On iterative methods for Incompressible fluids with mass diffusion

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Abstract

In this paper, we study the strong solutions of incompressible fluids with mass diffusion. We use an iterative method in order to approach the strong solutions and some convergence rates for this scheme are obtained, depending on weak, strong and more regular norms (latter only for strictly positive times).

Key words: Fluids with mass diffusion, Strong solutions, Iterative method, Convergence rates.

1 Introduction

We use an iterative process in order to approximate solutions for a nonhomogeneous Navier-Stokes model with mass diffusion.

The general argument is: firstly to obtain *a priori* estimations for the scheme sequence $(\rho^n, \mathbf{u}^n, p^n)$ (independent on n); afterwards, to show that $(\rho^n, \mathbf{u}^n, p^n)$ is a Cauchy-sequence in an appropriate Banach space and finally to pass to the

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limit, in order to prove that the limit $(\rho, \mathbf{u}, p)$ is the solution of the problem. By the way, some error estimates are also obtained.

1.1 The model

Let $(\rho, \mathbf{u}, p)$ be a solution of the initial-boundary problem in $Q_T = \Omega \times (0,T)$ (being $\Omega \subset \mathbb{R}^3$ a bounded regular domain) with boundary $\Sigma_T = \partial\Omega \times (0,T)$:

$$\begin{cases} \rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} - \lambda((\mathbf{u} \cdot \nabla)\nabla \rho + (\nabla \rho \cdot \nabla)\mathbf{u}) + \nabla p = \rho \mathbf{f} \text{ in } Q_T \\ \text{div } \mathbf{u} = 0 \text{ in } Q_T, \qquad \mathbf{u}|_{\Sigma_T} = 0, \qquad \mathbf{u}(0) = \mathbf{u}_0 \text{ in } \Omega, \\ \rho_t - \lambda \Delta \rho + \mathbf{u} \cdot \nabla \rho = 0 \text{ in } Q_T, \\ \left.\dfrac{\partial \rho}{\partial \mathbf{n}}\right|_{\Sigma_T} = 0, \qquad \rho(0) = \rho_0 \text{ in } \Omega. \end{cases} \tag{1}$$

Data of problem are: initial data $(\rho_0, \mathbf{u}_0)$, external forces $\mathbf{f}$, viscosity and mass diffusion coefficients $\mu, \lambda > 0$.

In this paper, we will always assume the hypothesis

$$0 < m \le \rho_0 \le M \quad \text{in } \Omega. \tag{2}$$

An interesting open problem is to extend the results of this paper to the case $m = 0$, i.e. assuming only $0 \le \rho_0 \le M$ in Ω.

The extension of the results of this paper for the complete model, with λ^2 terms (considered for instance in [2]), will be studied in a forthcoming paper.

1.2 Space functions and equivalent norms

We introduce standard spaces of the Navier-Stokes framework:

$$\begin{aligned} H &= \{\mathbf{u} : \mathbf{u} \in L^2(\Omega)^3, \ \text{div } \mathbf{u} = 0, \ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ V &= \{\mathbf{u} : \mathbf{u} \in H^1(\Omega)^3, \ \text{div } \mathbf{u} = 0, \ \mathbf{u} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

The norms $\|\mathbf{u}\|_{H^1}$ and $\|\nabla \mathbf{u}\|_{L^2}$ are equivalents in V, and $\|\mathbf{u}\|_{H^2}$ and $\|\Delta \mathbf{u}\|_{L^2}$ are equivalents in $H^2(\Omega) \cap V$.

On the other hand, for the density, let us consider the afin space

$$H_N^k(\Omega) = \left\{ \rho \in H^k(\Omega) : \frac{\partial \rho}{\partial \mathbf{n}} = 0 \text{ sobre } \partial\Omega, \ \int\limits_{\Omega} \rho(\mathbf{x}) = \int\limits_{\Omega} \rho_0(\mathbf{x}) \right\}$$

where $k = 2$ or 3. Obviously, $H_N^k(\Omega) = \overline{\rho_0} + H_{N,0}^k(\Omega)$, where $\overline{\rho_0} = (1/|\Omega|) \int_\Omega \rho_0(\mathbf{x}) d\mathbf{x}$ and

$$H_{N,0}^k(\Omega) = \left\{ \rho \in H^k(\Omega) : \frac{\partial \rho}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega, \ \int\limits_{\Omega} \rho(\mathbf{x}) = 0 \right\}.$$

Hence, $H_{N,0}^k(\Omega)$ ($k = 2$ or $k = 3$) is a closed subspace of $H_N^k(\Omega)$. Consequently, the norms $\|\rho\|_{H^2}$ and $\|\Delta\rho\|_{L^2}$ are equivalents in $H_N^2(\Omega)$ and $\|\rho\|_{H^3}$ and $\|\nabla\Delta\rho\|_{L^2}$ are equivalents in $H_N^3(\Omega)$.

1.3 Known results

Derivation of this model and physical discussion of equations (1) can be seen in Frank-Kamenestskii [6], Antoncev, Kazhikov and Monakhov [1], Prouse [12]. We observe that this model includes as a particular case the classical Navier-Stokes system, which has been very much studied (see, for instance, the classical books by Ladyzhenskaya [11] and Temam [17]).

Kazhikov and Smagulov [10] established, using a semi-Galerkin method, the local existence of weak and strong solutions under certain assumptions about the viscosity and diffusion coefficients. Also via this method, Salvi [13] proved the existence of weak solution in a non-cylindrical domain. On the other hand, Secchi in [16] studied the case $\Omega = \mathbb{R}^3$, proving without any hypothesis on the diffusive and viscosity coefficients, local existence and uniqueness of strong solutions, using a point fixed argument.

For a more complete model (including order λ^2 terms in the momentum equations), Beirão da Veiga [2], Secchi [15], established the local existence of strong solutions by using linearisation and fixed point argument. In the work [2] restrictions on the diffusive and viscosity coefficients are not imposed. The paper [15] imposed λ/μ small enough, in order to show the existence and uniqueness of an unique global solution in the 2-dimensional case. Moreover, it is showed the convergence, as $\lambda \to 0$, of a subsequence whose limit is a weak solutions of the non-homogeneous Navier-Stokes problem. A more practical semi-Galerkin method is being used by Damázio, Guillén-González and Rojas-Medar [5].

1.4 The iterative scheme

Assuming $\mathbf{u}_0 \in V$, $\rho_0 \in H^2_N(\Omega)$ verifying (2) and $\mathbf{f} \in L^2(0,T;L^2(\Omega)^3)$, we are going to consider the (unique) strong solution $(\rho, \mathbf{u}, p)$ of (1) defined in some (maybe small) time interval $(0,T)$ ([2]):

$$\rho \in L^2(0,T;H^3_N(\Omega)) \cap C([0,T];H^2_N(\Omega)), \quad \rho_t \in L^2(0,T;H^1(\Omega)), \tag{3}$$

$$\mathbf{u} \in L^2(0,T;H^2(\Omega)^3) \cap C([0,T];V), \quad \mathbf{u}_t \in L^2(0,T;H), \tag{4}$$

$$p \in L^2(0,T;H^1(\Omega)), \tag{5}$$

verifying PDE equations a.e. in Q_T, boundary conditions and initial conditions for $\rho, \mathbf{u}$ in the sense of spaces $H^2_N(\Omega)$ and V respectively.

Now, we define the iterative scheme what we will consider in this work:

Initialization: Let $\mathbf{u}^0(t) = \mathbf{u}_0$ for each $t \in [0,T]$.

Step $n \geq 1$: First, given $\mathbf{u}^{n-1}$ to find ρ^n such that

$$\rho^n_t - \lambda\Delta\rho^n + (\mathbf{u}^{n-1}\cdot\nabla)\rho^n = 0, \quad \rho^n|_{t=0} = \rho_0 \quad \text{and} \quad \frac{\partial\rho^n}{\partial\mathbf{n}}\Big|_{\Sigma_T} = 0. \tag{6}$$

Afterwards, given $\mathbf{u}^{n-1}$ and ρ^n, to find $(\mathbf{u}^n, p^n)$ such that

$$\begin{aligned} &\rho^n\mathbf{u}^n_t + (\rho^n\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^n - \mu\Delta\mathbf{u}^n + \nabla p^n \\ &\qquad -\lambda((\mathbf{u}^n\cdot\nabla)\nabla\rho^n + (\nabla\rho^n\cdot\nabla)\mathbf{u}^n)) = \rho^n\mathbf{f}, \\ &\mathrm{div}\ \mathbf{u}^n = 0, \quad \mathbf{u}^n|_{\Sigma_T} = 0, \quad \mathbf{u}^n|_{t=0} = \mathbf{u}_0. \end{aligned} \tag{7}$$

We have reduced the nonlinear coupled system (1) into a sequence of linear decoupled systems (6)-(7). Existence, regularity and uniqueness of approximate solution $(\rho^n, \mathbf{u}^n, p^n)$ can be obtained (see [2] for instance).

1.5 Main results of this paper

In this paper, we will denote by (f,g) to the inner product in $L^2(\Omega)$, by $|f|$ the $L^2(\Omega)$-norm and by $|f|_p$ the $L^p(\Omega)$-norm ($1 \leq p \leq +\infty$). Any other norm in a space $X(\Omega)$ defined in Ω will be denoted by $\|f\|_X$. Finally, for a cartesian product space $X \times Y$, we will consider the maximum norm $||(x,y)||_{X\times Y} = \max\{||x||_X, ||y||_Y\}$.

Our goal is this paper is double: to prove that $(\rho^n, \mathbf{u}^n, p^n)$ is a Cauchy sequence in a suitable Banach space which converges towards the strong solution $(\rho, \mathbf{u}, p)$ of problem (1), and to give some estimates of the convergence rates.

More precisely, in this paper we will prove the following three main results, which corresponding with the convergence rates respect to the weak norms, strong norms and higher regular norms (latter only for strictly positive times), see (19) for definition of bound $G(n)$:

Theorem 1.1 *Under hypotheses of Theorem 3.2 (see in Section 3), one has the existence (and uniqueness) of the strong solution $(\rho, \boldsymbol{u})$ of problem (1), which is obtained as the limit (in weak norms) of the sequence $(\rho^n, \boldsymbol{u}^n)$. Moreover, the following error estimates hold for all $t \in [0, T]$:*

$$\|(\rho^n - \rho, \boldsymbol{u}^n - \boldsymbol{u})(t)\|^2_{H^1 \times L^2} \le G(n), \tag{8}$$

$$\int_0^t \Big(\|(\rho^n - \rho, \boldsymbol{u}^n - \boldsymbol{u})(\tau)\|^2_{H^2 \times H^1} + |(\rho^n_t - \rho_t)(\tau)|^2 \Big) d\tau \le G(n). \tag{9}$$

Theorem 1.2 *Under hypotheses of Theorem 3.2 (see in Section 3), one has convergence (in strong norms) of the sequence $(\rho^n, \boldsymbol{u}^n, p^n)$ towards the strong solution $(\rho, \boldsymbol{u}, p)$ of problem (1). Moreover, the following error estimates hold for all $t \in [0, T]$:*

$$\|(\rho^n - \rho, \boldsymbol{u}^n - \boldsymbol{u})(t)\|^2_{H^2 \times H^1} + \int_0^t \|(\rho^n_t - \rho_t, \boldsymbol{u}^n_t - \boldsymbol{u}_t)\|^2_{H^1 \times L^2} d\tau \le G(n) \tag{10}$$

$$\int_0^t \|(\rho^n - \rho, \boldsymbol{u}^n - \boldsymbol{u}, p^n - p)\|^2_{H^3 \times H^2 \times H^1} d\tau \le G(n). \tag{11}$$

Theorem 1.3 *Under hypotheses of Theorem 3.3 (see in Section 3), one has higher regularity (only for strictly positive times) of the strong solution $(\rho, \boldsymbol{u}, p)$ of problem (1), which is obtained as the limit of the sequence $(\rho^n, \boldsymbol{u}^n, p^n)$. Moreover, if we define $\sigma(t) = \min\{t, 1\}$, the following error estimates hold for all $t \in [0, T]$:*

$$\sigma(t)\|(\rho^n_t - \rho_t, \boldsymbol{u}^n_t - \boldsymbol{u}_t)(t)\|^2_{H^1 \times L^2} \le G(n-1). \tag{12}$$

$$\int_0^t \sigma(\tau)\|(\rho^n_t - \rho_t, \boldsymbol{u}^n_t - \boldsymbol{u}_t)(\tau)\|^2_{H^2 \times H^1} d\tau \le G(n-1), \tag{13}$$

$$\sigma(t)\|(\rho^n - \rho, \boldsymbol{u}^n - \boldsymbol{u}, p^n - p)(t)\|^2_{H^3 \times H^2 \times H^1} \le G(n-1), \tag{14}$$

$$\int_0^t \sigma(\tau)\|(\rho^n-\rho,\boldsymbol{u}^n-\boldsymbol{u},p^n-p)(\tau)\|^2_{H^4\times H^3\times H^2}d\tau \le G(n-1). \tag{15}$$

Notice that error estimates in weak norms given in Theorem 1.1 are the same that in strong norms given in Theorem 1.2 (under the same hypotheses). But, error estimates for regular norms given by Theorem 1.3 are valid only for strictly positive times and the bound change from $G(n)$ to $G(n-1)$ (and more hypotheses on data are necessary).

2 Some estimates of Gronwall's type

The following well known Gronwall's Lemma will be frequently used:

Lemma 2.1 (Gronwall) *Let a,b,c,d be positive $L^1(0,T)$ functions verifying the differential inequality: a.e. $t\in(0,T)$,*

$$a'(t)+b(t)\le c(t)a(t)+d(t)$$

then, for any $t\in(0,T)$:

$$a(t)+\int_0^t b(s)ds \le \left(a(0)+\int_0^t d(s)ds\right)\exp\left(\int_0^t c(s)ds\right).$$

Now, we present a more specific estimate of Gronwall's type, which it will be used in the sequel, in order to obtain either scheme estimates or error estimates.

Lemma 2.2 (Gronwall with recurrence) *Let $(a_n),(b_n)$ two sequence of positive $L^1(0,T)$ functions such that*

$$a_n(0)=A\in\mathbb{R},\quad b_n(t)\ge P\,a_n(t)\ \ a.e.\ t\in(0,T) \tag{16}$$

with $P>0$ a constant (independent of t) and verifying the differential inequality: a.e. $t\in(0,T)$,

$$a_n'(t)+b_n(t)\le c_n(t)a_n(t)+d_n(t)a_{n-1}(t) \tag{17}$$

where $(c_n),(d_n)$ are two sequence of positive functions, bounded in $L^1(0,T)$ and $L^2(0,T)$ respectively. Then, there exists two constants $D>0$ and $E>0$

independent on n (depending on $\|c_n\|_{L^1(0,T)}$ and $\|d_n\|_{L^2(0,T)}$) such that for any $t \in (0,T)$ and for any $n \geq 1$, one has:

$$a_n(t) + \int_0^t b_n(s)\,ds \;\leq E\left(A\,\mathrm{e}^{D\,t/2} + \|a_0\|_{L^\infty(0,t)}\left[\mathrm{e}^{-2\,P\,t}\frac{(D\,t)^n}{n!} + \frac{(D\,t)^{n+1}}{(n+1)!}\right]^{1/2}\right).$$

Remark 2.3 *Notice that, for $t \in (0,T)$ with $T > 0$ fixed, one has that for any $n \geq n_0$ (with $n_0 = n_0(T)$),*

$$\left[\mathrm{e}^{-2\,P\,t}\frac{(D\,t)^n}{n!} + \frac{(D\,t)^{n+1}}{(n+1)!}\right]^{1/2} \leq 2\,\mathrm{e}^{-P\,t}\left[\frac{(D\,t)^n}{n!}\right]^{1/2}.$$

Therefore, under hypothesis of previous Lemma, one has in particular that for all $n \geq n_0$,

$$a_n(t) + \int_0^t b_n(s)\,ds \;\leq E\left(A\,\mathrm{e}^{D\,t/2} + \|a_0\|_{L^\infty(0,t)}\mathrm{e}^{-P\,t}\left[\frac{(D\,t)^n}{(n)!}\right]^{1/2}\right).$$

PROOF. Using (16) in (17) and applying Gronwall's Lemma to the inequality

$$(\mathrm{e}^{Pt}a_n)' \leq c_n\mathrm{e}^{Pt}a_n + \mathrm{e}^{Pt}d_n a_{n-1}$$

(recalling that $a_n(0) = A$) one has the estimate:

$$\begin{aligned}
\mathrm{e}^{P\,t}a_n(t) &\leq \left(A + \int_0^t \mathrm{e}^{P\,s}d_n(s)\,a_{n-1}(s)\,ds\right)\exp\left(\int_0^t c_n\right)\\
&\leq C\left(A + \left[\int_0^t |\mathrm{e}^{P\,s}a_{n-1}(s)|^2 ds\right]^{1/2}\right).
\end{aligned}$$

Therefore, if we define $\widetilde{a}_n(t) = |\mathrm{e}^{P\,t}a_n(t)|^2$, one has

$$\widetilde{a}_n(t) \leq D\left(A^2 + \int_0^t \widetilde{a}_{n-1}(s)\,ds\right)$$

hence, by means of an induction argument (applying Fubini's Theorem), we arrive at

$$\begin{aligned}
\widetilde{a}_n(t) &\leq D\,A^2\left(1 + Dt + \cdots + \frac{(C\,t)^{n-1}}{(n-1)!}\right) + D^n\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\widetilde{a}_0(s)\,ds\\
&\leq D\,A^2\,\mathrm{e}^{D\,t} + D^n\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}\mathrm{e}^{2\,P\,s}a_0^2(s)\,ds
\end{aligned}$$

Integrating by parts,

$$\widetilde{a}_n(t) \le D\,A^2\,\mathrm{e}^{D\,t} + D^n \|a_0\|^2_{L^\infty(0,t)} \left(\left[-\frac{(t-s)^n}{n!} \mathrm{e}^{2\,P\,s} \right]_{s=0}^{s=t} + 2\,P \int_0^t \frac{(t-s)^n}{n!} \mathrm{e}^{2\,P\,s} \right)$$

$$\le D\,A^2\,\mathrm{e}^{D\,t} + \|a_0\|^2_{L^\infty(0,t)} \left(\frac{(D\,t)^n}{n!} + 2\,\frac{P}{D}\,\mathrm{e}^{2\,P\,t} \frac{(D\,t)^{n+1}}{(n+1)!} \right)$$

hence we obtain

$$a_n(t)^2 \le D\,A^2\,\mathrm{e}^{D\,t} + \|a_0\|^2_{L^\infty(0,t)} \left(\mathrm{e}^{-2\,P\,t} \frac{(D\,t)^n}{n!} + 2\,\frac{P}{D} \frac{(D\,t)^{n+1}}{(n+1)!} \right).$$

Finally, applying Gronwall's Lemma to (17) (using again that $a_n(0) = A$):

$$\int_0^t b_n(s)\,ds \le \left(A + \int_0^t d_n(s) a_{n-1}(s) \right) \exp \int_0^t c_n(s) ds$$

$$\le C \left(A + \|d_n\|_{L^2(0,t)} \|a_{n-1}\|_{L^2(0,t)} \right).$$

Therefore, applying previous estimates for a^2_{n-1}, one has

$$\int_0^t b_n(s)\,ds \le C \left(A\Big(1 + \mathrm{e}^{D\,t/2}\Big) + \|A\|_{L^\infty(0,t)} \left[\int_0^t \mathrm{e}^{-2\,P\,s} \frac{(D\,s)^{n-1}}{(n-1)!} + 2\,\frac{P}{D} \int_0^t \frac{(D\,s)^n}{n!} \right]^{1/2} \right)$$

hence, integrating by parts

$$\int_0^t \mathrm{e}^{-2\,P\,s} \frac{(D\,s)^{n-1}}{(n-1)!} \le \frac{1}{D} \mathrm{e}^{-2\,P\,t} \frac{(D\,t)^n}{(n)!} + 2 \frac{P}{D^2} \frac{(D\,t)^{n+1}}{(n+1)!}$$

hence we can finish the proof of this Lemma.

Remark 2.4 *Arguing as in the proof of previous Lemma, but without hypothesis $b_n(t) \ge P\,a_n(t)$, one arrives at the following estimate for all $n \ge 1$:*

$$a_n(t) + \int_0^t b_n(s)\,ds \;\le E \left(A\,\mathrm{e}^{D\,t/2} + \|a_0\|_{L^\infty(0,t)} \left[\frac{(D\,t)^n}{n!} \right]^{1/2} \right).$$

Remark 2.5 *In this paper, we will use the previous Lemma in two situations, in order to obtain:*

(1) either scheme estimates, using in particular that

$$a_n(t) + \int_0^t b_n(s)\,ds \;\le C$$

(2) or error estimates. For this case, $A = 0$ and then

$$a_n(t) + \int_0^t b_n(s)\,ds \ \le G(n) \tag{18}$$

where

$$G(n) = C \begin{cases} \left[\mathrm{e}^{-2\,P\,t}(D\,t)^n/n! + (D\,t)^{n+1}/(n+1)!\right]^{1/2} & \forall\, n \ge 1, \ \textit{or} \\ \left[(D\,t)^n/n!\right]^{1/2} & \forall\, n \ge 1, \ \textit{or} \\ \mathrm{e}^{-P\,t}\left[(D\,t)^n/n!\right]^{1/2} & \forall\, n \ge n_0(T). \end{cases} \tag{19}$$

Here and in the sequel, we will denote by C different constants, always independent on n.

3 Scheme estimates

In this section, the task is to prove some estimates (uniformly respect to n) for the sequence $(\rho^n, \mathbf{u}^n, p^n)$. In particular, passing to the limit when $n \to +\infty$, we will obtain the (unique) strong solution $(\rho, \mathbf{u}, p)$ of (1) in $(0, T)$.

Usually, the following classical "interpolation and Sobolev" inequalities will be used:

$$|(f,g)| \le |f|_p |f|_q, \quad \forall\, p,q: \ 1/p + 1/q = 1,$$
$$|f|_3 \le |f|_2^{1/2} |f|_6^{1/2} \le C|f|^{1/2} \|f\|_{H^1}^{1/2}.$$

In particular

$$|f \cdot g| \le |f|_3 |g|_6 \le C|f|^{1/2} \|f\|_{H^1}^{1/2} \|g\|_{H^1}.$$

Moreover, we will use the following more specific interpolation inequality ([7]):

$$|f|_\infty \le C \|f\|_{H^1}^{1/2} \|f\|_{H^2}^{1/2} \tag{20}$$

In particular

$$|f \cdot g| \le |f|_\infty |g|_2 \le C \|f\|_{H^1}^{1/2} \|f\|_{H^2}^{1/2} |g|.$$

The existence (regularity and uniqueness) of solution $(\rho^n, \mathbf{u}^n, p^n)$ for the scheme (6)-(7), can be obtained (see [2] for instance). Moreover, the "weak maximum principle" applied to the ρ^n-problem (6) jointly with the hypothesis (2) imply

$$0 < m \le \rho^n(x,t) \le M \quad \text{in } Q_T \tag{21}$$

hence in particular,

$$\rho^n \text{ is bounded in } L^\infty(0,T;L^\infty(\Omega)).$$

On the other hand, from the corresponding energy inequalities of scheme problems (6) and (7), one can proves ([1,13]) the following weak estimates whatever $\lambda < 2\mu/(M-m)$: If $\mathbf{u}_0 \in H$, $\rho_0 \in H^1_N(\Omega)$ verifying (2) (but m could be zero) and $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^3)$, then

$$\begin{aligned} \rho^n &\text{ is bounded in } L^2(0,T;H^2_N(\Omega)) \cap L^\infty(0,T;H^1_N(\Omega)),\\ \sqrt{\rho^n}\mathbf{u}^n &\text{ is bounded in } L^\infty(0,T;L^2(\Omega)),\\ \mathbf{u}^n &\text{ is bounded in } L^2(0,T;H^1(\Omega)). \end{aligned}$$

However, these last estimates will be not necessary in the sequel. Only, we are going to use (21). Therefore, the hypothesis of ([1,13]) imposing λ small enough, $\lambda < 2\mu/(M-m)$, will not be imposed in this paper. Indeed, we will see directly that under certain smallness restrictions on the data (or equivalently on the final time), strong and more regular estimates hold.

First, we will prove the following auxiliary result

Lemma 3.1 *There exists some positive constants β, C_1, C_2, C_3 (depending on m, M, μ, Ω but independent on n and λ) such that, for any $n \geq 1$*

$$\lambda \frac{d}{dt}|\Delta\rho^n|^2 + \frac{\lambda^2}{4}|\nabla\Delta\rho^n|^2 + \frac{1}{2}|\nabla\rho^n_t|^2 \leq \frac{C_1}{\lambda}(\mu|\nabla \boldsymbol{u}^{n-1}|^2)^2\lambda|\Delta\rho^n|^2,$$

$$\begin{aligned} &\mu\frac{d}{dt}|\nabla\boldsymbol{u}^n|^2 + m|\boldsymbol{u}^n_t|^2 + \beta\Big(|\Delta\boldsymbol{u}^n|^2 + |\nabla p^n|^2\Big) \leq C_2|\boldsymbol{f}|^2\\ &+C_3\Big((\mu|\nabla\boldsymbol{u}^{n-1}|^2)^2 + \lambda^{1/2}(\lambda^{1/2}|\Delta\rho^n|)(\lambda|\nabla\Delta\rho^n|)\Big)\mu|\nabla\boldsymbol{u}^n|^2 \end{aligned}$$

PROOF. Multiplying the density equation (6) by $-\Delta\rho^n_t$, integrating by parts in Ω (boundary terms vanish) and using Young's inequalities in an appropriate form, we obtain:

$$\lambda\frac{d}{dt}|\Delta\rho^n|^2 + |\nabla\rho^n_t|^2 \leq C|\nabla(\mathbf{u}^{n-1}\cdot\nabla\rho^n)|^2.$$

On the other hand, taking gradient of the density equation (6),

$$\lambda^2|\nabla\Delta\rho^n|^2 \leq 2|\nabla\rho^n_t|^2 + 2|\nabla(\mathbf{u}^{n-1}\cdot\nabla\rho^n)|^2$$

Balancing between two previous inequalities and using $|\nabla(\mathbf{u}^{n-1}\cdot\nabla\rho^n)|^2 \leq C\|\mathbf{u}^{n-1}\|^2_{H^1}\|\rho^n\|_{H^2}\|\rho^n\|_{H^3}$, one arrives at

$$\lambda\frac{d}{dt}|\Delta\rho^n|^2 + \frac{\lambda^2}{4}|\nabla\Delta\rho^n|^2 + \frac{1}{2}|\nabla\rho^n_t|^2 \leq C\|\mathbf{u}^{n-1}\|^2_{H^1}\|\rho^n\|_{H^2}\|\rho^n\|_{H^3}.$$

Then, recalling equivalent norms and using again Young's inequalities, the first inequality of this Lemma holds.

For the second inequality, taking $\mathbf{u}_t^n$ as tests function in (7), one has:

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}|\nabla \mathbf{u}^n|^2 + \frac{3}{4}m\,|\mathbf{u}_t^n|^2 \\
&\quad\le C\Big(|\mathbf{f}|^2 + |(\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^n|^2 + \lambda^2|(\mathbf{u}^n\cdot\nabla)\nabla\rho^n + (\nabla\rho^n\cdot\nabla)\mathbf{u}^n|^2\Big) \\
&\quad\le C\Big(|\mathbf{f}|^2 + |\mathbf{u}^{n-1}|_6^2|\nabla\mathbf{u}^n|_3^2 + \lambda^2\Big(|\mathbf{u}^n|_6^2|\nabla^2\rho^n|_3^2 + |\nabla\rho^n|_\infty^2|\nabla\mathbf{u}^n|^2\Big)\Big) \\
&\quad\le C|\mathbf{f}|^2 + \varepsilon|\Delta\mathbf{u}^n|^2 + C_\varepsilon|\nabla\mathbf{u}^{n-1}|^4|\nabla\mathbf{u}^n|^2 + C\lambda^2|\Delta\rho^n|\,|\nabla\Delta\rho^n||\nabla\mathbf{u}^n|^2
\end{aligned}$$

In order to estimate the $H^2(\Omega)$-norm for the velocity $\mathbf{u}^n$ and the $H^1(\Omega)$-norm for the pressure p^n, we use that $(\mathbf{u}^n, p^n)$ is the solution of a stationary Stokes equations (considering in (7) all additional terms on the right hand side). Then, the classical $H^2\times H^1$ regularity results of the Stokes equations ([8]) and similar bounds as above for the L^2-norm of all additional terms, yield:

$$\begin{aligned}
|\Delta\mathbf{u}^n|^2 + |\nabla p^n|^2 \le{}& C|\mathbf{u}_t^n|^2 + C|\mathbf{f}|^2 + \varepsilon|\Delta\mathbf{u}^n|^2 \\
&+ C_\varepsilon|\nabla\mathbf{u}^{n-1}|^4|\nabla\mathbf{u}^n|^2 + C\lambda^2|\Delta\rho^n|\cdot|\nabla\Delta\rho^n|\cdot|\nabla\mathbf{u}^n|^2
\end{aligned}$$

Making an appropriate "balance" between the two previous inequalities (see [2]), one can arrive to the second inequality of this Lemma. □

As consequence of previous Lemma, by means of an standard induction argument jointly with Gronwall's Lemma, we arrive at

Theorem 3.2 *Assume $\boldsymbol{u}_0 \in V$, $\rho_0 \in H_N^2(\Omega)$ verifying (2) and $\mathbf{f} \in L^2(Q_T)^3$, such that the following hypotheses hold: there exists $K_1, K_2 > 0$ such that*

$$\lambda|\Delta\rho_0|^2\exp\left(\frac{C_1}{\lambda}K_1^2T\right) \le K_2, \tag{22}$$

$$\left(\mu|\nabla\boldsymbol{u}_0|^2 + C_3\int_0^T|\boldsymbol{f}|^2\right)\exp\Big(C_2(K_1^2T^{1/2} + \lambda^{1/2}K_2)T^{1/2}\Big) \le K_1, \tag{23}$$

then, for any $n \ge 1$ and for all $t \in (0,T)$:

$$\lambda|\Delta\rho^n(t)|^2 + \int_0^t\left(\frac{\lambda^2}{4}|\nabla\Delta\rho^n(\tau)|^2 + |\nabla\rho_t^n(\tau)|^2\right)d\tau \le K_2,$$

$$\mu|\nabla\boldsymbol{u}^n(t)|^2 + \int_0^t\Big(m|\boldsymbol{u}_t^n(\tau)|^2 + \beta\Big(|\Delta\boldsymbol{u}^n(\tau)|^2 + |\nabla p^n(\tau)|^2\Big)\Big)d\tau \le K_1.$$

Taking into account equivalent norms in V, $H^2 \cap V$, H^2_N and H^3_N, estimates of previous theorem imply

$$(\rho^n, \mathbf{u}^n) \text{ is bounded in } L^\infty(H^2 \times H^1) \cap L^2(H^3 \times H^2) \tag{24}$$

$$(\rho^n_t, \mathbf{u}^n_t) \text{ is bounded in } L^2(H^1 \times L^2) \tag{25}$$

$$p^n \text{ is bounded in } L^2(0, T; H^1) \tag{26}$$

Notice that hypotheses (22)-(23) are either smallness restrictions on the data $(\mathbf{f}, \mathbf{u}_0)$ (but not on ρ_0) if we take K_2 small enough, or smallness conditions on the final time T.

Now, and without additional restriction hypothesis, we will obtain higher regularity estimates when data are more regular. Last estimates are only verified for strictly positive times.

Theorem 3.3 *Assume* $\boldsymbol{u}_0 \in H^2(\Omega) \cap V$, $\rho_0 \in H^3_N(\Omega)$ *verifying (2) and* $\mathbf{f} \in L^2(0, T; L^2(\Omega)^3)$ *with* $\mathbf{f}_t \in L^2(0, T; H^{-1}(\Omega)^3)$, *such that restrictive hypotheses (22)-(23) hold, then one has the following estimations:*

$$(\rho^n_t, \boldsymbol{u}^n_t) \textit{ is bounded in } L^\infty(H^1 \times L^2) \cap L^2(H^2 \times H^1) \tag{27}$$

In addition, assuming $\mathbf{f} \in L^\infty(0, T; L^2(\Omega)^3) \cap L^2(0, T; H^1(\Omega)^3)$,

$$(\rho^n, \boldsymbol{u}^n, p^n) \textit{ is bounded in } L^\infty(H^3 \times H^2 \times H^1) \cap L^2(H^4 \times H^3 \times H^2) \tag{28}$$

Moreover, if $\mathbf{f}_t \in L^2(0, T; L^2(\Omega)^3)$ *and* $\mathbf{f} \in L^\infty(0, T; H^1(\Omega)^3) \cap L^2(0, T; H^2(\Omega)^3)$,

$$\sqrt{\sigma(t)}(\rho^n_t, \boldsymbol{u}^n_t) \textit{ is bounded in } L^\infty(H^2 \times H^1) \cap L^2(H^3 \times H^2) \tag{29}$$

$$\sqrt{\sigma(t)}(\rho^n_{tt}, \boldsymbol{u}^n_{tt}) \textit{ is bounded in } L^2(H^1 \times L^2) \tag{30}$$

$$\sqrt{\sigma(t)}(\rho^n, \boldsymbol{u}^n, p^n) \textit{ is bounded in } L^\infty(H^4 \times H^3 \times H^2) \cap L^2(H^5 \times H^4 \times H^3) \tag{31}$$

where $\sigma(t) = \min\{1, t\}$.

PROOF. Here, we will do an outline of the proof. First, deriving respect to t, the problem verified by $\mathbf{v} = \mathbf{u}^n_t$, $q = p^n_t$ and $\eta = \rho^n_t$ is considered. Then, (27) is obtained for weak estimates of this problem and (28) from (27) and regularity results for the Poisson problem associated to ρ^n and the Stokes problem associated to $(\mathbf{u}^n, p^n)$. Finally, (29)-(30) are obtained for strong estimates of the $(\mathbf{v}, q, \eta)$-problem and (31) from (29) and regularity results for the Poisson problem associated to ρ^n and the Stokes problem associated to $(\mathbf{u}^n, p^n)$.

4 Error estimates

We use the following notation

$$\mathbf{u}^{(n,s)} = \mathbf{u}^{n+s} - \mathbf{u}^n, \quad p^{(n,s)} = p^{n+s} - p^n \quad \text{and} \quad \rho^{(n,s)} = \rho^{n+s} - \rho^n.$$

The following problems are verified by these variables:

$$\rho_t^{(n,s)} - \lambda\Delta\rho^{(n,s)} = (\mathbf{u}^{(n-1,s)} \cdot \nabla)\rho^{n+s} + (\mathbf{u}^{n-1} \cdot \nabla)\rho^{(n,s)}, \tag{32}$$

$$\left.\frac{\partial\rho^{(n,s)}}{\partial\mathbf{n}}\right|_{\Sigma_T} = 0, \quad \rho^{(n,s)}|_{t=0} = 0. \tag{33}$$

and

$$\begin{aligned}\rho^n\mathbf{u}_t^{(n,s)} &- \mu\Delta\mathbf{u}^{(n,s)} + \nabla p^{(n,s)} \\ &= -\rho^{(n,s)}\mathbf{u}_t^{n+s} + \rho^{(n,s)}\mathbf{f} - (\rho^{(n,s)}\mathbf{u}^{n-1+s} \cdot \nabla)\mathbf{u}^{n+s} \\ &\quad -(\rho^n\mathbf{u}^{(n-1,s)} \cdot \nabla)\mathbf{u}^{n+s} - (\rho^n\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{(n,s)} \\ &\quad +\lambda(\mathbf{u}^{(n,s)} \cdot \nabla)\nabla\rho^{n+s} + \lambda(\mathbf{u}^n \cdot \nabla)\nabla\rho^{(n,s)} \\ &\quad +\lambda(\nabla\rho^{(n,s)} \cdot \nabla)\mathbf{u}^{n+s} + \lambda(\nabla\rho^n \cdot \nabla)\mathbf{u}^{(n,s)}.\end{aligned} \tag{34}$$

$$\text{div } \mathbf{u}^{(n,s)} = 0, \tag{35}$$

$$\mathbf{u}^{(n,s)}|_{\Sigma_T} = 0, \quad \mathbf{u}^{(n,s)}|_{t=0} = 0. \tag{36}$$

4.1 Proof of Theorem 1.1

Multiplying the equation (32) by $-\Delta\rho^{(n,s)}$ and integrating (by parts) on Ω, and "balancing" the result with an estimation of $|\rho_t^{(n,s)}|^2$, we obtain

$$\begin{aligned}\frac{d}{dt}|\nabla\rho^{(n,s)}|^2 &+ \lambda|\Delta\rho^{(n,s)}|^2 + \frac{1}{4\lambda}|\rho_t^{(n,s)}|^2 \\ &\leq \frac{C}{\lambda}\Big(|(\mathbf{u}^{(n-1,s)} \cdot \nabla)\rho^{n+s}|^2 + |(\mathbf{u}^{n-1} \cdot \nabla)\rho^{(n,s)}|^2\Big) \\ &\leq \frac{C}{\lambda}\Big(|\mathbf{u}^{(n-1,s)}|^2|\nabla\rho^{n+s}|_\infty^2 + |\mathbf{u}^{n-1}|_\infty^2|\nabla\rho^{(n,s)}|^2\Big)\end{aligned}$$

$$\leq \frac{C}{\lambda}\Big(|\Delta\rho^{n+s}|\,|\nabla\Delta\rho^{n+s}|\,|\mathbf{u}^{(n-1,s)}|^2 + |\nabla\mathbf{u}^{n-1}|\,|\Delta\mathbf{u}^{n-1}|\,|\nabla\rho^{(n,s)}|^2\Big)$$

Multiplying the velocity equation (34) by $\mathbf{u}^{(n,s)}$, integrating in Ω, using the equality (which is deduced using the equation (6)),

$$\Big(\rho^n\mathbf{u}_t^{(n,s)} + (\rho^n\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{(n,s)} - \lambda(\nabla\rho^n\cdot\nabla)\mathbf{u}^{(n,s)}, \mathbf{u}^{(n,s)}\Big) = \frac{1}{2}\frac{d}{dt}|(\rho^n)^{\frac{1}{2}}\mathbf{u}^{(n,s)}|^2$$

we have

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}|(\rho^n)^{\frac{1}{2}}\mathbf{u}^{(n,s)}|^2 + \mu|\nabla\mathbf{u}^{(n,s)}|^2 \\
&= -(\rho^{(n,s)}\mathbf{u}_t^{n+s}, \mathbf{u}^{(n,s)}) + (\rho^{(n,s)}\mathbf{f}, \mathbf{u}^{(n,s)}) \\
&\quad -((\rho^{(n,s)}\mathbf{u}^{n-1+s}\cdot\nabla)\mathbf{u}^{n+s}, \mathbf{u}^{(n,s)}) - ((\rho^n\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{u}^{n+s}, \mathbf{u}^{(n,s)}) \\
&\quad +\lambda((\mathbf{u}^{(n,s)}\cdot\nabla)\nabla\rho^{n+s}, \mathbf{u}^{(n,s)}) + \lambda((\mathbf{u}^n\cdot\nabla)\nabla\rho^{(n,s)}, \mathbf{u}^{(n,s)}) \\
&\quad +\lambda((\nabla\rho^{(n,s)}\cdot\nabla)\mathbf{u}^{n+s}, \mathbf{u}^{(n,s)})
\end{aligned}$$

We estimate the right-hand side of the above equality as follows:

$$\begin{aligned}
|(\rho^{(n,s)}\mathbf{u}_t^{n+s}, \mathbf{u}^{(n,s)})| &\leq |\rho^{(n,s)}|_6|\mathbf{u}_t^{n+s}|_2|\mathbf{u}^{(n,s)}|_3 \\
&\leq C_\varepsilon|\mathbf{u}_t^{n+s}|^{4/3}\Big(|\nabla\rho^{(n,s)}|^2 + |\mathbf{u}^{(n,s)}|^2\Big) + \varepsilon|\nabla\mathbf{u}^{(n,s)}|^2
\end{aligned}$$

$$\begin{aligned}
|(\rho^{(n,s)}\mathbf{f}, \mathbf{u}^{(n,s)})| &\leq |\rho^{(n,s)}|_6|\mathbf{f}|_2|\mathbf{u}^{(n,s)}|_3 \\
&\leq C_\varepsilon|\mathbf{f}|^{4/3}\Big(|\nabla\rho^{(n,s)}|^2 + |\mathbf{u}^{(n,s)}|^2\Big) + \varepsilon|\nabla\mathbf{u}^{(n,s)}|^2
\end{aligned}$$

$$\begin{aligned}
|((\rho^{(n,s)}\mathbf{u}^{n-1+s}\cdot\nabla)\mathbf{u}^{n+s}, \mathbf{u}^{(n,s)})| &\leq |\rho^{(n,s)}|_6|\mathbf{u}^{n-1+s}|_6|\nabla\mathbf{u}^{n+s}|_2|\mathbf{u}^{(n,s)}|_6 \\
&\leq C_\varepsilon|\nabla\mathbf{u}^{n-1+s}|^2|\nabla\mathbf{u}^{n+s}|^2|\nabla\rho^{(n,s)}|^2 + \varepsilon|\nabla\mathbf{u}^{(n,s)}|^2
\end{aligned}$$

$$\begin{aligned}
|((\rho^n\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{u}^{n+s}, \mathbf{u}^{(n,s)})| &\leq |\rho^n|_\infty|\mathbf{u}^{(n-1,s)}|_2|\nabla\mathbf{u}^{n+s}|_3|\mathbf{u}^{(n,s)}|_6 \\
&\leq C_\varepsilon|\nabla\mathbf{u}^{n+s}|\,|\Delta\mathbf{u}^{n+s}|\,|\mathbf{u}^{(n-1,s)}|^2 + \varepsilon|\nabla\mathbf{u}^{(n,s)}|^2
\end{aligned}$$

$$\begin{aligned}
|\lambda((\mathbf{u}^{(n,s)}\cdot\nabla)\nabla\rho^{n+s}, \mathbf{u}^{(n,s)})| &\leq \lambda|\mathbf{u}^{(n,s)}|_2|\Delta\rho^{n+s}|_3|\mathbf{u}^{(n,s)}|_6 \\
&\leq C_\varepsilon\lambda^2|\Delta\rho^{n+s}|\,|\nabla\Delta\rho^{n+s}|\,|\mathbf{u}^{(n,s)}|^2 + \varepsilon|\nabla\mathbf{u}^{(n,s)}|^2
\end{aligned}$$

$$
\begin{aligned}
|\lambda((\mathbf{u}^n\cdot\nabla)\nabla\rho^{(n,s)},\mathbf{u}^{(n,s)})| &= |\lambda(\nabla\mathbf{u}^n\cdot(\nabla\mathbf{u}^{(n,s)})^t,\rho^{(n,s)})| \\
&\leq \lambda|\nabla\mathbf{u}^n|_3|\nabla\mathbf{u}^{(n,s)})|_2|\rho^{(n,s)}|_6 \\
&\leq C_\varepsilon\lambda^2|\nabla\mathbf{u}^n|\,|\Delta\mathbf{u}^n|\,|\nabla\rho^{(n,s)}|^2+\varepsilon|\nabla\mathbf{u}^{(n,s)}|^2
\end{aligned}
$$

(the above equality is obtained integrating two times by parts)

$$
\begin{aligned}
|\lambda((\nabla\rho^{(n,s)}\cdot\nabla)\mathbf{u}^{n+s},\mathbf{u}^{(n,s)})| &\leq \lambda|\nabla\rho^{(n,s)}|_2|\nabla\mathbf{u}^{n+s}|_3|\mathbf{u}^{(n,s)}|_6 \\
&\leq C_\varepsilon\lambda^2|\nabla\mathbf{u}^{n+s}|\,|\Delta\mathbf{u}^{n+s}|\,|\nabla\rho^{(n,s)}|^2+\varepsilon|\nabla\mathbf{u}^{(n,s)}|^2
\end{aligned}
$$

Adding the previous inequalities and choosing ε small enough, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(|(\rho^n)^{1/2}\mathbf{u}^{(n,s)}|^2+|\nabla\rho^{(n,s)}|^2\right)+\mu|\nabla\mathbf{u}^{(n,s)}|^2+\lambda|\Delta\rho^{(n,s)}|^2+\frac{1}{4\lambda}|\rho_t^{(n,s)}|^2 \\
&\leq \psi_{n,s}(t)|\mathbf{u}^{(n-1,s)}|^2+\varphi_{n,s}(t)\Big(|(\rho^n)^{1/2}\mathbf{u}^{(n,s)}|^2+|\nabla\rho^{(n,s)}|^2\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_{n,s}(t) &= C\Big(\frac{1}{\lambda}|\Delta\rho^{n+s}|\,|\nabla\Delta\rho^{n+s}|+|\nabla\mathbf{u}^{n+s}|\,|\Delta\mathbf{u}^{n+s}|\Big) \\
\varphi_{n,s}(t) &= C\Big(\frac{1}{\lambda}|\nabla\mathbf{u}^{n-1}|\,|\Delta\mathbf{u}^{n-1}|+|\mathbf{u}_t^{n+s}|^{4/3}+|\mathbf{f}|^{4/3}+|\nabla\mathbf{u}^{n-1+s}|^2|\nabla\mathbf{u}^{n+s}|^2 \\
&\qquad +\lambda^2(|\Delta\rho^{n+s}|\,|\nabla\Delta\rho^{n+s}|+|\nabla\mathbf{u}^n|\,|\Delta\mathbf{u}^n|\;+|\nabla\mathbf{u}^{n+s}|\,|\Delta\mathbf{u}^{n+s}|)\Big).
\end{aligned}
$$

From strong estimates of $(\rho^n,\mathbf{u}^n,p^n)$ given in Theorem 3.2 (see (24)-(25)), $(\psi_{n,s})$ is bounded in $L^2(0,T)$ and $(\varphi_{n,s})$ is bounded in $L^{3/2}(0,T)$. Therefore, Lemma 2.2 implies (recalling that $|\mathbf{u}^{(n,s)}(0)|=0$ and $|\nabla\rho^{(n,s)}(0)|=0$) the rates estimates of Theorem 1.1.

Notice that $\|\psi_{n,s}\|_{L^2(0,T)}$ and $\|\varphi_{n,s}\|_{L^{3/2}(0,T)}\uparrow+\infty$ as $\lambda\downarrow 0$. Consequently, estimates given in Theorem 1.1 are dependent on λ (and degenerates as $\lambda\downarrow 0$). In our opinion, the asymptotic behaviour as $\lambda\downarrow 0$ remains as an interesting open problem.

4.2 Proof of Theorem 1.2

Multiplying the density error equation (32) by $-\Delta\rho_t^{(n,s)}$ and "balancing" the result with an estimate of $|\nabla\rho_t^{(n,s)}|^2$ (making an analogous argument as in Lemma 3.1), we obtain

$$
\frac{d}{dt}\,|\Delta\rho^{(n,s)}|^2+\lambda|\nabla\Delta\rho^{(n,s)}|^2+\frac{1}{4\lambda}|\nabla\rho_t^{(n,s)}|^2
$$

$$
\begin{aligned}
&\leq \frac{C}{\lambda}\Big(|(\nabla \mathbf{u}^{(n-1,s)} \cdot \nabla)\rho^{n+s}|^2 + |\mathbf{u}^{(n-1,s)} \cdot \nabla^2 \rho^{n+s}|^2 \\
&\qquad + |(\nabla \mathbf{u}^{n-1} \cdot \nabla)\nabla \rho^{(n,s)}|^2 + |\mathbf{u}^{n-1}\nabla^2 \rho^{(n,s)}|^2\Big) \\
&\leq \frac{C}{\lambda}\Big(|\Delta\rho^{n+s}|\,|\nabla\Delta\rho^{n+s}|\,|\nabla \mathbf{u}^{(n-1,s)}|^2 + |\nabla \mathbf{u}^{n-1}|\,|\Delta \mathbf{u}^{n-1}|\,|\Delta\rho^{(n,s)}|^2\Big).
\end{aligned}
$$

Multiplying the velocity equation (34) by $\mathbf{u}_t^{(n,s)}$ and balancing with the $H^2 \times H^1$ regularity of Stokes problem verified by $(\mathbf{u}^{(n,s)}, p^{(n,s)})$ (arguing again as in Lemma 3.1), we have

$$
\begin{aligned}
&m\,|\mathbf{u}_t^{(n,s)}|^2 + \mu \frac{d}{dt}|\nabla \mathbf{u}^{(n,s)}|^2 + \beta(|\Delta \mathbf{u}^{(n,s)}|^2 + |\nabla p^{(n,s)}|^2) \\
&\leq C\Big(|\rho^{(n,s)}\mathbf{u}_t^{n+s}|^2 + |\rho^{(n,s)}\mathbf{f}|^2 + |(\rho^{(n,s)}\mathbf{u}^{n-1+s}\cdot\nabla)\mathbf{u}^{n+s}|^2 \\
&\quad + |(\rho^n \mathbf{u}^{(n-1,s)} \cdot \nabla)\mathbf{u}^{n+s}|^2 + |(\rho^n \mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{(n,s)}|^2 \\
&\quad + \lambda^2(|(\nabla\rho^{(n,s)}\cdot\nabla)\mathbf{u}^{n+s}|^2 + |(\nabla\rho^n\cdot\nabla)\mathbf{u}^{(n,s)}|^2 \\
&\qquad + |(\mathbf{u}^{(n,s)}\cdot\nabla)\nabla\rho^{n+s}|^2 + |(\mathbf{u}^n\cdot\nabla)\nabla\rho^{(n,s)}|^2)\Big)
\end{aligned}
$$

Now, we estimate the right-hand side of the above equality as follows (using $L^\infty(0,T)$ estimates for $(\rho^n, \mathbf{u}^n, p^n)$ given in Theorem 3.2):

$$|\rho^{(n,s)}\mathbf{u}_t^{n+s}|^2 \leq |\rho^{(n,s)}|_\infty^2 |\mathbf{u}_t^{n+s}|^2 \leq C|\mathbf{u}_t^{n+s}|^2 |\Delta\rho^{(n,s)}|^2$$

$$|\rho^{(n,s)}\mathbf{f}|^2 \leq C|\Delta\rho^{(n,s)}|^2|\mathbf{f}|^2$$

$$|(\rho^{(n,s)}\mathbf{u}^{n-1+s}\cdot\nabla)\mathbf{u}^{n+s}|^2 \leq |\rho^{(n,s)}|_\infty^2|\mathbf{u}^{n-1+s}|_\infty^2|\nabla\mathbf{u}^{n+s}|^2 \leq C|\Delta\mathbf{u}^{n-1+s}|\,|\Delta\rho^{(n,s)}|^2$$

$$|(\rho^n\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{u}^{n+s}|^2 \leq |\rho^n|_\infty^2|\mathbf{u}^{(n-1,s)}|_6^2|\nabla\mathbf{u}^{n+s}|_3^2 \leq C|\Delta\mathbf{u}^{n+s}|\,|\nabla\mathbf{u}^{(n-1,s)}|^2$$

$$|(\rho^n\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{(n,s)}|^2 \leq |\rho^n|_\infty^2|\mathbf{u}^{n-1}|_\infty^2|\nabla\mathbf{u}^{(n,s)}|^2 \leq C|\Delta\mathbf{u}^{n-1}|\,|\nabla\mathbf{u}^{(n,s)}|^2$$

$$\lambda^2|(\nabla\rho^{(n,s)}\cdot\nabla)\mathbf{u}^{n+s}|^2 \leq \lambda^2|\nabla\rho^{(n,s)}|_6^2|\nabla\mathbf{u}^{n+s}|_3^2 \leq C|\Delta\mathbf{u}^{n+s}|\,|\Delta\rho^{(n,s)}|^2$$

$$\lambda^2|(\nabla\rho^n\cdot\nabla)\mathbf{u}^{(n,s)}|^2 \leq \lambda^2|\nabla\rho^n|_\infty^2|\nabla\mathbf{u}^{(n,s)}|^2 \leq C|\rho^n|_{H^3}|\nabla\mathbf{u}^{(n,s)}|^2$$

$$\lambda^2|(\mathbf{u}^{(n,s)}\cdot\nabla)\nabla\rho^{n+s}|^2 \leq \lambda^2|\mathbf{u}^{(n,s)}|_6^2|\nabla^2\rho^{n+s}|_3^2 \leq C|\rho^{n+s}|_{H^3}|\nabla\mathbf{u}^{(n,s)}|^2$$

$$\lambda^2|(\mathbf{u}^n\cdot\nabla)\nabla\rho^{(n,s)}|^2\leq\lambda^2|\mathbf{u}^n|^2_\infty|\nabla^2\rho^{(n,s)}|^2\leq C|\Delta\mathbf{u}^n|\,|\Delta\rho^{(n,s)}|^2$$

Adding all the previous inequalities, we have

$$\begin{aligned}&\frac{d}{dt}(\mu|\nabla\mathbf{u}^{(n,s)}|^2+|\Delta\rho^{(n,s)}|^2)+\frac{1}{4\lambda}|\nabla\rho_t^{(n,s)}|^2+\lambda|\nabla\Delta\rho^{(n,s)}|^2\\&+m|\mathbf{u}_t^{(n,s)}|^2+\beta\Big(|\Delta\mathbf{u}^{(n,s)}|^2+|\nabla p^{(n,s)}|^2\Big)\\&\leq\eta_1(t)|\nabla\mathbf{u}^{(n-1,s)}|^2+\eta_2(t)|\Delta\rho^{(n,s)}|^2+\eta_3(t)|\nabla\mathbf{u}^{(n,s)}|^2.\end{aligned}\tag{37}$$

where

$$\begin{aligned}\eta_1(t)&=C\Big(|\Delta\mathbf{u}^{n+s}(t)|+\|\rho^{n+s}(t)\|_{H^3}\Big),\\\eta_2(t)&=C\Big(|\mathbf{u}_t^{n+s}(t)|^2+|\Delta\mathbf{u}^{n-1+s}(t)|+|\Delta\mathbf{u}^{n+s}(t)|\\&\qquad+|\Delta\mathbf{u}^n(t)|+|\mathbf{f}(t)|^2+|\Delta\mathbf{u}^{n-1}(t)|\Big),\\\eta_3(t)&=C\Big(|\Delta\mathbf{u}^{n-1}|+\|\rho^n(t)\|_{H^3}+\|\rho^{n+s}(t)\|_{H^3}\Big)\end{aligned}$$

From strong estimates of Theorem 3.2 (see (24)-(25)), sequences η_1 and η_3 are bounded in $L^2(0,T)$ and η_2 is bounded in $L^1(0,T)$. Therefore, since $|\nabla\mathbf{u}^{(n,s)}(0)|=0$ and $|\Delta\rho^{(n,s)}(0)|=0$, applying Lemma 2.2 we obtain the rates estimates of Theorem 1.2.

Corollary 4.1 *Under hypothesis of Theorem 1.2, one has for each $t\in(0,T)$,*

$$|(\rho_t^n-\rho_t)(t)|^2\leq G(n-1).$$

PROOF. From the error density equation (32) and using $L^\infty(0,T)$ estimates given in Theorem 3.2, we have

$$|\rho_t^{(n,s)}|\leq C\Big(|\nabla\mathbf{u}^{(n-1,s)}|+|\Delta\rho^{(n,s)}|\Big)$$

hence applying Theorem 1.2, we can finish the proof.

4.3 Proof of Theorem 1.3

Differentiating the error equation of velocity (34) with respect to t, taking $\mathbf{u}_t^{(n,s)}$ as test function and using the equality

$$(\rho^n\mathbf{u}_{tt}^{(n,s)}+(\rho^n\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}_t^{(n,s)}-\lambda(\nabla\rho^n\cdot\nabla)\mathbf{u}_t^{(n,s)},\mathbf{u}_t^{(n,s)})=\frac{1}{2}\frac{d}{dt}|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}|^2,$$

we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}|^2+\mu|\nabla\mathbf{u}_t^{(n,s)}|^2\\
=&-(\rho_t^n\mathbf{u}_t^{(n,s)},\mathbf{u}_t^{(n,s)})-(\rho_t^{(n,s)}\mathbf{u}_t^{n+s},\mathbf{u}_t^{(n,s)})+(\rho^{(n,s)}\mathbf{u}_{tt}^{n+s},\mathbf{u}_t^{(n,s)})\\
&-((\rho_t^{(n,s)}\mathbf{u}^{n-1+s}\cdot\nabla)\mathbf{u}^{n+s},\mathbf{u}_t^{(n,s)})-((\rho^{(n,s)}\mathbf{u}_t^{n-1+s}\cdot\nabla)\mathbf{u}^{n+s},\mathbf{u}_t^{(n,s)})\\
&-((\rho^{(n,s)}\mathbf{u}^{n-1+s}\cdot\nabla)\mathbf{u}_t^{n+s},\mathbf{u}_t^{(n,s)})-((\rho_t^n\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{u}^{n+s},\mathbf{u}_t^{(n,s)})\\
&-((\rho^n\mathbf{u}_t^{(n-1,s)}\cdot\nabla)\mathbf{u}^{n+s},\mathbf{u}_t^{(n,s)})-((\rho^n\mathbf{u}^{(n-1,s)}\cdot\nabla)\mathbf{u}_t^{n+s},\mathbf{u}_t^{(n,s)})\\
&-((\rho_t^n\mathbf{u}^{n-1}\cdot\nabla)\mathbf{u}^{(n,s)},\mathbf{u}_t^{(n,s)})-((\rho^n\mathbf{u}_t^{n-1}\cdot\nabla)\mathbf{u}^{(n,s)},\mathbf{u}_t^{(n,s)})\\
&+\lambda((\mathbf{u}_t^{(n,s)}\cdot\nabla)\nabla\rho^{n+s},\mathbf{u}_t^{(n,s)})\\
&+\lambda((\mathbf{u}^{(n,s)}\cdot\nabla)\nabla\rho_t^{n+s},\mathbf{u}_t^{(n,s)})+\lambda((\mathbf{u}_t^n\cdot\nabla)\nabla\rho^{(n,s)},\mathbf{u}_t^{(n,s)})\\
&+\lambda((\mathbf{u}^n\cdot\nabla)\nabla\rho_t^{(n,s)},\mathbf{u}_t^{(n,s)})+\lambda((\nabla\rho_t^{(n,s)}\cdot\nabla)\mathbf{u}^{n+s},\mathbf{u}_t^{(n,s)})\\
&+\lambda((\nabla\rho^{(n,s)}\cdot\nabla)\mathbf{u}_t^{n+s},\mathbf{u}_t^{(n,s)})+\lambda((\nabla\rho_t^n\cdot\nabla)\mathbf{u}^{(n,s)},\mathbf{u}_t^{(n,s)})\\
&+(\rho_t^{(n,s)}\mathbf{f},\mathbf{u}_t^{(n,s)})+(\rho^{(n,s)}\mathbf{f}_t,\mathbf{u}_t^{(n,s)}).
\end{aligned}
$$

Estimating in a similar manner as in Theorem 3.3, we have

$$
\begin{aligned}
&\frac{d}{dt}\,|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}|^2+\mu|\nabla\mathbf{u}_t^{(n,s)}|^2\\
&\le C|\nabla\rho_t^{(n,s)}|^2(|\mathbf{f}|^2+|\Delta\mathbf{u}^n|+1)+C|\mathbf{u}_t^{(n,s)}|^2(|\nabla\rho_t^n|+1)\\
&+C|\Delta\rho^{(n,s)}|^2(|\mathbf{u}_{tt}^{n+s}|^2+|\nabla\mathbf{u}_t^n|^2+|\nabla\mathbf{u}_t^{n+s}|^2+\|\mathbf{f}_t\|_{H^{-1}}^2)\\
&+C|\nabla\mathbf{u}^{(n,s)}|^2(|\Delta\rho_t^{n+s}|^2+|\Delta\rho_t^n|^2+1)\\
&+C|\nabla\rho_t^{n+s}|\,|\Delta\mathbf{u}^{(n-1,s)}|\,|\nabla\mathbf{u}^{(n-1,s)}|+C|\Delta\mathbf{u}^{n+s}|\,|\mathbf{u}_t^{(n-1,s)}|^2.
\end{aligned}
$$

On the other hand, differentiating the error density equation (32) with respect to t, multiplying by $\Delta\rho_t^{(n,s)}$, integrating on Ω and estimating in a similar manner as in Theorem 1.1, we have

$$
\begin{aligned}
\frac{d}{dt}|\nabla\rho_t^{(n,s)}|^2+\lambda|\Delta\rho_t^{(n,s)}|^2\le\,& C\Big(|(\mathbf{u}_t^{(n-1,s)}\cdot\nabla)\rho^{n+s}|^2+|(\mathbf{u}^{(n-1,s)}\cdot\nabla)\rho_t^{n+s}|^2\\
&+|(\mathbf{u}_t^{n-1}\cdot\nabla)\rho^{(n,s)}|^2+|(\mathbf{u}^{n-1}\cdot\nabla)\rho_t^{(n,s)}|^2\Big)\\
\le\,& C\Big(\|\rho^{n+s}\|_{H^3}|\mathbf{u}_t^{(n-1,s)}|^2+|\nabla\mathbf{u}^{(n-1,s)}|\,|\Delta\mathbf{u}^{(n-1,s)}|\,|\nabla\rho_t^{n+s}|^2\\
&+|\nabla\mathbf{u}_t^{n-1}|\,|\Delta\rho^{(n,s)}|^2+|\Delta\mathbf{u}^{n-1}|\,|\nabla\rho_t^{(n,s)}|^2\Big).
\end{aligned}
$$

Adding the two previous inequalities, we obtain

$$
\frac{d}{dt}\left[|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}|^2+|\nabla\rho_t^{(n,s)}|^2\right]+\mu|\nabla\mathbf{u}_t^{(n,s)}|^2+\lambda|\Delta\rho_t^{(n,s)}|^2
$$

$$\leq C|\mathbf{u}_t^{(n,s)}|^2(|\nabla\rho_t^n|+1)+C(|\Delta\mathbf{u}^{n-1}|+|\Delta\mathbf{u}^n|+|\mathbf{f}|^2+1)|\nabla\rho_t^{(n,s)}|^2$$
$$+C|\Delta\rho^{(n,s)}|^2(|\mathbf{u}_{tt}^{n+s}|^2+|\nabla\mathbf{u}_t^n|^2+|\nabla\mathbf{u}_t^{n+s}|^2+|\nabla\mathbf{u}_t^{n-1}|+\|\mathbf{f}_t\|_{H^{-1}}^2)$$
$$+C|\nabla\mathbf{u}^{(n,s)}|^2(|\Delta\rho_t^{n+s}|^2+|\Delta\rho_t^n|^2+1)$$
$$+C|\nabla\rho_t^{n+s}|\,|\nabla\mathbf{u}^{(n-1,s)}|\,|\Delta\mathbf{u}^{(n-1,s)}|+C\|\rho^{n+s}\|_{H^3}|\mathbf{u}_t^{(n-1,s)}|^2$$

Notice that estimates will be for positive times, because of the term $|u_t^{n+s}|^2$, which appears from the nonlinear term ρu_t. Therefore, the function $\sigma(t)$ must be introduced.

Multiplying by $\sigma(t)=\min\{1,t\}$, recalling that $m|\mathbf{u}_t^{(n,s)}|^2\leq|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}|^2\leq M|\mathbf{u}_t^{(n,s)}|^2$ and $\sigma'(t)\leq 1$, we get

$$\frac{d}{dt}\Big[\sigma(t)\Big(|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}|^2+|\nabla\rho_t^{(n,s)}|^2\Big)\Big]+\sigma(t)\Big[\mu|\nabla\mathbf{u}_t^{(n,s)}|^2+\lambda|\Delta\rho_t^{(n,s)}|^2\Big]$$
$$\leq C(\|\rho^n\|_{H^3}^2+1+|\Delta\mathbf{u}^{n-1}|+|\Delta\mathbf{u}^n|+|\mathbf{f}|^2)\Big[\sigma(t)\Big(|\mathbf{u}_t^{(n,s)}|^2+|\nabla\rho_t^{(n,s)}|^2\Big)\Big]$$
$$+\sigma'(t)\Big[\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}|^2+|\nabla\rho_t^{(n,s)}|^2\Big]$$
$$+C\sigma(t)|\Delta\rho^{(n,s)}|^2(|\mathbf{u}_{tt}^{n+s}|^2+|\nabla\mathbf{u}_t^n|^2+|\nabla\mathbf{u}_t^{n+s}|^2+|\nabla\mathbf{u}_t^{n-1}|^2+\|\mathbf{f}_t\|_{H^{-1}}^2)$$
$$+C\sigma(t)|\nabla\mathbf{u}^{(n,s)}|^2(|\Delta\rho_t^{n+s}|^2+|\Delta\rho_t^n|^2+1)$$
$$+C\sigma(t)|\nabla\rho_t^{n+s}|\,|\Delta\mathbf{u}^{(n-1,s)}|\,|\nabla\mathbf{u}^{(n-1,s)}|+C\sigma(t)\|\rho^{n+s}\|_{H^3}|\mathbf{u}_t^{(n-1,s)}|^2$$
$$\leq a_n(t)\Big[\sigma(t)\Big(|\sqrt{\rho^n}\mathbf{u}_t^{(n,s)}|^2+C|\nabla\rho_t^{(n,s)}|^2\Big)\Big]+b_n(t)+c_n(t)$$

where $a_n(t)$ is bounded in $L^1(0,T)$, $\|b_n\|_{L^1(0,t)}\leq G(n)$ and $\|c_n\|_{L^1(0,t)}\leq G(n-1)$ thanks to Theorem 1.2, Theorem 3.2 and Theorem 3.3. Then, using Gronwall's Lemma, taking into account that $\sigma(0)=0$, we obtain (12)-(13).

From the $H^2\times H^1$ regularity of Stokes problem verified by $(\mathbf{u}^{(n,s)},p^{(n,s)})$, we obtain (bounding as in proof of Theorem 1.1)

$$\sigma(t)\ \Big(\ |\Delta\mathbf{u}^{(n,s)}(t)|^2+|\nabla p^{(n,s)}(t)|^2\Big)\leq C\sigma(t)\Big(|\mathbf{u}_t^{(n,s)}(t)|^2$$
$$+\eta_1(t)|\nabla\mathbf{u}^{(n-1,s)}(t)|^2+\eta_2(t)|\Delta\rho^{(n,s)}(t)|^2+\eta_3(t)|\nabla\mathbf{u}^{(n,s)}(t)|^2\Big)$$

Thus, by using estimates (10) and (12) we obtain (14) for the $(\mathbf{u},p)$-errors.

On the other hand, taking gradient in (32), we have

$$\sigma(t)|\nabla\Delta\rho^{(n,s)}|^2\leq C\sigma(t)\Big(|\nabla\rho_t^{(n,s)}|^2+|\Delta\mathbf{u}^{n-1}|\,|\Delta\rho^{(n,s)}|^2$$
$$+|\nabla\Delta\rho^{n+s}|\,|\nabla\mathbf{u}^{(n-1,s)}|^2\Big)$$

Thus, by using estimates (10) and (12) we obtain (14) for the ρ-error.

Finally, estimate (15) can be proved with analogous arguments, using now the $H^3 \times H^2$ regularity of Stokes problem verified by $(\mathbf{u}^{(n,s)}, p^{(n,s)})$ and the H^4 regularity of Poisson problem verified by $\rho^{(n,s)}$.

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