A NOTE ON THE NUMBER OF NODAL SOLUTIONS OF AN ELLIPTIC EQUATION WITH SYMMETRY

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ABSTRACT. We consider the semilinear problem $-\Delta u + \lambda u = |u|^{p-2}u$ in Ω, $u = 0$ on $\partial\Omega$ where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $2 < p < 2^* = 2N/(N-2)$. We show that if Ω is invariant by a nontrivial orthogonal involution then, for $\lambda > 0$ sufficiently large, the equivariant topology of Ω is related with the number of solutions which change sign exactly once.

1. INTRODUCTION

Consider the problem

$$(P_\lambda) \qquad -\Delta u + \lambda u = |u|^{p-2}u \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $2 < p < 2^* = 2N/(N-2)$. It is well known that it possesses infinitely many solutions. However, when we require some properties of the sign of the solutions, the problem seems to be more complicated. In the paper [1], Benci and Cerami showed that, if λ is sufficiently large, then (P_λ) has at least $\text{cat}(\Omega)$ positive solutions, where $\text{cat}(\Omega)$ denotes the Ljusternik-Schnirelmann category of Ω in itself. Since the work [1], multiplicity results for (P_λ) involving the category have been intensively studied (see [2, 3, 4] for subcritical, and [5, 6, 7] for critical nonlinearities).

In the aforementioned works, the authors considered positive solutions. In [8], Bartsch obtained infinite nodal solutions for (P_λ), that is, solutions which change sign. Motivated by this work and for a recent paper of Castro and Clapp [9], we are interested in relating the topology of Ω with the number of solutions which change sign exactly once. This means that the solution u is such that $\Omega \setminus u^{-1}(0)$ has exactly two connected components, u is positive in one of them and negative in the other. We deal with the problem

$$(P_\lambda^\tau) \qquad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ u(\tau x) = -u(x), & \text{for all } x \in \Omega, \end{cases}$$

where $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is a linear orthogonal transformation such that $\tau \neq \text{Id}$, $\tau^2 = \text{Id}$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain such that $\tau\Omega = \Omega$. Our main result can be stated as follows.

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Theorem 1.1. *For any $p \in (2, 2^*)$ fixed there exists $\overline{\lambda} = \overline{\lambda}(p)$ such that, for all $\lambda > \overline{\lambda}$, the problem (P_λ^τ) has at least τ-$\mathrm{cat}_\Omega(\Omega \setminus \Omega^\tau)$ pairs of solutions which change sign exactly once.*

Here, $\Omega^\tau = \{x \in \Omega : \tau x = x\}$ and τ-cat is the G_τ-equivariant Ljusternik-Schnirelmann category for the group $G_\tau = \{\mathrm{Id}, \tau\}$. There are several situations where the equivariant category turns out to be larger than the nonequivariant one. The classical example is the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ with $\tau = -\mathrm{Id}$. In this case $\mathrm{cat}(\mathbb{S}^{N-1}) = 2$ whereas τ-$\mathrm{cat}(\mathbb{S}^{N-1}) = N$. Thus, as an easy consequence of Theorem 1.1 we have

Corollary 1.2. *Let Ω be symmetric with respect to the origin and such that $0 \notin \Omega$. Assume further that there is an odd map $\varphi : \mathbb{S}^{N-1} \to \Omega$. Then, for any $p \in (2, 2^*)$ fixed there exists $\overline{\lambda} = \overline{\lambda}(p)$ such that, for all $\lambda > \overline{\lambda}$, the problem (P_λ) has at least N pairs of odd solutions which change sign exactly once.*

The above results complement those of [9] where the authors considered the critical semilinear problem

$$-\Delta u = \lambda u + |u|^{2^*-2}u,\ u \in H_0^1(\Omega),\ u(\tau x) = -u(x) \text{ in } \Omega,$$

and obtained the same results for $\lambda > 0$ small enough. It also complement the aforementioned works that deal only with positive solutions. We finally note that Theorem 1.1 also holds if $\lambda \geq 0$ is fixed and the exponent p is sufficiently close to 2^* (see Remark 3.2).

2. NOTATIONS AND SOME TECHNICAL RESULTS

Throughout this paper, we denote by H the Hilbert space $H_0^1(\Omega)$ endowed with the norm $\|u\| = \left\{\int_\Omega |\nabla u|^2 \mathrm{d}x\right\}^{1/2}$. The involution τ of Ω induces an involution of H, which we also denote by τ, in the following way: for each $u \in H$ we define $\tau u \in H$ by

$$(\tau u)(x) = -u(\tau x). \tag{2.1}$$

We denote by $H^\tau = \{u \in H : \tau u = u\}$ the subspace of τ-invariant functions.

Let $E_\lambda : H \to \mathbb{R}$ be given by

$$E_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\, \mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p\, \mathrm{d}x,$$

and its associated Nehari manifold

$$\mathcal{N}_\lambda = \{u \in H \setminus \{0\} : \langle E_\lambda'(u), u\rangle = 0\} = \left\{u \in H \setminus \{0\} : \|u\|^2 + \lambda|u|_2^2 = |u|_p^p\right\}$$

where $|u|_s$ denote the $L^s(\Omega)$-norm for $s \geq 1$. In order to obtain τ-invariant solutions, we will look for critical points of E_λ restricted to the τ-invariant Nehari manifold

$$\mathcal{N}_\lambda^\tau = \{u \in \mathcal{N}_\lambda : \tau u = u\} = \mathcal{N}_\lambda \cap H^\tau,$$

by considering the following minimization problems

$$m_\lambda = \inf_{u \in \mathcal{N}_\lambda} E_\lambda(u) \text{ and } m_\lambda^\tau = \inf_{u \in \mathcal{N}_\lambda^\tau} E_\lambda(u).$$

For any τ-invariant bounded domain $\mathcal{D} \subset \mathbb{R}^N$ we define $E_{\lambda,\mathcal{D}}$, $\mathcal{N}_{\lambda,\mathcal{D}}$, $\mathcal{N}^\tau_{\lambda,\mathcal{D}}$, $m_{\lambda,\mathcal{D}}$ and $m^\tau_{\lambda,\mathcal{D}}$ in the same way by taking the above integrals over $\mathcal{D}$ instead Ω. For simplicity, we use only $m_{\lambda,r}$ and $m^\tau_{\lambda,r}$ to denote $m_{\lambda,B_r(0)}$ and $m^\tau_{\lambda,B_r(0)}$, respectively.

Lemma 2.1. *For any $\lambda \geq 0$, we have that $2m_\lambda \leq m_\lambda^\tau$.*

Proof. Note that, if $u \in H^\tau$ is positive in some subset $A \subset \Omega$, we can use (2.1) to conclude that u is negative in $\tau(A)$. Thus, for any given $u \in \mathcal{N}_\lambda^\tau$, we have that $u^+, u^- \in \mathcal{N}_\lambda$, where $u^\pm = \max\{\pm u, 0\}$. Hence $E_\lambda(u) = E_\lambda(u^+) + E_\lambda(u^-) \geq 2m_\lambda$, and the result follows. □

Lemma 2.2. *If u is a critical point of E_λ restricted to $\mathcal{N}_\lambda^\tau$, then $E'_\lambda(u) = 0$ in the dual space of H.*

Proof. By the Lagrange multiplier rule, there exits $\theta \in \mathbb{R}$ such that

$$\langle E'_\lambda(u) - \theta J'_\lambda(u), \phi \rangle = 0,$$

for all $\phi \in H^\tau$, where $J_\lambda(u) = \|u\|^2 + \lambda |u|_2^2 - |u|_p^p$. Since $u \in \mathcal{N}_\lambda^\tau$, we have

$$0 = \langle E'_\lambda(u), u \rangle - \theta \langle J'_\lambda(u), u \rangle = \theta(p-2)|u|_p^p.$$

This implies $\theta = 0$ and therefore $\langle E'_\lambda(u), \phi \rangle = 0$ for all $\phi \in H^\tau$. The result follows from the principle of symmetric criticality [10] (see also [11, Theorem 1.28]). □

By standard regularity theory we know that if u is a solution of (P_λ), then it is of class C^1. We say it changes sign k times if the set $\{x \in \Omega : u(x) \neq 0\}$ has $k+1$ connected components. By (2.1), if u is a nontrivial solution of problem (P_λ^τ) then it changes sign an odd number of times.

Lemma 2.3. *If u is a solution of problem (P_λ^τ) which changes sign $2k-1$ times, then $E_\lambda(u) \geq km_\lambda^\tau$.*

Proof. The set $\{x \in \Omega : u(x) > 0\}$ has k connected components $A_1, \ldots, A_k$. Let $u_i(x) = u(x)$ if $x \in A_i \cup \tau A_i$ and $u_i(x) = 0$, otherwise. We have that

$$0 = \langle E'_\lambda(u), u_i \rangle = \int_\Omega (\nabla u \nabla u_i + \lambda u u_i - |u|^{p-2} u u_i)\, \mathrm{d}x = \|u_i\|^2 + \lambda |u_i|_2^2 - |u_i|_p^p.$$

Thus, $u_i \in \mathcal{N}_\lambda^\tau$ for all $i = 1, \ldots, k$, and $E_\lambda(u) = E_\lambda(u_1) + \cdots + E_\lambda(u_k) \geq km_\lambda^\tau$, as desired. □

We recall now some facts about equivariant Ljusternik-Schnirelmann theory. An involution on a topological space X is a continuous function $\tau_X : X \to X$ such that τ_X^2 is the identity map of X. A subset A of X is called τ_X-invariant if $\tau_X(A) = A$. If X and Y are topological spaces equipped with involutions τ_X and τ_Y respectively, then an equivariant map is a continuous function

$f : X \to Y$ such that $f \circ \tau_X = \tau_Y \circ f$. Two equivariant maps $f_0, f_1 : X \to Y$ are equivariantly homotopic if there is an homotopy $\Theta : X \times [0,1] \to Y$ such that $\Theta(x,0) = f_0(x)$, $\Theta(x,1) = f_1(x)$ and $\Theta(\tau_X(x),t) = \tau_Y(\Theta(x,t))$, for all $x \in X, t \in [0,1]$.

Definition 2.4. *The equivariant category of an equivariant map $f : X \to Y$, denoted by (τ_X, τ_Y)-cat(f), is the smallest number k of open invariant subsets $X_1, \ldots, X_k$ of X which cover X and which have the property that, for each $i = 1, \ldots, k$, there is a point $y_i \in Y$ and a homotopy $\Theta_i : X_i \times [0,1] \to Y$ such that $\Theta_i(x,0) = f(x)$, $\Theta_i(x,1) \in \{y_i, \tau_Y(y_i)\}$ and $\Theta_i(\tau_X(x),t) = \tau_Y(\Theta_i(x,t))$ for every $x \in X_i$, $t \in [0,1]$. If no such covering exists we define (τ_X, τ_Y)-cat$(f) = \infty$.*

If A is a τ_X-invariant subset of X and $\iota : A \hookrightarrow X$ is the inclusion map we write

$$\tau_X\text{-cat}_X(A) = (\tau_X, \tau_X)\text{-cat}(\iota) \text{ and } \tau_X\text{-cat}(X) = \tau_X\text{-cat}_X(X).$$

The following properties can be verified.

Lemma 2.5. (i) *If $f : X \to Y$ and $h : Y \to Z$ are equivariant maps then*

$$(\tau_X, \tau_Z)\text{-cat}(h \circ f) \le \tau_Y\text{-cat}(Y).$$

(ii) *If $f_0, f_1 : X \to Y$ are equivariantly homotopic, then (τ_X, τ_Y)-cat$(f_0) = (\tau_X, \tau_Y)$-cat(f_1).*

Let V be a Banach space, M be a C^1-manifold of V and $I : V \to \mathbb{R}$ a C^1-functional. We recall that I restricted to M satisfies de Palais-Smale condition at level c ((PS)$_c$ for short) if any sequence $(u_n) \subset M$ such that $I(u_n) \to c$ and $\|I'(u_n)\|_* \to 0$ contains a convergent subsequence. Here we are denoting by $\|I'(u)\|_*$ the norm of the derivative of the restriction of I to M (see [11, Section 5.3]).

Let $\tau_a : V \to V$ be the antipodal involution $\tau_a(u) = -u$ on the vector space V. Equivariant Ljusternik-Schnirelmann category provides a lower bound for the number of pairs $\{u, -u\}$ of critical points of an even functional, as stated in the following abstract result (see [12, Theorem 1.1], [13, Theorem 5.7]).

Theorem 2.6. *Let $I : M \to \mathbb{R}$ be an even C^1-functional on a complete symmetric $C^{1,1}$-submanifold M of some Banach space V. Assume that I is bounded below and satisfies* (PS)$_c$ *for all $c \le d$. Then, if $I^d = \{u \in M : I(u) \le d\}$, the functional I has at least τ_a-cat$_{I^d}(I^d)$ antipodal pairs $\{u, -u\}$ of critical points with $I(\pm u) \le d$.*

3. Proofs of the results

Given $r > 0$, we define the sets

$$\Omega_r^+ = \{x \in \Omega : \text{dist}(x, \Omega) < r\} \text{ and } \Omega_r^- = \{x \in \Omega : \text{dist}(x, \partial\Omega \cup \Omega^\tau) \ge r\}.$$

Throughout the rest of the paper we fix $r > 0$ sufficiently small in such way that the inclusion maps $\Omega_r^- \hookrightarrow \Omega \setminus \Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences. Without loss of generality we suppose that $B_r(0) \subset \Omega$.

We now note that, in [1], Benci and Cerami considered the minimization problem

$$\widetilde{m}_\lambda = \inf \left\{ \int_\Omega (|\nabla u|^2 + \lambda u^2)\,\mathrm{d}x : u \in H, \int_\Omega |u|^p\,\mathrm{d}x = 1 \right\}.$$

An easy calculation show that $m_\lambda = \left(\frac{p-2}{2p}\right) \widetilde{m}_\lambda^{p/(p-2)}$. Therefore, if we denote by $\beta : H \setminus \{0\} \to \mathbb{R}^N$ the barycenter map given by

$$\beta(u) = \frac{\int_\Omega x \cdot |\nabla u(x)|^2\,\mathrm{d}x}{\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x},$$

we can rephrase [1, Lemma 3.4] as

Lemma 3.1. *For any fixed $p \in (2, 2^*)$ there exist $\overline{\lambda} = \overline{\lambda}(p)$ such that,*

(i) $m_{\lambda,r} < 2m_\lambda$,

(ii) *if $u \in \mathcal{N}_\lambda$ and $E_\lambda(u) \le m_{\lambda,r}$, then $\beta(u) \in \Omega_r^+$,*

for all $\lambda > \overline{\lambda}$.

We are now ready to present the proof of our main result.

Proof of Theorem 1.1. Let $p \in (2, 2^*)$ and $\overline{\lambda}$ be given by the Lemma 3.1. For any $\lambda > \overline{\lambda}$, since $2 < p < 2^*$, the even functional E_λ satisfies the Palais-Smale condition at any level $c \in \mathbb{R}$. Thus, we can apply Theorem 2.6 to obtain τ_a-cat$(\mathcal{N}_\lambda^\tau \cap E_\lambda^{2m_{\lambda,r}})$ pairs $\pm u_i$ of critical points of E_λ restricted to $\mathcal{N}_\lambda^\tau$ verifying

$$E_\lambda(\pm u_i) \le 2m_{\lambda,r} < 4m_\lambda \le 2m_\lambda^\tau,$$

where we used Lemma 3.1(i) and Lemma 2.1. It follows from Lemmas 2.2 and 2.3 that $\pm u_i$ are solutions of (P_λ^τ) which change sign exactly once.

It suffices now to check that

$$\tau_a\text{-cat}(\mathcal{N}_\lambda^\tau \cap E_\lambda^{2m_{\lambda,r}}) \ge \tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau).$$

With this aim, we claim that there exist two maps

$$\Omega_r^- \xrightarrow{\alpha_\lambda} \mathcal{N}_\lambda^\tau \cap E_\lambda^{2m_{\lambda,r}} \xrightarrow{\gamma_\lambda} \Omega_r^+$$

such that $\alpha_\lambda(\tau x) = -\alpha_\lambda(x)$, $\gamma_\lambda(-u) = \tau\gamma_\lambda(u)$, and $\gamma_\lambda \circ \alpha_\lambda$ is equivariantly homotopic to the inclusion map $\Omega_r^- \hookrightarrow \Omega_r^+$.

Assuming the claim and recalling that the maps $\Omega_r^- \hookrightarrow \Omega \setminus \Omega^\tau$ and $\Omega \hookrightarrow \Omega_r^+$ are equivariant homotopy equivalences, we can use Lemma 2.5 to get

$$\tau_a\text{-cat}(\mathcal{N}_\lambda^\tau \cap E_\lambda^{2m_{\lambda,r}}) \ge \tau\text{-cat}_{\Omega_r^+}(\Omega_r^-) = \tau\text{-cat}_\Omega(\Omega \setminus \Omega^\tau).$$

In order to prove the claim we follow [9]. Let $v_\lambda \in \mathcal{N}_{\lambda,B_r(0)}$ be a positive radial function such that $E_{\lambda,B_r(0)}(v_\lambda) = m_{\lambda,r}$. We define $\alpha_\lambda : \Omega_r^- \to \mathcal{N}_\lambda^\tau \cap E_\lambda^{2m_{\lambda,r}}$ by

$$\alpha_\lambda(x) = v_\lambda(\cdot - x) - v_\lambda(\cdot - \tau x). \tag{3.1}$$

It is clear that $\alpha_\lambda(\tau x) = -\alpha_\lambda(x)$. Furthermore, since v_λ is radial and τ is an isometry, we have that $\alpha_\lambda(x) \in H^\tau$. Note that, for every $x \in \Omega_r^-$, we have $|x - \tau x| \geq 2r$ (if this is not true, then $\overline{x} = (x + \tau x)/2$ satisfies $|x - \overline{x}| < r$ and $\tau\overline{x} = \overline{x}$, contradicting the definition of Ω_r^-). Thus, we can check that $E_\lambda(\alpha_\lambda(x)) = 2m_{\lambda,r}$ and $\alpha_\lambda(x) \in \mathcal{N}_\lambda^\tau$. All this considerations show that α_λ is well defined.

Given $u \in \mathcal{N}_\lambda^\tau \cap E_\lambda^{2m_{\lambda,r}}$ we can use (2.1) and the τ-invariance of Ω to conclude that $u^+ \in \mathcal{N}_\lambda$ and $2E_\lambda(u^+) = E_\lambda(u) \leq 2m_{\lambda,r}$. Hence, $u^+ \in \mathcal{N}_\lambda \cap E_\lambda^{m_{\lambda,r}}$ and it follows from Lemma 3.1(ii) that $\gamma_\lambda : \mathcal{N}_\lambda^\tau \cap E_\lambda^{2m_{\lambda,r}} \to \Omega_r^+$ given by $\gamma_\lambda(u) = \beta(u^+)$ is well defined. A simple calculation shows that $\gamma_\lambda(-u) = \tau\gamma_\lambda(u)$. Moreover, using (3.1) and the fact that v_λ is radial we get

$$\gamma_\lambda(\alpha_\lambda(x)) = \frac{\int_{B_r(x)} y \cdot |\nabla v_\lambda(y - x)|^2 \, dy}{\int_{B_r(x)} |\nabla v_\lambda(y - x)|^2 \, dy} = \frac{\int_{B_r(0)} (y + x) \cdot |\nabla v_\lambda(y)|^2 \, dy}{\int_{B_r(0)} |\nabla v_\lambda(y)|^2 \, dy} = x,$$

for any $x \in \Omega_r^-$. This concludes the proof. □

Remark 3.2. *Arguing along the same lines of the above proof and using a version of Lemma 4.2 in* [1] *instead of Lemma 3.1, we can check that Theorem 1.1 also holds if $\lambda \geq 0$ is fixed and the exponent p is sufficiently close to 2^*.*

Proof of Corollary 1.2. Let $\tau : \mathbb{R}^N \to \mathbb{R}^N$ be given by $\tau(x) = -x$. It is proved in [9, Corollary 3] that our assumptions imply τ-cat$(\Omega) \geq N$. Since $0 \notin \Omega$, $\Omega^\tau = \varnothing$. It suffices now to apply Theorem 1.1. □

References

[1] V. Benci and G. Cerami, The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems, *Arch. Rational Mech. Anal.* **114** (1991), 79–93.

[2] V. Benci, G. Cerami and D. Passaseo, On the number of positive solutions of some nonlinear elliptic problems, Nonlinear Analysis, A tribute in honour of G. Prodi, *Quaderno Scuola Norm. Sup.*, Pisa, 1991, 93–107

[3] V. Benci and G. Cerami, Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology, *Cal. Var. Partial Differential Equations* **2** (1994), 29–48.

[4] M. Clapp, On the number of positive symmetric solutions of a nonautonomous semilinear elliptic problem, *Nonlinear Anal.* **42** (2000), 405–422.

[5] O. Rey, A multiplicity result for a variational problem with lack of compactness, *Nonlinear Anal.* **13** (1989), 1241–1249.

[6] M. Lazzo, Solutions positives multiples pour une équation elliptique non linéaire avec léxposant critique de Sobolev, *C. R. Acad. Sci. Paris* **314** (1992), 61–64.

[7] C.O. Alves and Y.H. Ding, Multiplicity of positive solutions to a p-Laplacian equation involving critical nonlinearity, *J. Math. Anal. Appl.* **279** (2003), 508–521.

[8] T. Bartsch, Critical point theory in partially ordered Hilbert spaces, *J. Funct. Anal.* **186** (2001), 117–152.

[9] A. Castro and M. Clapp, The effect of the domain topology on the number of minimal nodal solutions of an elliptic equation at critical growth in a symmetric domain, *Nonlinearity* **16** (2003), 579–590.

[10] R.S. Palais, The principle of symmetric criticality, *Comm. Math. Phys.* **69** (1979), 19-30.

[11] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[12] M. Clapp and D. Puppe, Critical point theory with symmetries, *J. Reine. Angew. Math.* **418** (1991), 1–29.

[13] M. Struwe, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer, Berlin, 1990.

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