Evolution equations driven by a fractional white noise in spaces of abstract stochastic distributions

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Abstract:

In this paper we study stochastic evolution equations driven by a fractional white noise with arbitrary Hurst parameter in infinite dimension. We establish the existence and uniqueness of a mild solution for a nonlinear equation with multiplicative noise under Lipschitz condition by using a fixed point argument in an appropriate inductive limit space. In the linear case with additive noise a strong solution is obtained. Those results are applied to stochastic parabolic partial differential equations perturbed by a fractional white noise.

Key words and phrases: fractional Brownian motion, stochastic partial differential equations, white noise analysis.

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1 Introduction

We recall that a fractional Brownian motion with parameter $H \in (0,1)$ is a centered Gaussian process $\{\beta_H(t); t \geq 0\}$ with the following covariance structure

$$\mathbb{E}\left(\beta_H(t)\beta_H(s)\right) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{1.1}$$

One can show that a fractional Brownian motion β_H is a semimartingale if, and only if, $H = \frac{1}{2}$, where in this case it is a standard Brownian motion (see among others Lin (1995) and Decreusefond and Ustunel (1999) for its basic properties). Therefore, the classical stochastic calculus can not be applied. On the other hand, it turns out that an efficient white noise theory can be constructed. This was done by Bender (2003) and Elliot and van der Hoek (2003) dealing with problems related to the construction of the fractional Brownian motion and its respective stochastic calculus under the framework of the white noise analysis. Based on the Lindstrom´s representation (see Lindstrom (1993)), Elliot and van der Hoek developed a fractional white noise theory under the white noise measure μ for any $H \in (0,1)$ providing a simple Wiener chaos decomposition for the fractional Brownian motion. At the same time, Bender also constructed a consistent fractional white noise theory based on the Mandelbrot-van Ness representation (see Mandelbrot and van Ness (1968)). Other approaches can also be considered in order to construct a stochastic calculus for the fractional Brownian motion (see among others Feyel and de la Pradelle (1996), Alós et al (2001), Alós and Nualart (2003), Carmona et al (2003), or Zhale (1998)).

A natural extension of these problems is to study stochastic partial differential equations driven by a fractional white noise. For example, Maslowsky and Nualart (2003) studied the following stochastic parabolic equation

$$\frac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x) + f(u(t,x)) + b(u(t,x))\frac{\partial B^H(t,x)}{\partial t} \tag{1.2}$$

where f and b are coefficients with some regularity properties, $\mathcal{L}$ is a $2m$-th order elliptic differential operator on $L^2(\mathbb{R}^d)$ and B^H is a fractional Brownian motion with nuclear covariance operator. By using a path-wise approach and semigroup methods, they obtained a result of existence and uniqueness for (1.2) only if $H > \frac{d}{4m}$ for $H > \frac{1}{2}$. Linear stochastic evolution equations with additive noise driven by a cilindrical fractional Brownian motion were studied by Duncan et al (2002) and Tindel et al (2003) in the Skorohod sense.

By using methods of white noise analysis, Hu et al (2004) deal with a linear heat equation on $[0,\infty) \times \mathbb{R}^d$ with additive fractional white noise of arbitrary Hurst parameter $H = (h_0, h_1, \ldots, h_d) \in (0,1)^{d+1}$. Hu (2001) studied a linear heat equation with multiplicative fractional white noise for $\frac{1}{2} < H < 1$.

The purpose of this paper is to study stochastic equations in infinite dimension driven by a fractional noise under the framework of the white noise analysis. We study the following stochastic differential equation

$$\frac{dX(t)}{dt} = AX(t) + FX(t) + BX(t)\diamondsuit\mathbb{W}_H(t) \tag{1.3}$$

on the spaces of abstract stochastic distributions. Here F is a Lipschitz function, A the generator of a C_0-semigroup, B a bounded operator, $\diamondsuit$ the Wick product and $\mathbb{W}_H$ is the infinite-dimensional fractional white noise. This paper extends that of Filinkov and Sorensen (2002) to the case where the evolution equation is nonlinear with multiplicative fractional white noise with arbitrary Hurst parameter. By taking advantage of the works of Bender (2003) and Elliot and van der Hoek (2003), we establish existence and uniqueness of (1.3) for any $H \in (0,1)$. The results of our paper allow one to solve equation (1.2) for any $H \in (0,1)$ and $d \geqslant 1$ on the Kondratiev spaces by using Hermite transforms and fixed point arguments.

This paper is organized as follows. Section 2 contains some preliminares on fractional white noise theory in infinite dimension and Wick calculus. In section 3 the main results are presented together with the applications for stochastic partial differential equations.

2 Preliminaries

2.1 Stochastic Distributions in Infinite Dimension

In this section we define the basic setting to study infinite-dimensional stochastic equations under the framework of the white noise analysis. For a detailed account on white noise analysis and in particular Kondratiev spaces, we refer to Holden et al (1994) and Huang and Yan (2000). We start by recalling some basic definitions and properties of the fractional white noise theory in finite dimension. The underlying probability space is the white noise space $(\Omega, \Im, \mu)$, where Ω is $S'(\mathbb{R})$, the space of tempered distributions, $\Im$ is the σ-field generated by the weak-star topology of $S'(\mathbb{R})$ and μ is the unique probability measure such that

$$\int_{S'} e^{i\langle\omega,\phi\rangle}\mu(d\omega) = e^{-\frac{1}{2}|\phi|^2},$$

for all $\phi \in S(\mathbb{R})$, where $|\phi|^2 = \|\phi\|^2_{L^2(\mathbb{R})}$ and $\langle\omega,\phi\rangle = \omega(\phi)$ is the action of $\omega \in S'(\mathbb{R})$ on $\phi \in L^2(\mathbb{R})$. We will work on the Gaussian probability space $\big(\Omega, \Im, \mu; L^2(\mathbb{R})\big)$ together with the isonormal Gaussian process given by $\langle\cdot,\phi\rangle$ with $\phi \in L^2(\mathbb{R})$.

Since we are interested in investigating stochastic equations in infinite dimension under the framework of the white noise analysis, it is natural to work on the Kondratiev spaces where the distributions are sequences of an underlying Hilbert space. Fillinkov and Sorensen (2002) developed the Hilbert-valued white noise analysis, using the approach of Holden et al (1994). We briefly recall some basic definitions from Filinkov and Sorensen (2002). Let $(E, \| \, \|_E)$ be a separable real Hilbert space with an orthonormal basis $(e_i)_{i=1}^{\infty}$. Sometimes we write $\| \, \|$ instead of $\| \, \|_E$. We write $L^2(\mu; E)$ for the space of the square-integrable functions in the Bochner sense. Consider the generalized Hermite polynomials as follows

$$\mathbf{H}_\alpha(\omega) := \prod_{i=1}^{\infty} h_{\alpha_i}\left(\langle \omega, \xi_i \rangle\right); \ \alpha \in \mathcal{N},$$

where $\{h_n; n \in \mathbb{N}_0\}$ are the Hermite polynomials, $\{\xi_n; n \in \mathbb{N}\}$ are the Hermite functions consisting an orthonormal basis of $L^2(\mathbb{R})$ and $\mathcal{N}$ is the set of sequences α in $\mathbb{N}_0$ such that $|\alpha| = \sum_{i=1}^{\infty} \alpha_i < \infty$.

Since $\{\mathbf{H}_\alpha\}_{\alpha \in \aleph}$ is an ortoghonal basis of $L^2(\mu)$ it follows that the family $\{\mathbf{H}_\alpha e_i\}_{\alpha \in \mathcal{N}, i \in \mathbb{N}}$ is an ortoghonal basis of $L^2(\mu; E)$ with the following Hilbertian ortoghonal sum

$$L^2(\mu; E) = \overset{\infty}{\underset{i=1}{\oplus}} \mathbb{H}_i \otimes E,$$

where $\mathbb{H}_k$ is the closed linear span of the set $\{\mathbf{H}_\alpha; \alpha \in \Lambda_k\}$ with $\Lambda_k = \{\alpha \in \mathcal{N}; |\alpha| = k\}$. Therefore, for each $f \in L^2(\mu; E)$ there exists an unique sequence $\{c_{i\alpha}; i \in \mathbb{N}, \alpha \in \mathcal{N}\}$ of real numbers such that

$$f = \sum_{\alpha \in \mathcal{N}} \sum_{i=1}^{\infty} c_{i\alpha} \mathbf{H}_\alpha e_i = \sum_{\alpha \in \mathcal{N}} c_\alpha \mathbf{H}_\alpha = \sum_{i=1}^{\infty} f_i e_i \quad \in L^2(\mu; E).$$

where $f_i = \sum_{\alpha \in \mathcal{N}} c_{i\alpha} \mathbf{H}_a \in L^2(\mu)$ and $c_\alpha = \sum_{i=1}^{\infty} c_{i\alpha} e_i \in E$. If $\gamma \in \left(\mathbb{R}^{\mathbb{N}}\right)_c$ we denote $(2\mathbb{N})^\gamma := \prod_i (2i)^{\gamma_i}$. Under this setting, it is straightfoward to contruct the Kondratiev spaces in infinite dimension as follows.

Definition 2.1. i) For $\rho \in [0, 1]$ and $q \in \mathbb{N}$, we define $S(E)_{\rho,q}$ as the space of the functions

$$f = \sum_{\alpha \in \aleph} c_\alpha \mathbf{H}_\alpha = \sum_{i=1}^{\infty} f_i e_i \quad \in L^2(\mu; E)$$

such that

$$\|f\|_{\rho,q}^2 := \sum_{\alpha \in \mathcal{N}} \|c_\alpha\|^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{\alpha q} = \sum_{i=1}^{\infty} |f_i|_{\rho,q}^2 < \infty,$$

where $| \ |_{\rho,q}$ denotes the norm of the finite-dimensional Kondratiev space of test function $(S)_{\rho,k}$.

ii) For $\rho \in [0, 1]$ and $q \in \mathbb{N}$, we define $S(E)_{-\rho,-q}$ as the space of sequences (formal expansions) in E

$$F = \sum_{\alpha \in \mathcal{N}} c_\alpha \mathbf{H}_\alpha = \sum_{i=1}^{\infty} F_i e_i \quad (c_\alpha \in E)$$

such that

$$\|F\|^2_{-\rho,-q} := \sum_{\alpha\in\mathcal{N}} \|c_\alpha\|^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-\alpha q} = \sum_{i=1}^{\infty} |F_i|^2_{-\rho,-q} < \infty,$$

where $|\ \ |_{-\rho,-q}$ is the norm of $(S)_{-\rho,-q}$, the dual space of $(S)_{\rho,q}$.

Similarly to the finite-dimensional case, we can identify $S(E)_{-\rho,-q}$ as the dual space of $S(E)_{\rho,q}$. We also put

$$S(E)_\rho = \bigcap_{k=1}^{\infty} S(E)_{\rho,k}\ , \qquad S(E)_{-\rho} = \bigcup_{k=1}^{\infty} S(E)_{-\rho,-k}.$$

We take on $S(E)_\rho$ and $S(E)_{-\rho}$ the projective limit topology and inductive limit topology, respectively, induced by the Hilbert spaces $S(E)_{\beta,r}$ where $\beta \in [-1,1]$ and $r \in \mathbb{Z}$. We can regard $S(E)_{-\rho}$ as the dual of $S(E)_\rho$ by the action

$$\langle F|\, f\rangle := \sum_{\alpha\in\mathcal{N}} \langle b_\alpha, c_\alpha\rangle_E \alpha!$$

where $F = \sum_{\alpha\in\mathcal{N}} b_\alpha \mathbf{H}_\alpha \in S(E)_{-\rho}$ and $f = \sum_{\alpha\in\mathcal{N}} c_\alpha \mathbf{H}_\alpha \in S(E)_\rho$. Moreover we have the following inclusions

$$S(E)_1 \subset S(E)_\rho \subset S(E)_0 \subset L^2(\mu;E) \subset S(E)_{-0} \subset S(E)_{-\rho} \subset S(E)_{-1}\,.$$

2.2 Infinite Dimensional Fractional White Noise and Wick Calculus

As mentioned in the Introduction, Bender (2003) and Elliot and van der Hoek (2003) obtained a representation of the fractional Brownian motion in terms of the indicator function. The main difference between these works lies on measurability questions of the fractional Brownian motion constructed by each one. It should be noted that since we are working under the white noise setting, these questions make no difference for us. In fact, both constructions can be easily connected as pointed out by Blinder. We choose here to work with the framework of Elliot and van der Hoek. The main idea is to relate the fractional Brownian motion with parameter $H \in (0,1)$ to the classical Brownian motion via the following space

$$L^2_H(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R}; |x|^{\frac{1}{2}-H} \hat{f}(x) \in L^2(\mathbb{R}) \right\},$$

where $\hat{}$ denotes the Fourier transform on $L^2(\mathbb{R})$. It can be shown that the Schwartz space of test functions and indicator functions are contained in this space. Now the following operator is fundamental.

Definition 2.2. The operator $M = M_H$ is defined on functions $f \in L^2_H(\mathbb{R})$ by

$$\widehat{(Mf)}(y) = |y|^{1/2-H}\,\hat{f}(y); \quad y \in \mathbb{R}.$$

By considering

$$\tilde{\beta}_H(t,\omega) = \left\langle \omega, M\chi_{[0,t]} \right\rangle,$$

and taking a continuous modification $\beta_H(t)$ of the above process we arrive at the classical fractional Brownian motion with parameter $H \in (0,1)$ on $(\Omega, \Im, \mu)$ (see Elliot and van der Hoek (2003) and Hu et al (2004) for the details). We just mention here that since $\beta_H(t) \in L^2(\mu)$ for all $t \in \mathbb{R}_+$ and $H \in (0,1)$ we can develop its Wiener-Itô chaos expansion given by

$$\beta_H(t) = \sum_{k=1}^{\infty} \int_0^t M\xi_k(s)ds\mathbf{H}_{\varepsilon_k}, \tag{2.1}$$

where $\varepsilon_k = (0, 0, \ldots, 1, \ldots)$ with 1 in the k-th entry and 0 otherwise. One can show that $t \mapsto \beta_H(t)$ is differentiable on the Hida space $(S)_{-0}$ where

$$\frac{d\beta_H(t)}{dt} = \sum_{k=1}^{\infty} M\xi_k(t)\mathbf{H}_{\varepsilon_k}.$$

In order to obtain an easy expansion for the fractional Brownian motion in infinite dimension, we make use of the following bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$

$$n(i,j) = j + \frac{(i+j)(i+j+1)}{2}.$$

This "diagonal counting" will facilitate many computations. When dealing with stochastic equations in infinite dimension it is natural to work with a genuine cilindrical fractional Brownian motion. Let β_i^H be a sequence of real and independent fractional Brownian motions with the following expansion

$$\beta_i^H(t) = \sum_{j=1}^{\infty} \int_0^t M\xi_j(s)ds\mathbf{H}_{\varepsilon_{n(i,j)}}. \tag{2.2}$$

Note that different β_n^H-s have disjoint families of $\mathbf{H}_{\varepsilon_{n(i,j)}}$ in their representations. The cilindrical fractional Brownian motion on E is defined as the sum

$$B_H(t) := \sum_{i=1}^{\infty} \beta_i^H(t)e_i. \tag{2.3}$$

Note that the sum in (2.3) does not converge in $L^2(\mu; E)$. However, it does in $S(E)_{-0}$. To see this, we will rewrite equation (2.2) as

$$\beta_i^H(t) = \sum_{k=1}^{\infty} b_{ik}^H(t)\mathbf{H}_{\varepsilon_k}; \quad b_{ik}^H(t) = \begin{cases} \int_0^t M\xi_j(s)ds; \ k = n(i,j) \\ 0; \ \text{otherwise.} \end{cases}$$

Therefore, we can write

$$B_H(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} b_{ik}^H(t)\mathbf{H}_{\varepsilon_k} e_i = \sum_{k=1}^{\infty} b_k^H(t)\mathbf{H}_{\varepsilon_k},$$

$$b_k^H(t) = \delta_{k,n(i,j)} \int_0^t M\xi_j(s)ds e_i.$$

Now if $q \geqslant 3$, $t \in \mathbb{R}_+$, and $H \in (0,1)$ then

$$\begin{aligned} \|B_H(t)\|_{-0,-q}^2 &= \sum_{k=1}^{\infty} \left\|b_k^H(t)\right\|^2 (\varepsilon_k!) \left(2\mathbb{N}\right)^{-q\varepsilon_k} \\ &\leqslant \sum_{k=1}^{\infty} \left| \int_0^t M\xi_k(s)ds \right|^2 (2k)^{-q} \\ &\leqslant t^2 2^{-q} C^2 \sum_{k=1}^{\infty} k^{\frac{4}{3}-H-q} < \infty. \end{aligned}$$

Here we have used the estimate of Elliot and van der Hoek (2003)

$$|M\xi_n(t)| \leqslant Cn^{\frac{2}{3}-\frac{H}{2}} \quad \text{for all } t \in \mathbb{R}. \tag{2.4}$$

The fractional white noise in E is defined as

$$\mathbb{W}_H(t) := \sum_{i=1}^{\infty} W_i^H(t)e_i = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} d_{ik}^H(t)\mathbf{H}_{\varepsilon_k} e_i = \sum_{k=1}^{\infty} d_k^H(t)\mathbf{H}_{\varepsilon_k},$$

where

$$W_i^H(t) := \sum_{k=1}^{\infty} d_{ik}^H(t)\mathbf{H}_{\varepsilon_k}; \quad d_{ik}^H(t) = \begin{cases} M\xi_j(t); & k = n(i,j) \\ 0; & \text{otherwise.} \end{cases}$$

and $d_k^H(t) = \delta_{k,n(i,j)} M\xi_j(t)e_i$.

Similar to the fractional Brownian motion, one can easily show that $\mathbb{W}_H(t) \in S(E)_{-0}$ for all $t \in \mathbb{R}_+$, $H \in (0,1)$. Moreover, using the fact that $Mg \in C^\infty(\mathbb{R})$ for all $g \in S(\mathbb{R})$, we have

$$\frac{dB_H(t)}{dt} = \mathbb{W}_H(t) \quad \text{in } S(E)_{-0}.$$

Following Filinkov and Sorensen (2002) we define the Wick product on $S\,(E)_{-1}$. Consider $F, G \in S\,(E)_{-1}$ having forms

$$F = \sum_{\alpha \in \mathcal{N}} c_\alpha \mathbf{H}_\alpha = \sum_{i=1}^{\infty} F_i e_i, \quad G = \sum_{\beta \in \mathcal{N}} b_\beta \mathbf{H}_\beta = \sum_{i=1}^{\infty} G_i e_i$$

where $c_\alpha, b_\beta \in E$ and $F_i, G_i \in (S)_{-1}$.

Definition 2.3. The Wick product of elements $F, G \in S\,(E)_{-1}$ is

$$F \diamondsuit G := \sum_{i=1}^{\infty} (F_i \diamondsuit G_i) e_i = \sum_{\gamma \in \mathcal{N}} \theta_\gamma \mathbf{H}_\gamma,$$

where $\theta_\gamma = \sum\limits_{i=1}^{\infty} \sum\limits_{\alpha+\beta=\gamma} c_{i\alpha} b_{i\beta} e_i \in E$.

Similar to the finite-dimensional case, the space $S\,(E)_{-1}$ is invariant under Wick products. Furthermore, we have the following result which is an immediate consequence of the definitions. For completeness, we give the details here.

Proposition 2.1. *If $F \in S\,(E)_{-0}$ then $F \diamondsuit \mathbb{W}_H(t) \in S\,(E)_{-0}$, for $t \in \mathbb{R}_+,\ H \in (0,1)$*

Proof. Let $F = \sum\limits_{\alpha \in \aleph} c_\alpha \mathbf{H}_\alpha = \sum\limits_{i=1}^{\infty} F_i e_i$ and $t \in \mathbb{R}_+$. By definition

$$F \diamondsuit \mathbb{W}_H(t) = \sum_{\gamma \in \mathcal{N}} \sum_{i=1}^{\infty} \left(\sum_{\alpha+\varepsilon_k=\gamma} c_{i\alpha} d_{ik}^H(t) \right) e_i \mathbf{H}_\gamma.$$

Take $r > 1, p \geqslant 3$ and $q \in \mathbb{N}\,\{1\}$ such that $\|F\|_{-0,-q} < \infty$. Then

$$\begin{aligned}
&\sum_{\gamma \in \mathcal{N}} \gamma!\,(2\mathbb{N})^{-(q+p+r)\gamma} \sum_{i=1}^{\infty} \left(\sum_{\alpha+\varepsilon_k=\gamma} \left| c_{i\alpha} d_{ik}^H(t) \right| \right)^2 \\
&\leqslant \sum_{\gamma \in \mathcal{N}} \gamma!\,(2\mathbb{N})^{-(q+p+r)\gamma} \sum_{i=1}^{\infty} \sum_{\alpha+\varepsilon_k=\gamma} |c_{i\alpha}|^2 \sum_{\alpha+\varepsilon_k=\gamma} \left| d_{ik}^H(t) \right|^2 \\
&\leqslant \sum_{\gamma \in \mathcal{N}} (2\mathbb{N})^{-r\gamma} \sum_{i=1}^{\infty} \sum_{\alpha+\varepsilon_k=\gamma} \alpha!(\alpha_k+1)\,|c_{i\alpha}|^2\,(2\mathbb{N})^{-q(\alpha+\varepsilon_k)} \sum_{\alpha+\varepsilon_k=\gamma} \left| d_{ik}^H(t) \right|^2 (2\mathbb{N})^{-p(\alpha+\varepsilon_k)} \\
&\leqslant \sum_{\gamma \in \mathcal{N}} (2\mathbb{N})^{-r\gamma} \sum_{i=1}^{\infty} \left(\sum_{\alpha \in \mathcal{N}} \alpha!\,|c_{i\alpha}|^2\,(2\mathbb{N})^{-q\alpha} \right) \left(\sum_{k=1}^{\infty} \left| d_{ik}^H(t) \right|^2 (2k)^{-p} \right) \\
&\leqslant \sum_{\gamma \in \mathcal{N}} (2\mathbb{N})^{-r\gamma}\, \|F\|_{-0,-q}^2 \sum_{k=1}^{\infty} 2^{-p} C k^{\frac{4}{3}-H-p} < \infty.
\end{aligned}$$

Here we have used the estimate (2.4) and the well known fact that $\sum\limits_{\gamma \in \mathcal{N}} (2\mathbb{N})^{-r\gamma} < \infty$ when $r > 1$. $\square$

We now proceed to consider integrals in the Petti sense for $S(E)_{-0}$- valued functions.

Definition 2.4. A function $Z : \mathbb{R} \to S(E)_{-0}$ is called $S(E)_{-0}$- integrable if

$$\langle Z(\cdot)|\, f\rangle \in L^1(\mathbb{R}) \quad \text{for all } f \in S(E)_0 \,.$$

Then the $S(E)_{-0}$-integral of $Z(\cdot)$ denoted by $\int_{\mathbb{R}} Z(t)dt$ is the unique $S(E)_{-0}$ -element such that

$$\langle \int_{\mathbb{R}} Z(t)dt \,|f\rangle = \int_{\mathbb{R}} \langle Z(t) \mid f\rangle \, dt; \quad f \in S(E)_0 \,.$$

Remark 2.1 The existence of the unique element $\int_{\mathbb{R}} Z(t)dt$ follows from the fact that $\int_{\mathbb{R}} \langle Z(t) \mid \cdot\rangle \, dt$ is a continuous linear function on $S(E)_0$. This easily follows from Proposition 8.1 in Hida et al (1993) and Definition 2.1.

Definition 2.5. Suppose that $F : \mathbb{R} \to S(E)_{-0}$ is such that $t \mapsto F(t)\diamondsuit \mathbb{W}_H(t)$ is $S(E)_{-0}$ -integrable. Then we define the abstract fractional Hitsuda-Skorohod integral of F by

$$\int_{\mathbb{R}} F(t)\delta\mathbb{W}_H(t) := \int_{\mathbb{R}} F(t)\diamondsuit\mathbb{W}_H(t)dt.$$

Proposition 2.2. *Consider* $F : \mathbb{R} \to S(E)_{-0}$ *given by* $F(t) = \sum_{\alpha\in\mathcal{N}} b_\alpha(t)\boldsymbol{H}_\alpha$ *such that* $\{b_\alpha(\cdot)\,; \alpha \in \mathcal{N}\}$ *satisfies the following hypothesis*

$$\sup_{\alpha\in\mathcal{N}} \left\{ \alpha!\,(2\mathbb{N})^{-\alpha q} \int_{\mathbb{R}} \|b_\alpha(t)\|^2 \, dt \right\} < \infty \quad \text{for some } q \in \mathbb{N}.$$

Then $t \mapsto F(t)$ *is Hitsuda-Skorohod integrable for any* $H \in (0,1)$*.*

Proof. The proof easily follows from the fundamental estimative (2.4) and Lemma 2.5.7 from Holden et al (1994). We ommit the details.

□

3 Stochastic Evolution Equation with Fractional White Noise.

In this section we study a stochastic differential equation of the following form:

$$\frac{dX(t)}{dt} = AX(t) + FX(t) + BX(t)\diamondsuit\mathbb{W}_H(t); \quad X(0) = \theta \in S(E)_{-1}, \tag{3.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup on E, F is a Lipschitz function, B is a linear operator and $\mathbb{W}_H$ is the fractional white noise on E. Next we recall the following definition taken from Filinkov and Sorensen (2002).

Definition 3.1. i) Let B be a bounded linear operator on E. If $G = \sum\limits_{\alpha \in \mathcal{N}} c_\alpha \mathbf{H}_\alpha \in S(E)_{-\rho}$, we define

$$BG := \sum_{\alpha \in \mathcal{N}} B c_\alpha \mathbf{H}_\alpha.$$

ii) Let $(A, D(A))$ be an unbounded linear operator densely defined on E. If $G = \sum\limits_{\alpha \in \mathcal{N}} c_\alpha \mathbf{H}_\alpha \in S(E)_{-\rho}$, is such that $c_\alpha \in D(A)$ for all $\alpha \in \mathcal{N}$ and

$$\sum_{\alpha \in \mathcal{N}} \|A c_\alpha\|^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-\alpha q} < \infty \text{ for some } q \in \mathbb{N},$$

then we write

$$AG := \sum_{\alpha \in \mathcal{N}} A c_\alpha \mathbf{H}_\alpha,$$

and we denote by $D(A)_{-\rho}$ the set of $G \in S(E)_{-\rho}$ safisfying the above property. Similarly, if $F : E \to E$ is a Lipschitz mapping with linear growth then we can define

$$FG := \sum_{\alpha \in \mathcal{N}} F c_\alpha \mathbf{H}_\alpha,$$

for $G = \sum\limits_{\alpha \in \mathcal{N}} c_\alpha \mathbf{H}_\alpha \in S(E)_{-1}$.

We want to study strong and generalized solutions of (3.1) in the largest space $S(E)_{-1}$. By the strong solution we mean a continuously differentiable $S(E)_{-1}$- valued function satysfing (3.1). By the generalized solution, we mean the so-called *mild* form of the equation given by

$$X(t) = S(t)\theta + \int_0^t S(t-s) F X(s) ds + \int_0^t S(t-s) B X(s) \diamondsuit \mathbb{W}_H(s) ds, \qquad (3.2)$$

where $\{S(t); 0 \leqslant t \leqslant T\}$ is the strongly continuous semigroup generated by A and the integrals above are considered in the inductive limit topology of $S(E)_{-1}$. It is easy to see, that a strong solution should be given by (3.2). The statement of our main existence and uniqueness theorem follows.

Theorem 3.1. *Let $(A, D(A))$ be an infinitesimal generator of a strongly continuous semigroup on E, $F : E \to E$ a Lipschitz function with linear growth and B a bounded operator on E such that there exists an orthonormal basis $(e_i)_{i=1}^\infty$*

and a sequence of real numbers $(\lambda_i)_{i=1}^{\infty}$ *such that* $Be_i = \lambda_i e_i$ *for all* $i \in \mathbb{N}$. *Then for any* $H \in (0,1)$ *and for each* $\theta \in S(E)_{-1}$ *there exists a unique mild solution* $(X(t))_{0 \leqslant t \leqslant T}$ *of (3.1) taking values on* $S(E)_{-1}$.

Remark 3.1 One may want to consider B as a nonlinear mapping. In fact, it is possible to prove Theorem 3.1 under other regularity assumptions. For example, suppose the existence of $\ell \in L^2_{loc}(\mathbb{R}_+)$ such that

i) $\|S(t)B(x)\|_E^2 \leqslant \left(1 + \|x\|_E^2\right) \ell^2(t); \quad t > 0,\ x \in E,$
ii) $\|S(t)Bx - S(t)By\|_E \leqslant \ell(t)\, \|x - y\|_E\,; \quad t > 0,\ x, y \in E.$

Then one can prove Theorem 3.1 by applying similar arguments used in this paper. We just have to ensure that $s \mapsto S(t-s)BX(s) \diamondsuit \mathbb{W}_H(s)$ is a continuous function when $X(\cdot)$ is continuous.

Remark 3.2 If $F(0) = 0$ in the above theorem, one can show that the solution $(X(t))_{0 \leqslant t \leqslant T}$ takes values on the Hida space $S(E)_{-0}$. This is easily proved by checking the proof of Theorem 3.1 with this hypothesis.

Remark 3.3 Other types of nonlinearities can also be considered for equation 3.1 with additive noise. For example, one may want to take for $F : D(F) \to E$ be a dissipative function. In this case, it is possible to prove Theorem 3.1 by imposing suitable dissipative hypotheses on the coefficients A and F. The strategy consists in approximating the solution of (3.2) by the Yosida approximations of F by Lipschitz functions.

If we consider equation (3.1) driven by an additive noise with $F \equiv 0$, we find a strong solution.

Theorem 3.2. *Let* $(A, D(A))$ *be an infinitesimal generator of a strongly continuous semigroup on* E *and* B *a bounded linear operator on* E. *Then for any* $H \in (0,1)$ *and for each* $\theta \in D(A)_{-1}$ *there exists a unique strong solution* $(X(t))_{0 \leqslant t \leqslant T}$ *of (3.1) taking values on* $S(E)_{-1}$ *given by*

$$X(t) = S(t)\theta + \int_0^t S(t-s)B\delta\mathbb{W}_H(s).$$

Before proving the theorems we now apply it to parabolic stochastic equations.

Example 3.3.

Let O be a bounded open subset of $\mathbb{R}^d$ with smooth boundary. Consider the following 2m-th order differential operator given by

$$L = \sum_{|\alpha| \leqslant m, |\beta| \leqslant m} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta} D^{\beta}),$$

where $a_{\alpha\beta}$ are smooth real functions on $\bar{O}$. We assume that L is strongly elliptic on $\bar{O}$. In this case, it is well known that the operator $-L$ defined on the Sobolev spaces $H^{2m}(O) \cap H_0^{2m}(O) \subset L^2(O)$ generates a strongly continuous semigroup on $L^2(O)$. Given $0 < T < \infty$ and $H \in (0,1)$ consider the following Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} + Lu(t,x) = f(u(t,x)) + b(u(t,x)) \diamondsuit W_H(t,x); (t,x) \in [0,T] \times O,$$

$$u(0,x) = u_0(x); \quad x \in \ O \subset \mathbb{R}^d, \tag{3.3}$$

where f is assumed to be a real valued Lipschitz function, b a suitable real function and W_H a fractional white-noise in time for fixed $x \in O$. It should be noted that since we are dealing with stochastic distributions the above mappings are acting on each element of their formal expansions. To apply Theorem 3.1, we only need to represent the system (3.3) in the infinite-dimensional form

$$\frac{dX(t)}{dt} = AX(t) + FX(t) + BX(t) \diamondsuit \mathbb{W}_H(t); \quad t \in [0,T], H \in (0,1),$$

$$X(0) = u_0 \in E, \tag{3.4}$$

in the standard way. We take $E = L^2(O)$ and set

$$(By)(x) := b(y(x)),$$

$$A\varphi := -L\varphi, \quad \varphi \in D(L); \quad F(y(\xi)) := f(y(\xi)), \quad y \in E; x, \xi \in O,$$

with b a real function which makes B a bounded operator on E satisfying the hypothesis of Theorem 3.1. Alternatively, B can be an integral operator of Hilbert-Schmidt type.

At this point, we would like to impose conditions on the coefficients of equation (3.3) to obtain a solution taking values in $L^2(\mu; E)$. The difficulty in finding these conditions is notorious: Due to the multiplicative noise with Wick product even if we impose on B a finite rank hypothesis such that $BY \in L^2(\mu; E)$ for all $Y \in S(E)_{-1}$ we can not ensure that

$$\int_0^t S(t-s)BX(s) \diamondsuit \mathbb{W}_H(s)ds \in L^2(\mu; E),$$

for all $t \in [0,T]$. This happens because of the irregularity of the Wick product on $L^p(\mu; E)$ spaces. In the linear case with additive noise we have a strong solution which is given by the generalized stochastic convolution. In this case, regularity properties can be studied in a suitable way. Let us discuss the particular case of the Laplacian on $O = \{x \in \mathbb{R}^d ; 0 < x_i < a_i, i = 1, \ldots, d\}$. In this case, we have $D(\Delta) = H^{2,2} \cap H_0^{1,2}$ where Δ has eigenvalues

$$\Upsilon = \left\{ -\sum_{i=1}^{d} \frac{k_i \pi}{a_i^2}; (k_1, \ldots, k_d) \in \mathbb{N}^d \right\},$$

with respective eigenvectors which is also an orthonormal basis of E

$$g_{k_1,\ldots,k_d}(x) = \prod_{i=1}^{d} \left(\frac{2}{a_i} \right)^{1/2} \left(sen \left(\frac{k_i \pi x_i}{a_i} \right) \right).$$

For simplicity we write $\left\{ -\widetilde{\lambda}_k \right\}_{k=1}^{\infty}$ and $\{\widetilde{e}_k\}_{k=1}^{\infty}$ to denote an ordering of the eigenvalues and eigenvectors, respectively. The Laplacian generates a C_0-semigroup given by

$$S(t)y = \sum_{k=1}^{\infty} \langle y,_k \rangle_E \exp(-\widetilde{\lambda}_k t) \widetilde{e}_k; \quad y \in E.$$

The solution of the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + W_H(t,x); \quad u(0,x) = \theta_0(x) \in D(\Delta),$$

is given by

$$X(t) = S(t)\theta_0 + \sum_{k=1}^{\infty} \int_0^t S(t-s) d_k^H(s) ds \mathbf{H}_{\varepsilon_k}.$$

Now it is easy to see that if $d = 1$ and $\frac{1}{3} < H < 1$, then $X(t) \in L^2(\mu; E)$ for all $t \in [0, T]$.

3.1 Proofs of Theorems 3.1 and 3.2

Now we aim at proving the existence and uniqueness Theorems 3.1 and 3.2. Due to the nonlinearity, the well-known strategy of taking the Hermite transform does not work out in Theorem 3.1. The difficulty here lies on the fact that we can not identify a deterministic equation with complex coefficients when dealing with the Hermite transform of nonlinear functions. In this case, the proof of Theorem 3.1 relies on the following extension of the Banach' s fixed point theorem taken from Deck (2002) which we enunciate here for sake of completeness. At first, we begin with a definition.

Definition 3.2. Let $J = \bigcup_{n=1}^{\infty} J_n$ be an inductive limit of Banach spaces with norms $\| \ \|_n$. We say that $\mathcal{K} : J \to J$ is a strict contraction on J if there exists $n_0 \in \mathbb{N}$ such that $\forall n > n_0$,
i) $\mathcal{K}(J_n) \subset J_n$;
ii) There exists $c_n \in [0,1)$ such that $\forall x, y \in J_n$, we have

$$\|\mathcal{K}x - \mathcal{K}y\|_n \leqslant c_n \|x - y\|_n .$$

By using the Banach structure of the intermediate spaces J_n, Deck (2002) proved that if $\mathcal{K} : J \to J$ is a strict contraction on J, then for each $u \in J$ there exists a unique $v_u \in J$ which is the solution of the fixed point problem

$$v = u + \mathcal{K}(v).$$

Let $C(0, T; S(E)_{-1,-q})$ be the Banach space of $S(E)_{-1,-q}$-valued continuous functions defined on $[0, T]$. We take on this space the usual topology given by the norm

$$\|F\|_{-1,-q,\infty} := \sup_{0 \leqslant t \leqslant T} \|F(t)\|_{-1.-q} ,$$

and we consider the following inductive limit of Banach spaces

$$C(0, T; S(E)_{-1}) = \bigcup_{q=1}^{\infty} C(0, T; S(E)_{-1,-q}).$$

Next we will make use of the Hermite transform on the space $S(E)_{-1}$. For a detailed account of this topic in finite and infinite dimension we refer to Holden et al (1994) and Filinkov and Sorensen (2002), respectively. We just fix here the basic notation which will be used throughout this paper. We denote by $E_{\mathbb{C}}$ the complexification of the Hilbert space E.

If $Y = \sum_{\alpha \in \mathcal{N}} c_\alpha \mathbf{H}_\alpha \in S(E)_{-1}$ then the Hermite transform of Y is defined as

$$\mathcal{H}Y(z) = \widetilde{Y}(z) := \sum_{\alpha \in \mathcal{N}} c_\alpha z^\alpha; \quad z \in \mathbb{C}^{\mathbb{N}},$$

when convergent in $E_{\mathbb{C}}$. Here we use the following notation: If $\alpha \in \mathcal{N}$ and $z \in \mathbb{C}^{\mathbb{N}}$ then we write $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \cdots$. Next we consider the following infinite-dimensional neighborhoods of 0 in $\mathbb{C}^{\mathbb{N}}$ as follows

$$\mathbb{K}_q := \left\{ z \in \mathbb{C}^{\mathbb{N}}; |z_i| < (2i)^{-q}, i \in \mathbb{N} \right\}; \quad q \in \mathbb{N}.$$

Lemma 3.3. *Let B be a bounded operator on E satisfying the hypothesis of Theorem 3.1, and fix $H \in (0,1)$. If $X \in C(0, T; S(E)_{-1})$ then for all $t \in [0, T]$*

$$s \mapsto S(t - s) B X(s) \diamondsuit \mathbb{W}_H(s) \in C(0, t; S(E)_{-1}) .$$

Proof. We will denote by C universal constants which may differ from line to line. Let $X(t) = \sum_{\alpha\in\mathcal{N}} c_\alpha(t)\mathbf{H}_\alpha$ and $BX(t) = \sum_{\alpha\in\mathcal{N}} Bc_\alpha(t)\mathbf{H}_\alpha$. First of all, we are going to use the hypothesis on B to express $BX(t)\diamondsuit W_H(t)$ in a suitable way. By hypothesis we can find a complete orthonormal system $(e_i)_{i=1}^\infty$ on E and a sequence of real numbers $(\lambda_i)_{i=1}^\infty$ such that $Be_i = \lambda_i e_i$ for all $i \in \mathbb{N}$. Then we have that

$$BX(t) = \sum_{\alpha\in\mathcal{N}} Bc_\alpha(t)\mathbf{H}_\alpha = \sum_{i=1}^{\infty} (BX)_i(t)e_i,$$

where $(BX)_i(t) := \sum_{\alpha\in\mathcal{N}} c_{i\alpha}(t)\lambda_i\mathbf{H}_\alpha \in (S)_{-1}$.
Therefore,

$$\begin{aligned} BX(t)\diamondsuit\mathbb{W}_H(t) &= \sum_{i=1}^{\infty}\left[(BX)_i(t)\diamondsuit W_i^H(t)\right]e_i \\ &= \sum_{i=1}^{\infty}\left[\sum_{\gamma\in\mathcal{N}}\left(\sum_{\alpha+\varepsilon_k=\gamma} c_{i\alpha}(t)\lambda_i d_{ik}^H(t)\right)\mathbf{H}_\gamma\right]e_i, \end{aligned}$$

where

$$\mathcal{H}\left[BX(t)\diamondsuit\mathbb{W}_H(t)\right](z) = \sum_{i=1}^{\infty}\left[\mathcal{H}(BX)_i(t,z).\mathcal{H}\left(W_i^H\right)(t,z)\right]e_i, \tag{3.5}$$

for each $z \in \mathbb{C}^{\mathbb{N}}$ such that $\mathcal{H}(BX)_i(t,z)$ and $\mathcal{H}\left(W_i^H\right)(t,z)$ exist. We can rewrite equation (3.5) as

$$\mathcal{H}\left[BX(t)\diamondsuit\mathbb{W}_H(t)\right](z) = \sum_{i=1}^{\infty}\left\langle B\tilde{X}(t,z), e_i\right\rangle_{E_{\mathbb{C}}} e_i.\widetilde{W}_i^H(t,z)$$

for each $z \in \mathbb{C}^{\mathbb{N}}$ such that $\mathcal{H}(BX)_i(t,z)$ and $\mathcal{H}\left(W_i^H\right)(t,z)$ exist. Note that $W_i^H(t,z) = \sum_{k=1}^{\infty} d_{ik}^H(t)z_k$ converges absolutely in $\mathbb{C}$ for each $z \in \mathbb{K}_q$ $(q \geqslant 2)$. Fix $q \geqslant 2$ and consider the following family of strongly continuous bounded linear operators $\{\mathcal{T}_z(t); 0 \leqslant t \leqslant T, z \in \mathbb{K}_q\}$ given by

$$\mathcal{T}_z(t)x := \sum_{i=1}^{\infty}\langle Bx, e_i\rangle_{E_{\mathbb{C}}} e_i.\sum_{k=1}^{\infty} d_{ik}^H(t)z_k, \tag{3.6}$$

where $\mathcal{T}_z(t)x$ converges absolutely on $E_{\mathbb{C}}$ for each $(t,z,x) \in [0,T]\times\mathbb{K}_q\times E_{\mathbb{C}}$, with

$$\left\|\mathcal{T}_z(t)x\right\|_{E_{\mathbb{C}}} \leqslant \|B\|\, C2^{-q}\,\|x\|_{E_{\mathbb{C}}}\sum_{k=1}^{\infty} k^{\frac{2}{3}-\frac{H}{2}-q},$$

for all $x \in E_{\mathbb{C}}$. We rewrite equation (3.6) as

$$\mathcal{T}_z(t)x = \sum_{k=1}^{\infty} \left\langle Bx\delta_{k,n(i,j)} M\xi_j(t) z_k, e_i \right\rangle_{E_{\mathbb{C}}} e_i.$$

Given $X \in C\left(0,T;S(E)_{-1}\right)$, there exists a $q' \in \mathbb{N}$ ($q' \geqslant 2$) such that

$$\sum_{\alpha \in \mathcal{N}} \|c_\alpha(s)\|_{E_{\mathbb{C}}} |z^\alpha| \leqslant \sup_{0 \leqslant v \leqslant T} \|X(v)\|_{-1,-q'} \left(\sum_{\alpha \in \mathcal{N}} (2\mathbb{N})^{-\alpha q'} \right)^{1/2} < \infty$$

for all $s \in [0,T]$ and $z \in \mathbb{K}_{q'}$. Then for each $(t,z) \in [0,T] \times \mathbb{K}_{q'}$ we have that

$$S(t-s)\mathcal{T}_z(s)\widetilde{X}(s,z) = \sum_{k=1}^{\infty} S(t-s) \left\langle B\widetilde{X}(s,z)\delta_{k,n(i,j)} M\xi_j(s) z_k, e_i \right\rangle_{E_{\mathbb{C}}} e_i \in E_{\mathbb{C}}.$$

We state that given $t \in [0,T]$,

$$s \mapsto S(t-s)\mathcal{T}_z(s)\widetilde{X}(s,z) \in E_{\mathbb{C}}$$

is continuous for each $z \in \mathbb{K}_{q'}$. To prove this, we write

$$g_k(s,z) := S(t-s) \left\langle B\widetilde{X}(s,z)\delta_{k,n(i,j)} M\xi_j(s) z_k, e_i \right\rangle_{E_{\mathbb{C}}} e_i.$$

Note that $\|g_k(s,z)\|_{E_{\mathbb{C}}} \leqslant Ck^{\frac{2}{3}-\frac{H}{2}-q'}$ $\forall (s,z) \in [0,t] \times \mathbb{K}_{q'}$, where

$$\sum_{k=1}^{\infty} k^{\frac{2}{3}-\frac{H}{2}-q'} < \infty.$$

By the M-Test of Weierstrass, $\sum_{k=1}^{\infty} g_k(\cdot,z)$ converges uniformly on $[0,t]$ for each $z \in \mathbb{K}_{q'}$, thus proving our statement.

Since $\mathcal{H}\left(BX\right)_i(s,z)$ and $\mathcal{H}\left(W_i^H\right)(s,z)$ exist on $[0,t] \times \mathbb{K}_{q'}$ for each $i \in \mathbb{N}$ it follows that

$$\mathcal{H}\left[S(t-s)BX(s) \diamondsuit \mathbb{W}_H(s)\right](z) = S(t-s)\mathcal{T}_z(s)\widetilde{X}(s,z).$$

We have thus proved that $\mathcal{H}\left[S(t-s)BX(s) \diamondsuit \mathbb{W}_H(s)\right](z)$ is continuous on $[0,t]$ and bounded on $[0,t] \times \mathbb{K}_{q'}$. By Theorem 1 of Filinkov and Sorensen (2002) we can conclude that

$$s \mapsto S(t-s)BX(s) \diamondsuit \mathbb{W}_H(s) \in S(E)_{-1}$$

is a continuous function. $\square$

Proof of Theorem 3.1

The strategy consists in proving that the mapping

$$\mathcal{K}(X)(t) := \int_0^t S(t-s)FX(s)ds + \int_0^t S(t-s)BX(s)\diamondsuit\mathbb{W}_H(s)ds$$

is a strict contraction on the inductive limit space $C\left(0,T;S(E)_{-1}\right)$. At first, we have to show that the above mapping is well-defined. Throughout this proof we denote by N a generic constant which my differ from line to line. For simplicity we sometimes write $C_{-1,-q}$ and C_{-1} instead of $C\left(0,T;S(E)_{-1,-q}\right)$ and $C\left(0,T;S(E)_{-1}\right)$, respectively. To check the continuity properties we will make use the Hermite transform. Fix $X \in C\left(0,T;S(E)_{-1}\right)$ with form

$$X(s) = \sum_{\alpha\in\mathcal{N}} c_\alpha(s)\mathbf{H}_\alpha.$$

It is easy to see that $s \mapsto S(t-s)FX(s) \in S(E)_{-1}$ is continuous for each $t \in [0,T]$. By Lemma 3.3 the mapping $s \mapsto S(t-s)BX(s)\diamondsuit\mathbb{W}_H(s) \in S(E)_{-1}$ is continuous and therefore

$$\int_0^t S(t-s)FX(s)ds = \sum_{\alpha\in\mathcal{N}} \int_0^t S(t-s)Fc_\alpha(s)ds\mathbf{H}_\alpha$$

$$\int_0^t S(t-s)BX(s)\diamondsuit\mathbb{W}_H(s)ds = \sum_{\gamma\in\mathcal{N}} \int_0^t S(t-s)\left(BX(s)\odot\mathbb{W}_H(s)\right)_\gamma ds\mathbf{H}_\gamma$$

where

$$\left(BX(s)\odot\mathbb{W}_H(s)\right)_\gamma := \sum_{i=1}^{\infty}\left[\sum_{\alpha+\varepsilon_k=\gamma} \langle c_\alpha(s), Be_i\rangle_E\, d_{ik}^H(s)\right] e_i \in E.$$

By using the Hermite transform, clearly the integrals above are continuous $S(E)_{-1}$-valued functions. Therefore, $\mathcal{K} : C_{-1} \to C_{-1}$ is well-defined.

Next we will show that if $X \in C_{-1,-q}$ then $\int_0^{\cdot} S(t-s)FX(s)ds \in C_{-1,-q}$ for $q \geqslant 2$. Fix $t \in [0,T]$. By using the linear growth condition and the fact that $\sup_{0\leqslant v\leqslant T} \|X(v)\|_{-1,-q} < \infty$, we arrive at the following estimative

$$\sum_{\alpha\in\mathcal{N}} \left\|\int_0^t S(t-s)Fc_\alpha(s)ds\right\|_E^2 (2\mathbb{N})^{-\alpha q} \leqslant N\sum_{\alpha\in\mathcal{N}}(2\mathbb{N})^{-\alpha q} + NT < \infty.$$

Therefore, we can conclude that $\int_0^t S(t-s)FX(s)ds \in S(E)_{-1.-q}$ for all $t \in [0,T]$. Next we state that if $X \in C_{-1,-q}$ with $q \geqslant 3$, then

$$\int_0^t S(t-s)BX(s)\diamondsuit\mathbb{W}_H(s)ds \in S(E)_{-1,-q}$$

for all $t\in[0,T]$. To see this, fix $t\in[0,T]$. We have that

$$\begin{aligned}
&\sum_{\gamma\in\mathcal{N}}\left\|\int_0^t S(t-s)\left(BX(s)\odot\mathbb{W}_H(s)\right)_\gamma ds\right\|_E^2 (2\mathbb{N})^{-\gamma q}\\
&\leqslant NT\sum_{\gamma\in\mathcal{N}}\int_0^t \left\|\left(BX(s)\odot\mathbb{W}_H(s)\right)_\gamma\right\|_E^2 (2\mathbb{N})^{-\gamma q}\\
&\leqslant NT\sum_{\gamma\in\mathcal{N}}\sum_{i=1}^{\infty}\sum_{\alpha+\varepsilon_k=\gamma}\left(\int_0^t \|c_\alpha(s)\|_E^2\left|d_{ik}^H(s)\right|^2 ds\right)(2\mathbb{N})^{-\gamma q}\\
&= NT\sum_{i=1}^{\infty}\sum_{\alpha,\varepsilon_k}\int_0^t \|c_\alpha(s)\|_E^2\left|d_{ik}^H(s)\right|^2 ds\,(2\mathbb{N})^{-(\alpha+\varepsilon_k)q}\\
&\leqslant NT\sum_{i=1}^{\infty}\sum_{\alpha,\varepsilon_k}\delta_{k,n(i,j)}j^{\frac{4}{3}-H}\int_0^t \|c_\alpha(s)\|_E^2 ds\,(2\mathbb{N})^{-(\alpha+\varepsilon_k)q}\\
&= NT\sum_{i=1}^{\infty}\left(\sum_{k=1}^{\infty}\delta_{k,n(i,j)}j^{\frac{4}{3}-H}(2k)^{-q}\right)\left(\sum_{\alpha\in\mathcal{N}}\int_0^t \|c_\alpha(s)\|_E^2 ds\,(2\mathbb{N})^{-\alpha q}\right)\\
&\leqslant NT^2\left(\sum_{k=1}^{\infty}k^{\frac{4}{3}-H-q}\right)\|X(v)\|_{-1,-q,\infty}^2<\infty.
\end{aligned}$$

Here we have used the estimate (2.4) and the fact that $X\in C_{-1,-q}$. Therefore, we can conclude that $\mathcal{K}\left(C_{-1,-q}\right)\subset C_{-1,-q}$ for all $q\geqslant 3$. Now fix $q\geqslant 3$ and consider $X,Y\in C_{-1,-q}$ with forms

$$X(t)=\sum_{\alpha\in\mathcal{N}}c_\alpha(t)\mathbf{H}_\alpha,\quad Y(t)=\sum_{\alpha\in\mathcal{N}}b_\alpha(t)\mathbf{H}_\alpha.$$

By the linearity of B and the distributive law of the Wick product on $S(E)_{-1}$ we have that

$$\begin{aligned}
\left[\mathcal{K}(X)-\mathcal{K}(Y)\right](t) &= \int_0^t S(t-s)\left[FX(s)-FY(s)\right]ds\\
&+\int_0^t S(t-s)B\left[X(s)-Y(s)\right]\diamondsuit\mathbb{W}_H(s)ds
\end{aligned}$$

Then

$$\begin{aligned}\|[\mathcal{K}(X)-\mathcal{K}(Y)](t)\|_{-1,-q} &\leqslant \left\| \int_0^t S(t-s)\left[FX(s)-FY(s)\right]ds \right\|_{-1,-q} \\ &+ \left\| \int_0^t S(t-s)B\left[X(s)-Y(s)\right]\diamondsuit \mathbb{W}_H(s)ds \right\|_{-1,-q}.\end{aligned} \tag{3.7}$$

We will analyse the two parts of the above inequality separately. Let us estimate the first one. By using the usual estimatives for the C_0-semigroup and the global Lipschitz hypothesis on F, we have that

$$\left\| \int_0^t S(t-s)\left[Fc_\alpha(s)-Fb_\alpha(s)\right]ds \right\|_E^2 \leqslant NT^2 \sup_{0\leqslant u\leqslant t} \|c_\alpha(u)-b_\alpha(u)\|_E^2,$$

where $c_\alpha(\cdot)$ and $b_\alpha(\cdot)$ are continuous functions on $[0,T]$. Therefore,

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T}\sum_{\alpha\in\mathcal{N}} \left\| \int_0^t S(t-s)\left[Fc_\alpha(s)-Fb_\alpha(s)\right]ds \right\|_E^2 (2\mathbb{N})^{-\alpha q} \\ &\leqslant NT^2 \sup_{0\leqslant t\leqslant T}\sum_{\alpha\in\mathcal{N}} \sup_{0\leqslant u\leqslant t}\|c_\alpha(u)-b_\alpha(u)\|_E^2 (2\mathbb{N})^{-\alpha q} \\ &= NT^2 \sup_{0\leqslant v\leqslant T}\sum_{\alpha\in\mathcal{N}} \|c_\alpha(u)-b_\alpha(u)\|_E^2 (2\mathbb{N})^{-\alpha q} = NT^2\|X-Y\|_{-1,-q,\infty}^2.\end{aligned}$$

Let us estimate the second part. By Lemma (3.3), given $t\in[0,T]$, we have that

$$\begin{aligned}&\int_0^t S(t-s)B\left[X(s)-Y(s)\right]\diamondsuit \mathbb{W}_H(s)ds \\ &= \sum_{\gamma\in\mathcal{N}} \int_0^t S(t-s)\left(B\left[X(s)-Y(s)\right]\odot \mathbb{W}_H(s)\right)ds\mathbf{H}_\gamma,\end{aligned}$$

where

$$\begin{aligned}&\sup_{0\leqslant t\leqslant T}\left\| \int_0^t S(t-s)B\left[X(s)-Y(s)\right]\diamondsuit \mathbb{W}_H(s)ds \right\|_{-1,-q} \\ &\leqslant NT2^{-\frac{q}{2}}\left(\sum_{k=1}^\infty k^{\frac{4}{3}-H-q}\right)^{1/2}\|X-Y\|_{-1,-q,\infty}.\end{aligned}$$

Therefore, by (3.7) and previous estimates we have that

$$\|\mathcal{K}(X)-\mathcal{K}(Y)\|_{-1,-q,\infty} \leqslant \left[NT + NT2^{-\frac{q}{2}}\left(\sum_{k=1}^{\infty} k^{\frac{4}{3}-H-q}\right)^{1/2}\right] \|X-Y\|_{-1,-q,\infty}.$$

Consequently, if

$$NT + NT2^{-\frac{q}{2}}\left(\sum_{k=1}^{\infty} k^{\frac{4}{3}-H-q}\right)^{1/2} \in [0,1) \tag{3.8}$$

for any $H \in (0,1)$ and $q \geqslant 3$ then the transformation $\mathcal{K}$ has unique fixed point X in C_{-1}. The extra condition (3.8) on T can be easily removed by considering the equation on intervals $\left[0,\tilde{T}\right], \left[\tilde{T},2\tilde{T}\right], \ldots$ with $\tilde{T}$ satisfying (3.8). Thus we have shown that $\mathcal{K}$ is a strict contraction on $C\left(0,T;S(E)_{-1}\right)$ for all $H \in (0,1)$ and therefore for each $\theta \in S(E)_{-1}$ there exists a unique $X \in C\left(0,T;S(E)_{-1}\right)$ such that

$$X(\cdot) = S(\cdot)\theta + \mathcal{K}(X)(\cdot)$$

This completes the proof of the theorem.

$\square$

Remark 3.3 If we consider the linear equation with multiplicative noise we can obtain a similar result of existence and uniqueness by using the classical method of taking the Hermite transform. In this case, we again have a mild solution. Moreover, note that the generalized expectation of the solution is given by

$$\mathbb{E}X(t) = S(t)\mathbb{E}\theta + \int_0^t S(t-s)F\mathbb{E}X(s)ds$$

considering that $\mathbb{E}\left[\int_0^{\cdot} S(t-s)BX(s)\diamondsuit\mathbb{W}_H(s)ds\right] = 0$.

Now we aim at proving Theorem 3.2. We begin with a lemma.

Lemma 3.4. *Let B be a bounded linear operator on E and $H \in (0,1)$. Then $t \mapsto B\widetilde{\mathbb{W}}_H(t,z)$ is continuously differentiable on $[0,T]$ for all $z \in \mathbb{K}_{2q}$ with $q \geqslant 4$.*

Proof. Let us denote by N universal constants which may differ from line to line. We begin by observing that $B\mathbb{W}_H(t) = \sum_{k=1}^{\infty} Bd_k^H(t)\mathbf{H}_{\varepsilon_k} \in S(E)_{-0,-r}$ for all $t \in [0,T]$ and $r \geqslant 3$, where $B\widetilde{\mathbb{W}}_H(t,z)$ is bounded on $[0,T] \times \mathbb{K}_r$ for $r \geqslant 3$. Furthermore,

$$\frac{dBd_k^H(t)}{dt} = \delta_{k,n(i,j)} \left(M\xi_j\right)'(t)Be_i$$

for each $t \in [0,T]$ and we write

$$\Phi(t,z) := \sum_{k=1}^{\infty} \kappa_k(t) z_k,$$

where $\kappa_k(t) := \delta_{k,n(i,j)} \left(M\xi_j\right)'(t)Be_i$. Note that there exists a constant $K > 0$ such that

$$\int_{-\infty}^{+\infty} |y|^{\frac{3}{2}-H} |\xi_n(y)|\, dy \leqslant K n^{\frac{7}{6}-\frac{H}{2}}; \quad n \in \mathbb{N}.$$

To see this we recall that there exist contants η and γ such that

$$|\xi_n(t)| \leqslant \begin{cases} \eta n^{-\frac{1}{12}}; & |t| \leqslant 2\sqrt{n}, \\ \eta \exp(-\gamma t^2); & |t| > 2\sqrt{n}. \end{cases}$$

See for example Thangavelu (1993) for the details. Fix $n \in \mathbb{N}$. We have that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |y|^{\frac{3}{2}-H} |\xi_n(y)|\, dy \\
&= \int_{-2\sqrt{n}}^{2\sqrt{n}} |y|^{\frac{3}{2}-H} |\xi_n(y)|\, dy + \int_{-\infty}^{-2\sqrt{n}} |y|^{\frac{3}{2}-H} |\xi_n(y)|\, dy + \int_{2\sqrt{n}}^{\infty} |y|^{\frac{3}{2}-H} |\xi_n(y)|\, dy \\
&\leqslant N n^{\frac{7}{6}-\frac{H}{2}} + 2N \int_{2\sqrt{n}}^{+\infty} |y|^{\frac{3}{2}-H} \exp(-\gamma y^2) dy.
\end{aligned}$$

Note that we can find $y_0 > 0$ such that if $y > y_0$ then $|y|^{\frac{3}{2}-H} \exp(-\gamma y^2) \leqslant \frac{1}{y^2}$. We may suppose that $y_0 > 2\sqrt{n}$. Under these conditions we have that

$$\begin{aligned}
\int_{2\sqrt{n}}^{+\infty} |y|^{\frac{3}{2}-H} \exp(-\gamma y^2) dy &\leqslant N \int_{2\sqrt{n}}^{y_0} |y|^{\frac{3}{2}-H}\, dy + \int_{2\sqrt{n}}^{+\infty} \frac{1}{y^2} dy \\
&\leqslant N y_0^{\frac{5}{2}-H} + \frac{1}{2\sqrt{n}}.
\end{aligned}$$

That is, $2N\int\limits_{2\sqrt{n}}^{+\infty}|y|^{\frac{3}{2}-H}\exp(-\gamma y^2)dy \leqslant Ny_0^{\frac{5}{2}-H}+N$ and therefore we can find a constant $K>0$ which does not depend on $H\in(0,1)$ such that

$$\int\limits_{-\infty}^{+\infty}|y|^{\frac{3}{2}-H}|\xi_n(y)|\,dy \leqslant Kn^{\frac{7}{6}-\frac{H}{2}}.$$

By definition

$$(M\xi_n)(t)=(-i)^{n-1}\int\limits_{-\infty}^{+\infty}e^{itx}|x|^{\frac{1}{2}-H}\xi_n(x)dx,$$

and therefore

$$(M\xi_n)'(t)=(-i)^{n-1}i\int\limits_{-\infty}^{+\infty}e^{itx}x\,|x|^{\frac{1}{2}-H}\xi_n(x)dx,$$

where

$$\left|(M\xi_n)'(t)\right| \leqslant \int\limits_{-\infty}^{+\infty}|y|^{\frac{3}{2}-H}|\xi_n(y)|\,dy \leqslant Kn^{\frac{7}{6}-\frac{H}{2}} \quad ; t\in\mathbb{R},\quad n\in\mathbb{N}.$$

By fixing $H\in(0,1)$ and taking $z\in\mathbb{K}_q$ for $q\geqslant 4$ we have that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\|\kappa_k(t)\|_{E_{\mathbb{C}}}|z_k| = \sum_{k=1}^{\infty}\|\kappa_k(t)\|_{E_{\mathbb{C}}}|z_k|(2k)^{-\frac{q}{2}}(2k)^{\frac{q}{2}}\\
&\leqslant\left(\sum_{k-1}^{\infty}\|\kappa_k(t)\|_{E_{\mathbb{C}}}^2(2k)^{-q}\right)^{1/2}\left(\sum_{k-1}^{\infty}|z_k|^2(2k)^{q}\right)^{1/2}\\
&\leqslant\left(\sum_{k=1}^{\infty}\delta_{k,n(i,j)}\|B\|^2\left|(M\xi_n)'(t)\right|^2(2k)^{-q}\right)^{1/2}\left(\sum_{k=1}^{\infty}(2k)^{-q}\right)^{1/2}\\
&\leqslant C\|B\|2^{-\frac{q}{2}}\left(\sum_{k=1}^{\infty}k^{-(H+q-\frac{7}{3})}\right)^{1/2}\left(\sum_{k=1}^{\infty}(2k)^{-q}\right)^{1/2}<\infty.
\end{aligned}$$

Therefore, $(z,t)\mapsto B\widetilde{\mathbb{W}}_H(t,z)$ and $\Phi(t,z)$ are well-defined on $[0,T]\times\mathbb{K}_q$ with $q\geqslant 4$ where $\Phi(t,z)$ is bounded on $[0,T]\times\mathbb{K}_q$. Then it follows that $B\widetilde{\mathbb{W}}_H(t,z)$ is differentiable on $[0,T]$ for each $z\in\bar{\mathbb{K}}_{2q}$ where

$$\frac{dB\widetilde{\mathbb{W}}_H(t,z)}{dt}=\Phi(t,z).$$

Moreover, as $\Phi(t,z)=\sum_{k=1}^{\infty}\kappa_k(t)z_k$ is bounded and $\kappa_k(\cdot)$ is continuous on $[0,T]$, then we have that $\Phi(t,z)$ is a continuous function for each $z\in\bar{\mathbb{K}}_{2q}$ with $q\geqslant 4$. Those statements follow from Propositions 19, 20 taken from Filinkov and Sorensen (2002). $\square$

Proof of Theorem 3.2

We state that

$$\widetilde{X}(t,z):=S(t)\widetilde{\theta}(z)+\int_0^t S(t-s)B\widetilde{\mathbb{W}}_H(s,z)ds$$

satisfies the following Cauchy problem

$$\frac{d\widetilde{X}(t,z)}{dt}=A\widetilde{X}(t,z)+B\widetilde{\mathbb{W}}_H(t,z);\quad (t,z)\in[0,T]\times\mathbb{K}_q \tag{3.9}$$

$$\widetilde{X}(0,z)=\widetilde{\theta}(z)\in D(A)$$

for each $z\in\mathbb{K}_q$ $(q\geqslant 8)$. This easily follows from Lemma 3.4. Moreover,

$$A\widetilde{X}(t,z)=S(t)B\widetilde{\mathbb{W}}_H(t,z)-B\widetilde{\mathbb{W}}_H(t,z)+\int_0^t S(s)\left(B\widetilde{\mathbb{W}}_H\right)'(z,t-s)ds$$

on $[0,T]\times\mathbb{K}_q$. We know from Lemma 3.4 that there exist constants N_1 and N_2 such that

$$\left\|B\widetilde{\mathbb{W}}_H(t,z)\right\|_{E_{\mathbb{C}}}\leqslant N_1,\quad \left\|\left(B\widetilde{\mathbb{W}}_H\right)'(t,z)\right\|_{E_{\mathbb{C}}}\leqslant N_2$$

for all $(t,z)\in[0,T]\times\mathbb{K}_q$ $(q\geqslant 8)$ and therefore, by the usual estimatives of the C_0-semigroup we can find constants $\bar{N}_1$ and $\bar{N}_2$ such that

$$\left\|S(t)B\widetilde{\mathbb{W}}_H(t,z)\right\|_{E_{\mathbb{C}}}\leqslant \bar{N}_1,\quad \left\|\int_0^t S(s)\left(B\widetilde{\mathbb{W}}_H\right)'(z,t-s)ds\right\|_{E_{\mathbb{C}}}\leqslant \bar{N}_2$$

for all $(t,z)\in[0,T]\times\mathbb{K}_q$ $(q\geqslant 8)$. Now we observe that

$$A\widetilde{X}(t,z)+B\widetilde{\mathbb{W}}_H(t,z)$$

is a continuous function on $[0,T]$ for each $z\in\mathbb{K}_q$, bounded on $[0,T]\times\mathbb{K}_q$ for $q\geqslant 8$ and with form $\sum_\alpha\varphi_\alpha z^\alpha$. By the so-called Characterization Theorem (see Filinkov and Sorensen (2002) and Holden et al (1994)) there exists a unique function $t\mapsto F(t)\in S(E)_{-1}$ such that

$$\mathcal{H}F(t)(z) = A\widetilde{X}(t,z) + B\widetilde{\mathbb{W}}_H(t,z) \text{ on } [0,T]\times\mathbb{K}_q.$$

By considering

$$X(t) := S(t)\theta + \int_0^t S(t-s)B\delta\mathbb{W}_H(s)$$

it follows by Proposition 20 from Filinkov and Sorensen (2002) that $X(\cdot)$ is the unique continuous differentiable function satisfying (3.9). This completes the proof.

$\square$

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References

[1] Alós, E., Mazet, O., Nualart, D., 2001. Stochastic calculus with respect to Gaussian processes. Ann. Probab. 29, 766-801.

[2] Alós, E., Nualart, D., 2003. Stochastic integration with respect to the fractional Brownian motion. Stoch. Stoch. Rep. 75, 129-152.

[3] Bender, C., 2003. An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. Stochastic Process. Appl. 104, 81-106.

[4] Carmona,P., Coutin, L., Montseny, G., 2003. Stochastic integration with respect to fractional Brownian motion. Ann.I.H.Poincar. 39-1, 27-68.

[5] Deck, T., 2002. Continuous dependence on initial data for solutions of non-linear stochastic evolution equations. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 5-3, 333-350.

[6] Decreusefond, L., stunel, A.S., 1999. Stochastic Analysis of the fractional Brownian motion. Potential Analysis. 10, 177-214.

[7] Duncan, T.E., Maslowski, B., Pasik-Duncan, B., 2002. Fractional Brownian Motion and Stochastic Equations in Hilbert Spaces. Stochastics and Dynamics. 2, 225-250.

[8] Elliot, R., van der Hoek , J., 2003. A general fractional white noise theory and applications to finance. Math. Finance. 338-2, 495-535.

[9] Feyel, D., de la Pradelle, A., 1996. Fractional integrals and Brownian processes. Potential Analysis. 10, 273-288.

[10] Filinkov,A., Sorensen, J., 2002. Differential Equations in spaces of abstract stochastic distributions. Stoch. Stoch. Rep. 72, 129-173.

[11] Hida, T., Kuo,H.H., Potthoff, J., Streit, L., 1993. White Noise: An Infinite Dimensional Calculus. Kluwer Academic Publishers.

[12] Holden, H., Oksendal, B., Uboe, J., Zhang, T., 1996. Stochastic Partial Diferential Equations: A Modeling, White Noise Functional Approach. Birkhuser. Boston.

[13] Huang Z., Yan, J., 2000. Introduction to Infinite Dimensional Stochastic Analysis. Kluwer Academic Publishers, Dordrecht.

[14] Hu, Y., Oksendal, B., Zhang, T., 2004. General fractional multiparameter white noise theory and stochastic partial differential equations. Commun. Partial Differential Equations, 29, 1-23.

[15] Hu, Y., 2001. Heat equation with fractional white noise potentials. Appl. Math. and Optmization. 43, 221-243.

[16] Lin, S.J., 1995. Stochastic analysis of fractional Brownian motions. Stoch. Stoch. Rep. 55, 121-140.

[17] Lindstrom, T., 1993. Fractional Brownian fields and integrals of white noise. Bull. London. Math. Soc. 25, 83-88.

[18] Mandelbrot, B., van Ness, J.W., 1968. Fractional Brownian motions, fractional noises and applications. SIAM Rev. 10, 422-437.

[19] Maslowsky, B., Nualart, D., 2003. Evolution equations driven by a fractional Brownian motion. J. of Functional Analysis. 124-4, 558-574.

[20] Thangavelu, S., 1993. Lectures on Hermite and Leguerre Expansions. Princeton University Press.

[21] Tindel, S., Tudor, C.A., Viens, F., 2003. Stochastic evolution equations with fractional Brownian motion. Probab. Theory Related Fields. 127, 186-204.

[22] Zhale, M., 1998. Integration with respect to fractal functions and stochastic calculus. I. Probab. Theory Related Fields. 111, 333-374.