It exposes the richness of irreducible automorphisms of handlebodies when compared with pseudo-Anosov automorphisms of surfaces, whose complexity is captured on the level of the fundamental group.

1.2 This research

We shall build a method for generating irreducible automorphisms of handlebodies of higher (even) genus.

In Section 2 we develop a particular case of the method. Although it will yield an automorphism of a genus two handlebody, which we chose for simplicity, our argument is different from that of [Bon83b] (see Example 2.1 and Proposition 2.2)

In Section 3 we generalize the construction of that first example and develop the general method. This is done in theorems 3.2 and 3.4. Their statements depend on some rather technical constructions, unsuited for this introduction. We also use it to build an example of an irreducible automorphism of a genus four handlebody.

The following two theorems will serve us as important tools. We refer the reader to [Pen88] and [BH92] respectively for details and precise definitions.

Theorem 1.4 (Penner, [Pen88]). *Let $\mathcal{C}$, $\mathcal{D}$ be two systems of closed curves in an orientable surface S with $\chi(S) < 0$. Assume that $\mathcal{C}$ and $\mathcal{D}$ intersect efficiently, do not have parallel components and fill S. Let $f\colon S \to S$ be a composition of Dehn twists: right twists along curves of $\mathcal{C}$ and left twist along curves of $\mathcal{D}$. If a twist along each curve appears at least once in the composition, then f is isotopic to a pseudo-Anosov automorphism of S.*

Theorem 1.5. *Let S be a compact surface with $\chi(S) < 0$ and precisely one boundary component. An automorphism $f\colon S \to S$ is pseudo-Anosov if and* *only if f_*^n is irreducible for all $n > 0$.*

As a final introductory remark, we observe that an ideal classification of automorphisms of handlebodies should identify in each isotopy class a representative which is "best" in some sense. Considering the classification of Theorem 1.3, this has been done for periodic and many reducible automorphisms [Bon83a, Oer02]. The problem of finding a best representative of an irreducible automorphism is addressed in [Oer02, Car03] but not yet solved.

We will adopt the following notation: given a topological space A (typically a manifold or sub-manifold), $\overline{A}$ will denote its topological closure, $\mathring{A}$ its interior and $|A|$ its number of connected components. If M is a manifold and $S \subseteq M$ a compact codimension 1 submanifold, we can "cut M open along S" obtaining M_S. More precisely, a Riemannian metric in M determines a path-metric in $M - S$, in which the distance between two points is the infimum of the lengths of paths in $M - S$ connecting them. We let M_S to be the completion of $M - S$ with this metric.

I thank Ulrich Oertel for many enlightening meetings and helpful suggestions in his role as dissertation advisor, and also for laying the foundations on which the research in this paper is built.

2 An example.

We develop a particular simple case of the general method.

Let H be a genus 2 handlebody. We will describe an automorphism of H as a composition of Dehn twists along two annuli and a disc. We shall prove that it is irreducible by showing that its restriction to ∂H is pseudo-Anosov and that, for an algebraic reason, there can be no closed reducing surface.

Example 2.1. We start with a pseudo-Anosov automorphism $\varphi\colon S \to S$ of the once punctured torus S. Such a φ will be defined as a composition of Dehn twists along two curves.

We will represent S as a cross, after identifying pairs of opposite sides as shown in Figure 1.

Let α_0, α_1 be simple closed curves as in the figure. It is easy to verify that the systems $\mathcal{C} = \{\alpha_0\}$ e $\mathcal{D} = \{\alpha_1\}$ satisfy the hypothesis of Theorem 1.4 (Penner). Let T_0^- be the left Dehn twist along α_0 and T_1^+ the right twist along α_1. We define:

$$\varphi = T_1^+ \circ T_0^-.$$

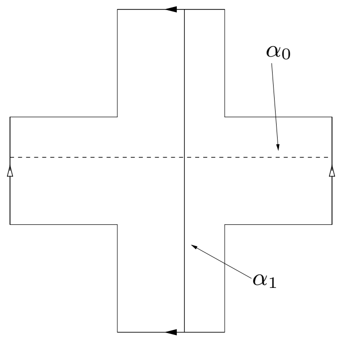

Figure 1: The oriented surface S and the curves α_0, α_1.

By Theorem 1.4, φ is pseudo-Anosov. Then, by Theorem 1.5, any positive power φ_*^n, of the induced homomorphism $\varphi_* \colon \pi_1(S) \to \pi_1(S)$ is irreducible. We note this fact for future use.

We now consider the handlebody $H = S \times I$ and lift φ to H, obtaining $\phi \colon H \to H$, a composition of twists along the annuli $A_0 = \alpha_0 \times I$, $A_1 = \alpha_1 \times I$ as in Figure 2.

Remark. For future use, we will think of the picture as being looked at "from above". More precisely, we orient H in such a way that the induced orientation in $S \times \{1\}$ coincides with the one inherited naturally from S.

Identifying $\pi_1(H)$ with $\pi_1(S)$ we have $\phi_* = \varphi_*$.

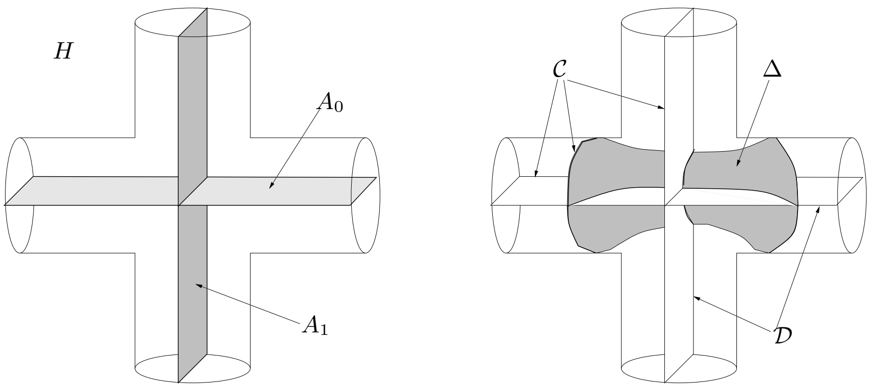

Figure 2: The automorphism f is defined as a composition of Dehn twists along the annuli A_0, A_1 and the disc Δ.

Finally, we will obtain the desired irreducible automorphism $f \colon H \to H$ by composing ϕ with a twist along a disc Δ, shown in Figure 2.

Let T_Δ^+ be the right Dehn twist along Δ. We define:

$$f = T_\Delta^+ \circ \phi.$$

Proposition 2.2. *The automorphism $f\colon H \to H$ is irreducible.*

Part of the proof will be done in the following general lemma.

Lemma 2.3. *Let $g\colon H \to H$ be an automorphism of a handlebody H such that ∂g is pseudo-Anosov. If g is reducible then, for some $n \in \mathbb{N}$, $g_*^n\colon \pi_1(H) \to \pi_1(H)$ is reducible.*

Proof. Let Q be a compression body invariant under g. Let $F \subseteq \partial_i Q$ be a component of the closed reducing surface and $J \subseteq \mathring{H}$ the handlebody bounded by F. Choosing a base point in J and omitting the obvious inclusion homomorphisms we claim that

$$\pi_1(H) = \pi_1(J) * G,$$

where G is not trivial. To see this, first consider the connected and nontrivial compression body $Q' = \overline{H - J}$, whose boundary decomposes as $\partial_i Q' = F$ and $\partial_e Q' = \partial H$. The compression body structure of Q' gives it as a product $F \times I$ to which 1-handles are attached. Regarding $F \times I \subseteq Q' \subseteq H$, we see that the handlebody $J' = (F \times I) \cup J$ deformation retracts to J (so $\pi_1(J') = \pi_1(J)$ through inclusion). But the compression body structure of Q' gives H as J' with 1-handles attached to $\partial J'$. Since $\partial J'$ is connected, we can moreover assume that these 1-handles are attached to a disc in $\partial J'$, which gives $\pi_1(H) = \pi_1(J') * G = \pi_1(J) * G$, where G is a free group (whose rank equals the number of 1-handles of Q'). Since Q' is not trivial, G is not trivial, proving the claim. Therefore $\pi_1(J)$ is a proper free factor of $\pi_1(H)$.

Let g^n be the first power of g preserving J. Isotoping g we assume moreover that the base point is fixed by g^n. From

$$g^n(J) = J$$

follows that $g_*^n(\pi_1(J))$ is conjugate to $\pi_1(J)$, hence the class of g_*^n in $Out\big(\pi_1(H)\big)$ is reducible. □

Proof of Proposition 2.2. We need to prove that $\partial f = f|_{\partial H}$ is pseudo-Anosov and that f does not admit closed reducing surfaces.

We start by verifying that ∂f is pseudo-Anosov. It is given as composition of Dehn twists: left twists along curves of

$$\mathcal{C} = \{\, (\alpha_0 \times \{1\})\,,\, (\alpha_1 \times \{0\})\,\}\,,$$

(see Figure 2) and right twists along curves of

$$\mathcal{D} = \{\, (\alpha_0 \times \{0\}) \,,\, (\alpha_1 \times \{1\}) \,,\, \partial\Delta \,\}\,.$$

We now note that $\mathcal{C}$, $\mathcal{D}$ satisfy the hypotheses of Theorem 1.4, hence ∂f is pseudo-Anosov.

We prove by contradiction that f admits no closed reducing surface. Suppose there is a closed reducing surface. By Lemma 2.3, there exists n such that f_*^n is reducible. But $f = (T_\Delta^+) \circ \phi$ and the twist (T_Δ^+) (along a disc) induces the identity in $\pi_1(H)$. Therefore, recalling that $\pi_1(H)$ is identified with $\pi_1(S)$, we have that $f_*^n = \phi_*^n = \varphi_*^n$, which was seen before to be irreducible for any n, a contradiction.

Therefore f is irreducible. □

3 A method for generating irreducible automorphisms.

The construction of Example 2.1 may be generalized to provide a method for generating a larger class of irreducible automorphisms of handlebodies (Theorems 3.2 and 3.4). This method partially solves a problem proposed in [Oer02].

Definition 3.1. We say that a pair $(\mathcal{C}, \mathcal{D})$ of curve systems in a compact, connected and orientable surface S with $\chi(S) < 0$ is a *Penner pair in S* if $\mathcal{C}$, $\mathcal{D}$ satisfy the hypotheses of Penner's Theorem 1.4 i.e.,

1. each $\mathcal{C}$, $\mathcal{D}$ is a finite collection of simple, closed and pairwise disjoint essential curves,
2. $\mathcal{C}$ and $\mathcal{D}$ intersect efficiently, do not have parallel components and *fill* S (i.e., the components of $S - (\mathcal{C} \cup \mathcal{D})$ are either contractible or deformation retract to ∂S).

Suppose that $(\mathcal{C}, \mathcal{D})$ is a Penner pair. An automorphism φ of S obtained from $\mathcal{C}$, $\mathcal{D}$ as in Theorem 1.4 is called a *Penner automorphism subordinate to* $(\mathcal{C}, \mathcal{D})$.

If $\partial S \neq \emptyset$ then a properly embedded and essential arc θ is called *dual to* $(\mathcal{C}, \mathcal{D})$ if θ intersects $\mathcal{C} \cup \mathcal{D}$ transversely and in exactly one point $p \notin \mathcal{C} \cap \mathcal{D}$.

Remark. Although not all Penner pairs admit dual arcs it is easy to construct pairs that do: such a pair $(\mathcal{C}, \mathcal{D})$ in S has the property that there are two adjacent components (not necessarily distinct) of $S-(\mathcal{C}\cup\mathcal{D})$ each containing some component of ∂S. If a pair does not have this property then we can remove discs from S and introduce dual arcs.

We constructed the irreducible automorphism in Example 2.1 by lifting a pseudo-Anosov automorphism of a surface to a product and composing it with a twist on a disc. The general method will be similar. Our interest in dual arcs is that we can use them to construct discs that will yield irreducible automorphisms.

Throughout this section we fix a compact, connected and oriented surface S with $\partial S \neq \emptyset$ and define $H = S \times I$, which is a handlebody. We identify S with $S \times \{1\} \subseteq H$, inducing orientation in H.

Given a Penner pair $(\mathcal{C}, \mathcal{D})$ in S and a dual arc θ, we build a disc Δ_θ in H in the following way. Let γ be the curve of $(\mathcal{C}, \mathcal{D})$ that θ intersects and assume without loss of generality that $\gamma \subseteq \mathcal{C}$. Let $D = \theta \times I \subseteq H$. Then ∂D intersects $\gamma_1 = \gamma \times \{1\}$ in a point. Now let Δ_θ be the *band sum* of D with itself along γ_1. This means that Δ_θ is obtained from D and γ_1 by the following construction: consider a regular neighborhood $N = N(D \cup \gamma_1)$. Then $\Delta_\theta = \overline{\partial N - \partial H}$ is a properly embedded disc.

Theorem 3.2. *Suppose that $\partial S \neq \emptyset$ has exactly one component. Let $(\mathcal{C}, \mathcal{D})$ be a Penner pair in S with dual arc θ and $\varphi\colon S \to S$ a Penner automorphism subordinate to $(\mathcal{C}, \mathcal{D})$. Let $\hat{\varphi}\colon H \to H$ be the lift of φ to the product $H = S \times I$ and $\Delta_\theta \subseteq H$ the disc constructed from the arc θ as above. Then there exists a Dehn twist $T_{\Delta_\theta}\colon H \to H$ along Δ_θ such that the composition*

$$\hat{\varphi} \circ T_{\Delta_\theta}\colon H \to H$$

is an irreducible automorphism of H.

The key to the proof is the verification that $\mathcal{C}$, $\mathcal{D}$ and $\partial\Delta_\theta$ induce a Penner pair in ∂H.

Lemma 3.3. *Let S, $(\mathcal{C}, \mathcal{D})$, θ, $H = S \times I$ and Δ_θ be as in the statement of Theorem 3.2. Let $\mathcal{C}_i = \mathcal{C} \times \{i\} \subseteq S_i = S \times \{i\}$ and $\mathcal{D}_i = \mathcal{D} \times \{i\} \subseteq S_i = S \times \{i\}$, defining $\mathcal{C}_0$, $\mathcal{D}_0 \subseteq S_0$ and $\mathcal{C}_1$, $\mathcal{D}_1 \subseteq S_1$. Under these conditions the following system of curves in ∂H:*

$$\begin{aligned} \mathcal{Q} =& \mathcal{D}_0 \cup \mathcal{C}_1 \cup \{\, \partial\Delta_\theta \,\}, \\ \mathcal{R} =& \mathcal{C}_0 \cup \mathcal{D}_1, \end{aligned}$$

determine a Penner pair $(\mathcal{Q}, \mathcal{R})$ in ∂H.

Proof. We start by making the obvious remarks that $\mathcal{C}_0$, $\mathcal{D}_0$, $\mathcal{C}_1$, $\mathcal{D}_1 \subseteq \partial H$ and $\mathcal{C}_0 \cap \mathcal{D}_1 = \emptyset$, $\mathcal{D}_0 \cap \mathcal{C}_1 = \emptyset$. Recall we are assuming that $\theta \cap (\mathcal{C} \cup \mathcal{D}) \subseteq \gamma \subseteq \mathcal{C}$. We verify that:

- $\partial\Delta_\theta \cap \mathcal{D}_0 = \emptyset$, because $(\theta \times \{0\}) \cap \mathcal{D}_0 = \emptyset$ and $\partial\Delta_\theta \cap S_0$ consists of two arcs parallel to $\theta \times \{0\}$,

- $\partial\Delta_\theta \cap \mathcal{C}_1 = \emptyset$, because $\partial\Delta_\theta \cap \gamma_1 = \emptyset$ by construction.

Therefore each $\mathcal{Q} = \mathcal{D}_0 \cup \mathcal{C}_1 \cup \{\partial\Delta\}$ and $\mathcal{R} = \mathcal{C}_0 \cup \mathcal{D}_1$ is a system of simple closed curves essential in ∂H. To conclude that $(\mathcal{Q}, \mathcal{R})$ is indeed a Penner pair we just need to verify that $\mathcal{Q} \cup \mathcal{R}$ fills ∂H.

A component of $S - (\mathcal{C} \cup \mathcal{D})$ either is a disc or an annulus that retracts to ∂S. Therefore a component of $\partial H - (\mathcal{C}_0 \cup \mathcal{D}_0 \cup \mathcal{C}_1 \cup \mathcal{D}_1)$ either is a disc, or an annulus A (that retracts to $\partial S \times I$). But $A \cap \partial\Delta_\theta$ is a union of four arcs essential in A, hence each component of $\partial H - (\mathcal{Q} \cup \mathcal{R})$ is a disc, showing that $\mathcal{Q} \cup \mathcal{R}$ fills ∂H, completing the proof. □

Instead of proving Theorem 3.2 we will prove the more general result below, which clearly implies the other. We note that twists on curves of $\mathcal{C}$, $\mathcal{D}$ in S lift to twists along annuli in H. We call these systems of annuli $\hat{\mathcal{C}}$, $\hat{\mathcal{D}}$ respectively. It thus makes sense to refer to directions of the twists along these vertical annuli (recall that H has orientation induced by $S \times \{1\} \subseteq H$).

Theorem 3.4. *Let $(\mathcal{C}, \mathcal{D})$, S, θ, H and Δ_θ be as in Theorem 3.2. Let f be a composition $f\colon H \to H$ of twists along the annuli of $\hat{\mathcal{C}}$, $\hat{\mathcal{D}}$ and the disc Δ_θ: in one direction along the annuli in $\hat{\mathcal{D}}$ and in the opposite direction along the annuli in $\hat{\mathcal{C}}$ and the disc Δ_θ. If each of these twists appear in the composition at least once f is irreducible.*

Proof. We will show initially that $f_*^n\colon \pi_1(H) \to \pi_1(H)$ is an irreducible automorphism of a free group for any $n \geq 0$ (hence there can be no closed reducing surface by Lemma 2.3) and then that $\partial f = f|_{\partial H}$ is pseudo-Anosov, thus completing the proof that f is irreducible.

We first identify $\pi_1(H)$ with $\pi_1(S)$, identifying S with $S \times \{1\} \subseteq H$. Let T_{Δ_θ} be a twist along Δ_θ. Since $(T_{\Delta_\theta})_*\colon \pi_1(H) \to \pi_1(H)$ is the identity (Δ_θ is a disc) the hypotheses on f imply that $f_* = \varphi_*$ for some Penner automorphism $\varphi\colon S \to S$ subordinate to $(\mathcal{C}, \mathcal{D})$. Penner automorphisms are pseudo-Anosov so, given that ∂S has a single component, it follows from Theorem 1.5 that φ_*^n is an irreducible automorphism of $\pi_1(S)$ for any $n \geq 0$. Therefore $f_*^n\colon \pi_1(H) \to \pi_1(H)$ is irreducible, proving that f does not admit closed reducing surfaces (Lemma 2.3).

We now prove that ∂f is pseudo-Anosov. Let $(\mathcal{Q}, \mathcal{R})$ be as in Lemma 3.3, therefore a Penner pair. By construction the twists that compose f restrict to ∂H as twists along curves of $\mathcal{Q}$ or $\mathcal{R}$.It is then straightforward to verify that ∂f is a Penner automorphism subordinate to $(\mathcal{Q}, \mathcal{R})$,hence pseudo-Anosov, completing the proof that f is irreducible.

□

Example 3.5. Consider S a genus 2 surface minus a disc, represented in Figure 3 as an octagon whose sides are identified according to the arrows.

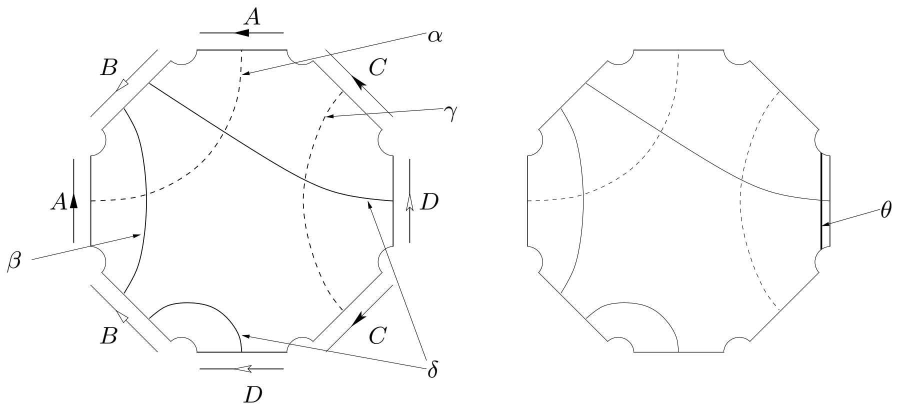

Figure 3: A Penner pair in S, with dual arc θ.

In the picture there are represented four further curves: α, β, γ and δ. Defining

$$\mathcal{C} = \{\, \beta, \delta \,\},$$
$$\mathcal{D} = \{\, \alpha, \gamma \,\},$$

it is easy to check that $(\mathcal{C}, \mathcal{D})$ is a Penner pair in S. The automorphism $\varphi\colon S \to S$ defined by

$$\varphi = T_\beta^- \circ T_\delta^- \circ T_\alpha^+ \circ T_\gamma^+$$

is, therefore, a Penner automorphism subordinate to the pair $(\mathcal{C}, \mathcal{D})$.

The pair $(\mathcal{C}, \mathcal{D})$ admits dual arcs. The picture shows one, labelled as θ. We consider the corresponding disc Δ_θ. Figure 4 shows $S_0 = S \times \{0\}$, $S_1 = S \times \{1\} \subseteq \partial H$ and how $\partial \Delta_\theta$ intersects them.

Figure 4 shows the pair $(\mathcal{Q}, \mathcal{R})$ obtained by Lemma 3.3 as well: $\mathcal{Q}$ consists on the solid curves, including $\partial \Delta_\theta$, while the dotted curves form $\mathcal{R}$.

Theorem 3.2 assures that, if $\hat{\varphi}\colon H \to H$ is the lift of φ to H, then

$$\hat{\varphi} \circ T_{\Delta_\theta}\colon H \to H$$

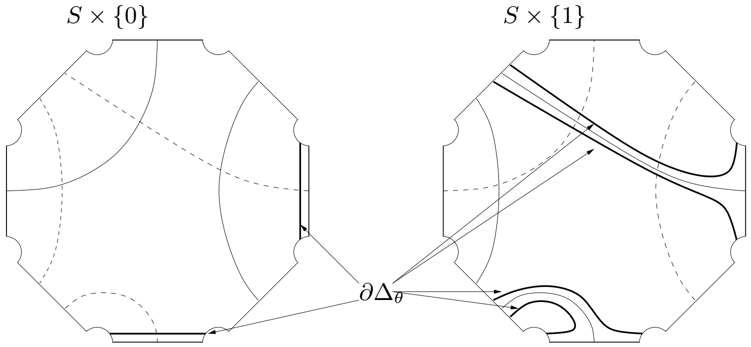

Figure 4: The curve $\partial\Delta_\theta$ in ∂H.

is an irreducible automorphism for a certain twist T_{Δ_θ} along Δ_θ.

A Pseudo-Anosov versus irreducibility

The following example is like those developed in [FL80]. It shows that an automorphism of a handlebody that restricts to the boundary as a pseudo-Anosov need not be irreducible.

Example A.1. Figure 5 a), b) represent the boundaries of two discs in a genus three handlebody H. It is easy to see that these boundaries yield a Penner pair in ∂H. Hence a composition f of twists to opposite directions along these discs yields ∂f as a pseudo-Anosov automorphism. One can see that there exists a torus that does not intersect the discs – therefore being invariant under f (Figure 5 c)). This torus is clearly a closed reducing surface for f, which is then reducible.

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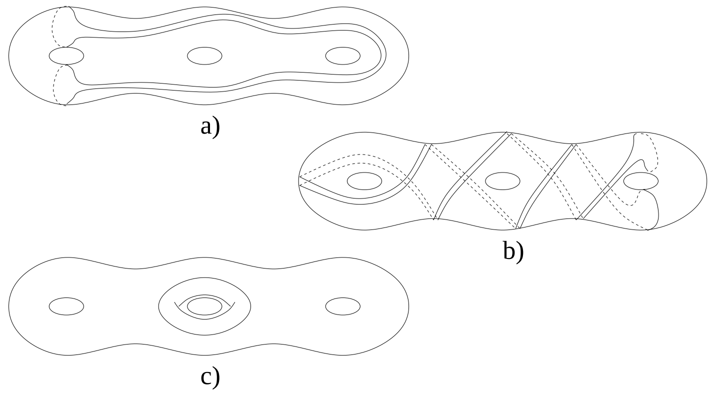

Figure 5: a) the curve C_0 bounding a disc $D_0 \subseteq H$; b) the curve C_1 bounding a disc $D_1 \subseteq H$; c) the invariant torus.

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