Chain Codes and Spherical Tits Buildings

Luciano Panek *and Marcelo Firer †

Abstract

To a n-dimensional vector space V over a finite field $\mathbb{F}_q$ it is possible to associate a structure of spherical Tits building. The chambers of such building are maximal flags: maximal sequences of nested subspaces. In the case $q = 2$, there is a unique $(n-1)$-dimensional 1-MDS code $C \subset V$. We show the existence of chambers associated to such a code that are chain type (in the sense of codes theory) and given a complete characterization of the connected components of the chain type chambers.

Key words: Hamming weights, chain codes, spherical Tits buildings.

1 Introduction

In 1991 Victor Wei introduced the concept of generalized minimum Hamming weights ([6]) motivated by several applications in cryptography, including the wire-tap channel of type II. With different motivation, similar properties of irreducible cyclic codes were studied by Helleseth, Kløve and Mykkeltveit in 1977 (see [1]).

Since generalized weights were introduced several results were obtained generalizing already known results in codes theory. Great part of these results can be found in the work [4] of Tsfasman and Vlădut where the generalized weights are calculated through the projective systems (see book [5]).

Given a k-dimensional linear codes $C \subset \mathbb{F}_q^n$, the *r-th minimum Hamming weights* is defined as

$$d_r(C) = \min\{\|D\| : D \subset C, \dim(D) = r\},$$

*Centro de Ciências Exatas, Universidade Estadual de Maringá, Av. Colombo 5790, 87020-900 - Maringá - PR, Brazil. Email: panek@uem.br

†IMECC - UNICAMP, Universidade Estadual de Campinas, Cx. Postal 6065, 13081-970 - Campinas - SP, Brazil. Email: mfirer@ime.unicamp.br

with

$$\|D\| = \# \bigcup_{v \in D} \mathrm{Supp}\,(v)\,,$$

where $\mathrm{Supp}\,(v)$, the *support of* v, is the number of non-zero coordinates of $v = (v_1, \ldots, v_n)$. A code C with *minimal weights hierarchy* $(d_1, \ldots, d_k)$ is called a $[n; k; d_1, \ldots, d_k]_q$- *code.*

Generalized weights naturally leads to the generalization of spectrum, the *r-th spectrum*:

$$E_i^{(r)}(C) = \{D : D \subset C, \dim(D) = r \text{ and } \|D\| = i\}\,.$$

As a consequence of the definition we have that

$$1 \leqslant d_1(C) < d_2(C) < \ldots < d_k(C) \leqslant n$$

([6, theorem 1]) and it follows that

$$r \leqslant d_r(C) \leqslant n - k + r$$

for all $r \in \{1, \ldots, k\}$ ([6, theorem 10]). A code such that $d_r(C) = n - k + r$ for a given $r \in \{1, \ldots, k\}$ is called *r-MDS code* (*r-maximum distance separable*). If for some r a code is r-MDS then it is also s-MDS for any $s \geqslant r$.

In this work we will study a special family of codes, introduced by Wei and Yang ([7]), called chain codes. Using a terminology usual in Projective Geometry, we say a sequence of linear subspaces

$$\{0\} = D_0 \subsetneq D_1 \subsetneq \ldots \subsetneq D_{k-1} \subsetneq D_k = C,$$

is a *flag in C*, or a *maximal flag* in case $\dim(C_i) = i$, for $i = 1, 2, ..., k$. A code C is called a *chain code* (or *code of chain type*) if there is a flag

$$D_1 \subset D_2 \subset \ldots \subset D_k = C,$$

with $\|D_r\| = d_r(C)$ for every $r = 1, 2, \ldots, k$, where $k = \dim(C)$. This is a particular but significant class of codes, since it includes the Hamming and the dual Hamming codes, Reed-Muller codes of all the orders, 1-MDS codes and the Golay codes ([7, theorem 6]). We observe that we can describe a maximal flag by giving an ordered base $\{v_1, ..., v_k\}$ such that for every $i = 1, 2, ..., k$, C_i is generated by $\{v_1, ..., v_i\}$. Similar construction can be made for other flags, not necessarily maximal.

The principal result of this work is Theorem 4.1, where we show that the set of flags associated to a chain code of codimension 1with fixed weight hierarchy is connected, in the sense there is a sequence of flags $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_m = \beta$, where all α_i has the same weight hierarchy and each (α_i, α_{i+1}) can be determined by ordered bases that differ only by a transposition.

This result is obtained through the characterization of the chain codes as chambers of a spherical Tits building, as we will see in the following section. This characterization, together with Theorem 4.1, enable us to count all the chain codes with a given weight hierarchy (Proposition 4.3 and Corollary 4.1).

In section 2 we introduce basic concepts related to Tits buildings and enunciate the proposed problems in terms of this structure. In the section 3 we study in details the structure of chain codes with weights hierarchy $(2, 3, \ldots, n)$, beginning the counting procedures, mainly Theorem 3.1. Finally, in the section 4 we present the main results in this work: considering the family of all flags associated to chain codes of codimension 1, we have that each connected component of this family is determined exclusively by the weights hierarchies $(1, 2, \ldots, n-1)$ and $(2, 3, \ldots, n)$ (Theorem 4.1 and Corollary 4.3).

2 Spherical Tits Buildings

We begin this section with generic definitions on abstract chamber systems and basic concepts of Tits buildings. We present only the concepts that are strictly necessary for this work, refering the reader to [3] for more details.

A *chamber system* over a set I is triple $\left(\Lambda, \overset{i}{-}, I\right)$, where Λ is a set, I is a set of indices and for each $i \in I$, $\overset{i}{-}$ is an equivalence relation in Λ. The elements of Λ are called *chambers* and if $\alpha \overset{i}{-} \beta$ we say that the chambers α and β are *i-adjacent*, or simply adjacent if we do not need to distinguish the adjacency type. A *gallery of length k* and *type $i_1 i_2 \ldots i_k$* joining two chambers α and β is a finite sequence of chambers $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \beta$ such that the chambers α_{j-1} and α_j are different but i_j-adjacent for each $j \in \{1, \ldots, k\}$. A *minimal gallery* joining α and β is a (not necessarily unique) gallery of minimal length joining the chambers. A subset $\Lambda' \subseteq \Lambda$ is said to be *connected* if any two chambers can be connected by a gallery. In this case, we define the *distance* $d(\alpha, \beta)$ between two chambers as the length of a *minimal gallery* joining the chambers. A subset $\Lambda' \subseteq \Lambda$ is called *convex* if every minimal gallery between any two chambers of Λ' is entirely contained in Λ'.

If every i_j belongs to some subset $J \subset I$, we say the gallery is a *J-gallery.* A chamber system Λ is *connected* (or *J-connected*) if any two chambers can be joined by gallery (or J-gallery). The J-connected components are called *residues of type J*, or simply *J-residues.* We denote the equivalence class (J-residue) of an element $\alpha \in \Lambda$ by $\operatorname{Res}(\alpha; J)$. The *rank* of chambers system over set I is the cardinality of I, and the *corank* of $J \subset I$ is the rank of $I \backslash J$. A *morphism* $\phi : \Lambda \to \Gamma$ between two chambers system over the same index set I is a map defined on the chambers that preserves i-adjacency for any $i \in I$.

An *isomorphism* is a morphism that possess inverse that is also a morphism.

A *Coxeter group* is a group W finitely generated by a set $\{r_1, ..., r_n\}$, subject only to the relations $(r_i r_j)^{m_{ij}} = 1$, where $m_{ij} \in \mathbb{N} \cup \{\infty\}$ and $m_{ii} = 1$, for any $i, j \in \{1, ..., n\}$. The matrix $(m_{ij})_{i,j=1}^{n}$ is called the *Coxeter matrix* of W and denoted for M_I. The generators of W define a structure of chambers systems over I in the group in a canonical way: two elements $w, w' \in W$ are said to be i-adjacent if and only if $w' = wr_i$. This systems is called *Coxeter complex*, the fundamental "bricks" that constitute a Tits building:

Definition 2.1 *Let $\mathbf{\Delta}$ be a chamber system and $\mathbf{\Sigma}$ a family of subsystems, all isomorphic to a given finite Coxeter complex, such that:*

(*i*) *For any two chambers there is $\Sigma \in \mathbf{\Sigma}$ containing both of them;*

(*ii*) *For each pair $\Sigma, \Sigma' \in \mathbf{\Sigma}$ with a chamber in common there is an isomorphism of chamber systems $\phi : \Sigma \to \Sigma'$ that fixes $\Sigma \cap \Sigma'$ pointwise;*

Then the pair $(\mathbf{\Delta}, \mathbf{\Sigma})$ is called a spherical Tits buildings *and the subsystems of $\mathbf{\Sigma}$* apartments*.*

Being the Coxeter complex finite, it can be realized as a complex structure on a metric sphere (see [2]).

Example 2.1 *Let $\mathbb{F}_q$ be a finite field with q elements and $\mathbb{F}_q^n$ the vector space of dimension n. We use the notation $(D_i)_{i=1}^{l}$ to represent the* flag of length l

$$\{0\} = D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_{l-1} \subset D_l = \mathbb{F}_q^n.$$

A maximal flag *is a flag of length n, and in this case $\dim (D_j) = j$. We consider the building $(\mathbf{\Delta}, \mathbf{\Sigma})$ defined as follows:*

$$\mathbf{\Delta} = A_{n-1}(q) = \left\{ (D_i)_{i=1}^{n-1} : \dim (D_j) = j, D_j \subset \mathbb{F}_q^n \right\}$$

$$\mathbf{\Sigma} = \left\{ \left\{ \left(\left\langle v_{\sigma(1)}, \ldots, v_{\sigma(i)} \right\rangle \right)_{i=1}^{n-1} : \sigma \in \mathbf{S}_n \right\} : \{v_1, \ldots, v_n\} \text{ base of } \mathbb{F}_q^n \right\}$$

where $\mathbf{S}_n$ is the symmetric group and $\left\langle v_{\sigma(1)}, \ldots, v_{\sigma(i)} \right\rangle$ is the space of $\mathbb{F}_q^n$ spanned by the vectors $\left\{ v_{\sigma(1)}, \ldots, v_{\sigma(i)} \right\}$. Fixed a base $\{v_1, \ldots, v_n\}$ of $\mathbb{F}_q^n$, an apartment of $\mathbf{\Delta}$ is the set of all chambers

$$\left(\left\langle v_{\sigma(1)}, \ldots, v_{\sigma(i)} \right\rangle \right)_{i=1}^{n-1}$$

with $\sigma \in \mathbf{S}_n$. Two chambers $(D_i)_{i=1}^{n-1}$ and $(D_i')_{i=1}^{n-1}$ in $\mathbf{\Delta}$ are i-adjacent if $D_j = D_j'$ for any $j \neq i$. The Coxeter group associated to the building $A_{n-1}(q)$ is isomorphic to the symmetric group $\mathbf{S}_n$.

Example 2.2 *All J-residue of a Tits building $\mathbf{\Delta}$ of the type M_I is a Tits building of type M_J (see [3, theorem 3.5]).*

3 Connecting Chambers by Galleries

We denote by $\Delta_q(d_1, d_2, \ldots, d_{n-1})$ the set of all maximal flags of $\mathbb{F}_q^n$ that achieve the weights hierarchy $(d_1, d_2, \ldots, d_{n-1})$. Each of those flags is called a *chambers of type* $(d_1, d_2, \ldots, d_{n-1})$.

Consider the particular case when the maximal non trivial subspace of the flag, $M^{n-1}(2) \subset \mathbb{F}_2^n$, is the $(n-1)$-dimensional 1-MDS code. Since $M^{n-1}(2)$ may be viewed as the set of all words $w \in \mathbb{F}_2^n$ with even weight, we have that

$$\Delta_2(2, 3, \ldots, n) \subset \mathrm{Res}(\alpha, \{1, \ldots, n-2\}).$$

Given $\alpha \in \Delta_2(2, 3, \ldots, n)$, the j-sphere of center α and ray 1 is the set of chambers in $\Delta_2(2, 3, \ldots, n)$ j-adjacent to α:

$$B_j(\alpha) = \left\{\beta \in \Delta_2(2, 3, \ldots, n) : d(\alpha, \beta) = 1 \text{ and } \beta \overset{j}{-} \alpha\right\}.$$

We will show those spheres are rather trivial. We start with a lemma:

Lemma 3.1 *Given $j \in \{1, \ldots, n-1\}$ consider $D_j \subset M^{n-1}(2)$ such that $\dim D_j = j$ and $\|D_j\| = j+1$. Then*

$$E_i^{(1)}(D_j) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ \binom{j+1}{i} & \text{if } i \text{ is even} \end{cases}.$$

Proof. Let $\{m_1, \ldots, m_{j+1}\} = \mathrm{Supp}(D_j)$. If $\pi : \mathbb{F}_2^n \to \mathbb{F}_2^{j+1}$ is the projection $\pi(x_1, \ldots, x_n) = \left(x_{m_1}, \ldots, x_{m_{j+1}}\right)$, then $\pi(D_j) \subset \mathbb{F}_2^{j+1}$ is a subcode of dimension j. Since $D_j \subset M^{n-1}(2)$, we find that the Hamming weight of any word in $\pi(D_j)$ is even. Then

$$\pi(D_j) = M^{j+1}(2),$$

and the result follows. $\square$

Let $U \subset V \subset \mathbb{F}_q^n$ be linear spaces, with dimensions r and t respectively. The number of s-dimensional subspaces of V containing U equals (see [4, lemma 2.2])

$$\begin{bmatrix} t-r \\ s-r \end{bmatrix} := \prod_{i=1}^{s-r} \frac{(q^{t-r} - q^{i-1})}{(q^{s-r} - q^{i-1})}.$$

Theorem 3.1 *Let $n > 2$ and $\alpha \in \Delta_2(2, 3, \ldots, n)$. Then*

$$\#B_i(\alpha) = \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i \neq 1 \end{cases}.$$

In particular, $B_1(\alpha) \cup \{\alpha\} = \mathrm{Res}(\alpha; \{1\})$.

Proof. Let $\alpha := (D_i)_{i=1}^{n-1} \in \Delta_2(2,3,\ldots,n)$. Since $\dim D_{i+1}/D_{i-1}$=2, it has exactly 3 subspaces of dimension 1. One of them corresponds to the projection of D_i and the other two defines the two chambers i-adjacent to $(D_i)_{i=1}^{n-1}$, but those are not necessarily contained in $\Delta_2(2,3,\ldots,n)$.

If $i=1$, it is enough to notice that for any $v \in D_2\backslash\{0\}$, $\|v\|=2$ and it this means that every chamber 1-adjacent to $(D_i)_{i=1}^{n-1}$ is in $\Delta_2(2,3,\ldots,n)$ and it follows that $\#B_1(\alpha)=2$.

If $i \neq 1$, we consider $\{v_1,\ldots,v_{i+1}\}$ a base of D_{i+1} such that $\{v_1,\ldots,v_{i-1}\}$ generates D_{i-1} and $\{v_1,\ldots,v_i\}$ generates D_i. Since D_{i+1} is a chain code, we can assume that

$$\#\left(\bigcup_{j=1}^{l}\operatorname{Supp}(v_j)\right)=l+1,$$

for every $l \in \{1,\ldots,i+1\}$. From Lemma 3.1, we find that D_i and D_{i+1} have respectively $\binom{i+1}{2}$ and $\binom{i+2}{2}$ words with weight 2. We claim there is $w_i \in D_{i+1}, w_i \neq v_i$ such that $\|w_i\|=2$ and

$$\#\left(\left(\bigcup_{j=1}^{i-1}\operatorname{Supp}(v_j)\right)\cup\operatorname{Supp}(w_i)\right)=i+1.$$

In fact, if $\operatorname{Supp}(\langle v_1,\ldots,v_{i+1}\rangle)=\{m_1,...,m_{i+2}\}$ there are $m_s \in \operatorname{Supp}(v_{i+1})$ and $m_r \in \operatorname{Supp}(\langle v_1,\ldots,v_i\rangle)$ such that $m_s \notin \operatorname{Supp}(\langle v_1,\ldots,v_i\rangle)$ and then, every $w_i \in D_{i+1}$ such that $\operatorname{Supp}(w_i)=\{m_r,m_s\}$ satisfies the desired conditions. So, we obtained two i-dimensional codes of chain type, let us say D_i and D_i'. The amount of distinct words of weight 2 in those codes equals

$$E_2^{(1)}(D_i)+E_2^{(1)}(D_i')-E_2^{(1)}(D_{i-1}).$$

Since the number of words of weight 2 in $D_{i+1}\backslash(D_i\cup D_i')$ is smaller then the number of words of weight 2 in D_i (or D_i') that are not in D_{i-1}, that is,

$$E_2^{(1)}(D_{i+1})-\left(E_2^{(1)}(D_i)+E_2^{(1)}(D_i')-E_2^{(1)}(D_{i-1})\right)<E_2^{(1)}(D_i)-E_2^{(1)}(D_{i-1}),$$

we conclude that just two i-adjacent chambers can be of chain type, because the number of distinct words of weight 2 in a code $D \subset M^{n-1}(2)$ such that $\dim D=i$ and $\|D\|=i+1$ equals $\binom{i+1}{2}$ (Lemma 3.1). □

From here on, we will withdraw the trivial case $\mathbb{F}_q^2$ and consider only codes in n-dimensional spaces with $n>2$.

If $B(\alpha)=\bigcup_j B_j(\alpha)$, then $\#B(\alpha)$ is called the *valency* of α, that is, the number of chambers in $\Delta_2(2,3,\ldots,n)$ adjacent to α. The next two corollaries follows trivially from the preceding theorem.

Corollary 3.1 *Let* $\alpha = (D_i)_{i=1}^{n-1} \in \Delta_2(2,3,\ldots,n)$*. Then*

$$\#\left(\bigcup_{j=1}^{r} B_j(\alpha)\right) = \begin{cases} r+1 & \textit{if} \quad r = 1,2,\ldots,n-2 \\ r & \textit{if} \quad r = n-1 \end{cases}.$$

Corollary 3.2 *Let* $\alpha \in \Delta_2(2,3,\ldots,n)$*. Then the valency of* α *is* $n-1$*.*

Lemma 3.2 *Let* $\alpha \in \Delta_2(2,3,\ldots,n)$ *and* $i \in \{2,3,\ldots,n-2\}$*. Then there is a unique chamber* $(D_1,\ldots,D_{n-1}) \in A_{n-1}(2)$ *that is* i*-adjacent to* α*, and such that*

$$\|D_i\| = i+2.$$

Proof. Suppose $D_i \subset D_{i+1} \subset M^{n-1}(2)$ are codes with $\|D_{i+1}\| = i+2$. Since $D_i \subset D_{i+1}$ we have that $\|D_i\| \leqslant \|D_{i+1}\| = i+2$. Since D_i is i-dimensional we cannot have $\|D_i\| \leqslant i$, because this means $\|D_i\| < d_i(M^{n-1}(2))$, contradicting the minimality of $d_i(M^{n-1}(2))$. It follows that $\|D_i\| = i+1$ or $\|D_i\| = i+2$. Since $\|D_i\| \in \{i+1, i+2\}$ and for $i \geq 2$ there are only 2 chambers i-adjacent to α, exactly one of them of type $(2,3,\ldots,n)$ (Theorem 3.1), we must have one of them satisfying the equation $\|D_i\| = i+2$. $\square$

Despite the fact it is very simple, Corollary 3.2 has an interesting consequence: given two chambers in the building

$$\left(D_1^1,\ldots,D_{n-1}^1\right), \left(D_1^2,\ldots,D_{n-1}^2\right) \in A_{n-1}(q)$$

such that $\dim\left(D_i^1 \cap D_i^2\right) = i-1$ for any $i \in \{2,3,\ldots,n-1\}$, they can be connected by a galleries of length $2n-3$ (the indexes under the lines indicate the adjacency type):

$$\begin{tikzcd}
(D_1^1, D_2^1, D_3^1, \ldots, D_{n-2}^1, D_{n-1}^1) \arrow[r, dash, "1"] & (D_2^1 \cap D_2^2, D_2^1, D_3^1, \ldots, D_{n-2}^1, D_{n-1}^1) \arrow[d, dash, "2"] \\
(D_2^1 \cap D_2^2, D_3^1 \cap D_3^2, \ldots, D_{n-1}^1 \cap D_{n-1}^2, D_{n-1}^1) \arrow[r, dash, "\cdots" description] \arrow[d, dash, "n-1"'] & (D_2^1 \cap D_2^2, D_3^1 \cap D_3^2, D_3^1, \ldots, D_{n-2}^1, D_{n-1}^1) \\
(D_2^1 \cap D_2^2, D_3^1 \cap D_3^2, \ldots, D_{n-1}^1 \cap D_{n-1}^2, D_{n-1}^2) \arrow[r, dash, "\cdots" description] & (D_2^1 \cap D_2^2, D_3^1 \cap D_3^2, D_3^2, \ldots, D_{n-2}^2, D_{n-1}^2) \arrow[d, dash, "2"] \\
(D_1^2, D_2^2, D_3^2, \ldots, D_{n-2}^2, D_{n-1}^2) \arrow[r, dash, "1"] & (D_2^1 \cap D_2^2, D_2^2, D_3^2, \ldots, D_{n-2}^2, D_{n-1}^2)
\end{tikzcd}$$

If the initial and final chambers of the gallery above are in $\Delta_2(2,3,\ldots,n)$, the whole gallery is contained in $\Delta_2(2,3,\ldots,n)$, as follows from the next theorem.

Theorem 3.2 *If* $(D_i^1)_{i=1}^{n-1}, (D_i^2)_{i=1}^{n-1} \in \Delta_2(2,3,\ldots,n)$ *and* $\dim(D_i^1 \cap D_i^2) = i-1$ *for any* $i \in \{2,3,\ldots,n-2\}$*, then*

$$\left\|D_r^1 \cap D_r^2\right\| = r$$

for any $r \in \{2,3,\ldots,n-2\}$.

Proof. The proof is by induction on r. For $r=2$, the result is trivial since every $0 \neq u \in D_2^i$ satisfies $\|u\| = 2$, for $i = 1,2$.

We assume now that $\|D_i^1 \cap D_i^2\| = i$ for every $i \in \{3,\ldots,r\}$ and suppose that $\left\|D_{r+1}^1 \cap D_{r+1}^2\right\| \neq r+1$. By Lemma 3.2 we get that $\left\|D_{r+1}^1 \cap D_{r+1}^2\right\| = r+2$. Let $\{i_1,\ldots,i_r\}$ be the support of $D_r^1 \cap D_r^2$. Since

$$\left(D_r^1 \cap D_r^2\right) \subset \left(D_{r+1}^1 \cap D_{r+1}^2\right) \subset D_{r+1}^j \ (j = 1,2),$$

we have that

$$\operatorname{Supp}\left(D_{r+1}^1 \cap D_{r+1}^2\right) = \{i_1,\ldots,i_r,l_1,l_2\}$$

and since $(D_r^1 \cap D_r^2) \subset D_{r+1}^j$ $(j=1,2)$ we find that

$$\operatorname{Supp}\left(D_{r+1}^j\right) = \left\{i_1,\ldots,i_r,i_{r+1}^j,i_{r+2}^j\right\} \ (j=1,2).$$

But $\left(D_{r+1}^1 \cap D_{r+1}^2\right) \subset D_{r+1}^j$ $(j=1,2)$, and we can assume with no loss of generality that $l_1 = i_{r+1}^j$ and $l_2 = i_{r+2}^j$. But then, $\operatorname{Supp}\left(D_{r+1}^1\right) = \operatorname{Supp}\left(D_{r+1}^2\right)$, and since $\left\|D_{r+1}^1\right\| = \left\|D_{r+1}^2\right\| = j+1$, it follows from Lemma 3.1 that $D_{r+1}^1 = D_{r+1}^2$, and $\dim\left(D_{r+1}^1 \cap D_{r+1}^2\right) = r+1$, contradicting the hypothesis of that dimension $\dim\left(D_{r+1}^1 \cap D_{r+1}^2\right) = r$. □

Corollary 3.3 *If* $(D_i^1)_{i=1}^{n-1}, (D_i^2)_{i=1}^{n-1} \in \Delta_2(2,3,\ldots,n)$, $\dim(D_i^1 \cap D_i^2) = i-1$ *for any* $i \in \{2,3,\ldots,n-3\}$ *and* $\#(\operatorname{Supp}(D_1^1) \cap \operatorname{Supp}(D_1^2)) = 1$*, then*

$$\left\|D_i^1 + D_i^2\right\| = i+2$$

for any $i \in \{1,2,\ldots,n-3\}$.

Proof. From Theorem 3.2 we know that support $\|D_i^1 \cap D_i^2\| = i$ for any $i \in \{2,3,\ldots,n-3\}$. Since $\|D_i^1 + D_i^2\| = \|D_i^1\| + \|D_i^2\| - \|D_i^1 \cap D_i^2\|$, we have that $\|D_i^1 + D_i^2\| = i+2$ for any $i \in \{2,3,\ldots,n-3\}$. For the case $i=1$, we notice that $\#\{\operatorname{Supp}(D_1^1) \cap \operatorname{Supp}(D_1^2)\} = 1$, and $\|D_1^1\| = \|D_1^2\| = 2$ and find that $\|D_1^1 + D_1^2\| = 3$. □

4 Connected Components

The main result of this work is presented in Theorem 4.1, where we characterize the connected components of chain codes. To prove that $\Delta_2(2,3,\ldots,n)$ is connected, we will show that the set of chambers of type $(2,3,\ldots,n)$ can be described (see Example 2.2 for details) as the disjoint union of apartments in the Tits Building

$$\mathrm{Res}\,(\alpha;\{1,2,\ldots,n-2\})\,,\alpha\in\Delta_2\,(2,3,\ldots,n)\,.$$

To understand the structure of those buildings defined by residues, we take a close look at the 4-dimensional case. We notice that $\Delta_2(2,3,4)$ is the union of the four apartments defined by the bases bellow:

$$\{(1,0,0,\underline{1})\,,(0,1,0,\underline{1})\,,(0,0,1,\underline{1})\}\,,\{(0,\underline{1},0,1)\,,(0,\underline{1},1,0)\,,(1,\underline{1},0,0)\}\,,$$
$$\{(0,1,\underline{1},0)\,,(1,0,\underline{1},0)\,,(0,0,\underline{1},1)\}\,,\{(\underline{1},0,0,1)\,,(\underline{1},1,0,0)\,,(\underline{1},0,1,0)\}$$

The underlined coordinates suggest the picture we wish to generalize: we chose a co-dimension 1 subspace defned by a non-zero coordinate and in this space we take a base formed by vectors with weight two. This will produce the apartments of $\Delta_2(2,3,\ldots,n)$ and constitute the foundation of the proof that $\Delta_2(2,3,\ldots,n)$ is connected.

Lemma 4.1 *Given* $(D_i)_{i=1}^{n-1}\in\Delta_2(2,3,\ldots,n)$ *there is a base* $\{v_1,\ldots,v_{n-1}\}$ *of* $M^{n-1}(2)$ *and* $l\in\{1,2,...,n\}$ *such that:*

(*i*) $\langle v_1,\ldots,v_i\rangle=D_i$ *for any* $i\in\{1,\ldots,n-1\}$;

(*ii*) $\mathrm{Supp}\,(v_i)\cap\ \mathrm{Supp}(v_j)=\{l\}$, *for any* $i,j\in\{1,\ldots,n-1\}$ *with* $i\neq j$.

Proof. Since $M^{n-1}(2)$ is of chain type, there is a base $\{w_1,\ldots,w_{n-1}\}$ of $M^{n-1}(2)$ such that

$$\#\left(\bigcup_{j=1}^{i}\mathrm{Supp}\,(w_j)\right)=d_i\left(M^{n-1}(2)\right)$$

and

$$\langle w_1,\ldots,w_i\rangle=D_i,$$

with $i\in\{1,\ldots,n-1\}$. We note that since $d_1\left(M^{n-1}(2)\right)=2$ we have that $\|w_1\|=2$. Lets say $\mathrm{Supp}\,(w_1)=\{i_1,i_2\}$ and $\mathrm{Supp}\,(w_j)=\{i_j,i_{j+1}\}$, $j\in\{2,3,\ldots,n-1\}$, with $i_j\in\mathrm{Supp}\,(w_{j-1})$ and $i_{j+1}\in\{1,\ldots,n-1\}\setminus\mathrm{Supp}\,(D_{j-1})$. Defining

$$v_1=w_1,$$
$$v_j=v_{j-1}+w_j,\ j=2,...,n$$

we find that

$$v_1 = w_1,\ \mathrm{Supp}\,(v_1) = \{i_1, i_2\}\,,$$
$$v_{l-1} = v_{l-2} + w_{l-1},\ \mathrm{Supp}\,(v_{l-1}) = \{i_1, i_l\}\,.$$

Then we put $l = i_1$ and find that $\mathrm{Supp}\,(v_i) \cap \mathrm{Supp}\,(v_j) = \{l\}$ for any $i, j \in \{1, \ldots, n-1\}$ with $i \neq j$ and the base $\{v_1, \ldots, v_{n-1}\}$ satisfies the requested conditions. $\square$

Let us define the vector $v_j^i \in \mathbb{F}_2^n$ as

$$v_j^i := \begin{cases} (0, \ldots, 0, 1_i, 0, \ldots, 0, 1_{j+1}, 0, \ldots, 0) & \text{if} \quad j \geqslant i \\ (0, \ldots, 0, 1_j, 0, \ldots, 0, 1_i, 0, \ldots, 0) & \text{if} \quad j < i \end{cases}$$

where the subindex indicate the corresponding coordinate we are assigning non-zero values . As $M^{n-1}\,(2)$ is an $(n-1)$-dimensional subspace containing all words of $\mathbb{F}_2^n$ with even weight, and since $\left\{v_1^i, v_2^i, \ldots, v_{n-1}^i\right\}$ is linearly independent, we find that for each $i \in \{1, ..., n\}$, the set $\left\{v_j^i | j = 1, ..., n-1\right\}$ is a base of $M^{n-1}\,(2)$. So it defines the apartment $\Sigma_i \subset \Delta_2\,(2, 3, \ldots, n)$:

$$\Sigma_i = \left\{ \left(\left\langle v_{\sigma(1)}^i, ..., v_{\sigma(j)}^i \right\rangle_{j=1}^{n-2}, M^{n-1}\,(2) \right) : \sigma \in \mathbf{S}_{n-1} \right\}.$$

Proposition 4.1 *With the notation above defined,*

$$\Delta_2\,(2, 3, \ldots, n) = \bigcup_{i=1}^{n} \Sigma_i.$$

Proof. By construction, $\bigcup_{i=1}^n \Sigma_i \subset \Delta_2\,(2, 3, \ldots, n)$ and it is left to prove that each chamber $\alpha = (D_i)_{i=1}^{n-1} \in \Delta_2\,(2, 3, \ldots, n)$ is contained in some of those apartments. Let us consider a base $\{v_1, \ldots, v_{n-1}\}$ of $M^{n-1}\,(2)$ such that $\langle v_1, \ldots, v_r \rangle = D_r$ and $\mathrm{Supp}\,(v_i) \cap \mathrm{Supp}\,(v_j) = \{l\}$, whenever $i \neq j$ and $r \in \{1, 2, \ldots n-1\}$ (existence guaranteed by Lemma 4.1). Setting

$$i_1 \in \mathrm{Supp}\,(v_1) \setminus \{l\}\,, \ldots, i_{n-1} \in \mathrm{Supp}\,(v_{n-1}) \setminus \{l\}$$

and

$$v^{i,j} = (0, \ldots, 0, 1_i, 0, \ldots, 0, 1_j, 0, \ldots, 0)\,,$$

we have that $\alpha = (D_i)_{i=1}^{n-1} = \left(\left\langle v^{l,i_1}, \ldots, v^{l,i_j} \right\rangle_{j=1}^{n-1}, M^{n-1}\,(2) \right) \in \Sigma_l$ and it follows that $\Delta_2\,(2, 3, \ldots, n) = \bigcup_{i=1}^n \Sigma_i$. $\square$

Proposition 4.2 *Let $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. Then $\#\,(\Sigma_i \cap \Sigma_j) = (n-2)!$.*

Proof. Consider two apartments Σ_i and Σ_j. Let $\sigma \in \mathbf{S}_{n-1}$ such that $\sigma(1) = j$, and suppose that $j < i$. Then

$$\left(\left\langle v^i_{\sigma(1)}, \ldots, v^i_{\sigma(l)} \right\rangle_{l=1}^{n-2}, M^{n-1}(2) \right) \in \Sigma_i.$$

Let $h \in \{2, 3, \ldots, n\}$. Since

$$v^i_{\sigma(1)} + v^i_{\sigma(h)} = \begin{cases} v^{j,\sigma(h)} & \text{if } \sigma(h) \in \{1, 2, \ldots, i-1\} \\ v^{j,\sigma(h)+1} & \text{if } \sigma(h) \in \{i, i+1, \ldots, n-1\} \end{cases},$$

we find that

$$\left(\left\langle v^i_{\sigma(1)}, \ldots, v^i_{\sigma(l)} \right\rangle_{l=1}^{n-2}, M^{n-1}(2) \right) \in \Sigma_j.$$

and therefore $\Sigma_i \cap \Sigma_j \neq \varnothing$. Since we have an total of $(n-2)!$ permutations of type $(1j\ldots) \in \mathbf{S}_{n-1}$, we conclude that $\#(\Sigma_i \cap \Sigma_j) = (n-2)!$. $\square$

Corollary 4.1 *The number of chambers in $\Delta_2(2, 3, \ldots, n)$ is $n!/2$.*

Proof. We note that

$$(\Sigma_l \cap \Sigma_i) \cap (\Sigma_l \cap \Sigma_j) = \varnothing,$$

if $i \neq j$ and $i, j \neq l$. So, if we assume that $i \neq j$ with $i, j \neq l$, we have that

$$(\Sigma_l \cap \Sigma_1) \cup (\Sigma_l \cap \Sigma_2) \cup \ldots \cup (\Sigma_l \cap \Sigma_{l-1})$$

is a disjoint union. Therefore

$$\begin{aligned} \#\Delta_2(2, 3, \ldots, n) &= \#(\Sigma_1 \cup \ldots \cup \Sigma_n) \\ &= \#\Sigma_1 + \sum_{i=2}^{n} \left(\#\Sigma_i - \sum_{j=1}^{i-1} \#(\Sigma_i \cap \Sigma_j) \right) \\ &= (n-1)! + (n-2)(n-1)! - \frac{(n-1)(n-2)}{2}(n-2)! \\ &= \frac{n!}{2}. \end{aligned}$$

$\square$

Corollary 4.2 *For any $n \geq 3$, $\Delta_2(2, 3, \ldots, n)$ is connected.*

Proof. Let $\alpha, \beta \in \Delta_2(2,3,\ldots,n)$. If $\alpha, \beta \in \Sigma_i$ for some i, then the chambers can be connected by a gallery, since each apartment Σ_i is convex ([3, theorem 3.8]). Suppose now that $\alpha \in \Sigma_i$ and $\beta \in \Sigma_j$ with $\alpha, \beta \notin \Sigma_i \cap \Sigma_j$. Since the intersection is not empty (Proposition 4.2), we can connect α to $\gamma \in \Sigma_i \cap \Sigma_j$ through a gallery in Σ_i, and β to γ through a gallery in Σ_j. Making the juxtaposition of these two galleries, we obtain a gallery joining α to β (passing through γ), entirely contained in $\Sigma_i \cup \Sigma_j \subset \Delta_2(2,3,\ldots,n)$. $\square$

Proposition 4.3 *The set of chambers $\Delta_q(1,2,\ldots,n-1)$ is an apartment of $A_{n-1}(q)$. In particular, $\Delta_q(1,2,\ldots,n-1)$ is convex.*

Proof. Consider the canonical base $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$,..., $e_n = (0,\ldots,0,1)$. It follows immediately from the definition that

$$\left\{ \left(\left\langle e_{\sigma(1)}, \ldots, e_{\sigma(i)} \right\rangle \right)_{i=1}^{n-1} : \sigma \in \mathbf{S}_n \right\} \subseteq \Delta_q(1,2,\ldots,n-1).$$

Let $\alpha = (D_1, \ldots, D_{n-1}) \in \Delta_q(1,2,\ldots,n-1)$ and choose a base $\{v_1, \ldots, v_{n-1}\}$ of D_{n-1} such that

$$\#\left(\bigcup_{j=1}^{i} \operatorname{Supp}(v_j) \right) = d_i(D_{n-1})$$

for any $i \in \{1, \ldots, n-1\}$. We have to prove there is a permutation $\sigma \in \mathbf{S}_n$ such that

$$\langle v_1, \ldots, v_i \rangle = \left\langle e_{\sigma(1)}, \ldots, e_{\sigma(i)} \right\rangle, \; i = 1, ..., n.$$

This means that, up to re-scaling by scalars, this ordered base is just a permutation of the canonical one.

Indeed, since $d_1(D_{n-1}) = 1$, we find that

$$v_1 = (0, \ldots, 0, k_1, 0, \ldots, 0), \; k_1 \neq 0,$$

so that $k_1^{-1} v_1 = e_j$ for some $j \in \{1, \ldots, n-1\}$. Since $\|\langle v_1, v_2 \rangle\| = 2$, we have that $\operatorname{Supp}(v_2) = \{j, l\}$ or $\operatorname{Supp}(v_2) = \{l\}$ (assuming, without loss of generality, that $l > j$). In the second of these possibilities we have that $v_2 = (0, \ldots, 0, k_2, 0, \ldots, 0)$, $k_2 \neq 0$, or in other words, that $k_2^{-1} v_2 = e_l$. In the first case, we find that

$$v_2 = (0, \ldots, 0, k_3, 0, \ldots, 0, k_4, 0, \ldots, 0)$$

and it follows that $(-k_3)\, e_j + k_4^{-1} v_2 = e_l$. Proceeding with this process in the same manner, we find that $\alpha \in \left\{ \left(\left\langle e_{\sigma(1)}, \ldots, e_{\sigma(i)} \right\rangle \right)_{i=1}^{n-1} : \sigma \in \mathbf{S}_n \right\}$.

The convexity follow of the fact that apartments are convex ([3, theorem 3.8]). $\square$

Let us notice now that there are many codes $C \subset \mathbb{F}_2^n$ of codimension 1 with weights hierarchy $\left(1, 2, \ldots, \widehat{m+1}, \ldots, n\right)$. Indeed, any such code may be described as the kernel of a linear functional $\varphi :\longrightarrow \mathbb{F}_2$; the code $M^{n-1}(2)$ is the kernel of $\varphi(v_1, \ldots, v_n) = v_1 + \ldots + v_n$. In the general case, a code of type $\left(1, 2, \ldots, \widehat{m+1}, \ldots, n\right)$ is the kernel of

$$\varphi(v_1, \ldots, v_n) = v_1 + \ldots + \widehat{v_{i_1}} + \ldots + \widehat{v_{i_m}} + \ldots + v_n. \tag{1}$$

With this we have a total of

$$n + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n-2} = 2\left(2^{n-1} - 1\right) - n$$

codes of codimension 1 in $\mathbb{F}_2^n$ that are neither of the type $(2, 3, \ldots, n)$ nor $(1, 2, \ldots, n-1)$.

We denote by $\Delta_2\{i_1, \ldots, i_m\}$ the set of chambers with weights hierarchy $\left(1, 2, \ldots, \widehat{m+1}, \ldots, n\right)$ that has the codimension 1 code defined as the kernel of a functional as in (1).

Theorem 4.1 *The set* $\Delta_2\left(1, 2, \ldots, \widehat{m+1}, \ldots, n\right)$ *is a disjoint union of the* J*-connected components* $\Delta_2 I$*, with* $I = \{i_1, \ldots, i_m\}$ *and* $J = \{1, 2, \ldots, \widehat{m}, \ldots, n-2\}$*. Consequently the set*

$$\bigcup_{(d_1, \ldots, d_{n-1})} \Delta_2(d_1, \ldots, d_{n-1})$$

has exactly $2^n - n$ *connected components.*

Proof. Each partial flag $(D_i)_{i=1}^m \in \Delta_2\{i_1, \ldots, i_m\}$ may be described as the sequence of subspaces defined by an ordered base $\{e_{i_1}, \ldots, e_{i_m}\}$ of D_m. In others word, the set of all such partial flags is J'-connected, for $J' = \{1, 2, \ldots, m-1\}$.

The partial flags $(D_i)_{i=m+1}^{n-1}$ in $\Delta_2\{i_1, \ldots, i_m\}$ are obtained from the vectors

$$(0, \ldots, 0, 1_{j_1}, 0, \ldots, 0, 1_{j_2}, 0, \ldots, 0),$$

where j_1 is the first non-zero position in $\{1, 2, \ldots, n\} \setminus I$ and $j_2 \in \{1, 2, \ldots, n\} \setminus I$, $j_2 \neq j_1$. The set of all such partial flags is J''-connected, $J'' = \{m+1, m+2, \ldots, n-2\}$. It follows that $\Delta_2\{i_1, \ldots, i_m\}$ is $J' \cup J''$-connected.

Finally, we prove that

$$\Delta_2\{i_1, \ldots, i_m\} \cup \Delta_2\{j_1, \ldots, j_m\},$$

is not connected, for $\{i_1, \ldots, i_m\} \neq \{j_1, \ldots, j_m\}$. Assuming so, there is an $j_r \notin \{i_1, \ldots, i_m\}$. As happens with connected spaces in topology, we will have only to prove that $\Delta_2 \{i_1, \ldots, i_m\} \cap \Delta_2 \{j_1, \ldots, j_m\} = \emptyset$.

Indeed, let $\alpha = (D_i)_{i=1}^m \in \Delta_2 \{i_1, \ldots, i_m\}$ and $\beta = \left(D_i'\right)_{i=1}^m \in \Delta_2 \{j_1, \ldots, j_m\}$, and suppose that α and β can be connected by a gallery in $\Delta_2 \{i_1, \ldots, i_m\} \cup \Delta_2 \{j_1, \ldots, j_m\}$. But adjacency in $A_{n-1}(q)$ is defined by permutations of the elements of a given base, so that a gallery joining α to β needs to change, at some place the subspace D_m by the subspace D_m'. But in order to do so, we must have $\{e_{i_1}, \ldots, e_{i_m}, e_{j_r}\} \subset D_{n-1} \cap D_{n-1}'$. But in this case, the subspace $\langle e_{i_1}, \ldots, e_{i_m}, e_{j_r}\rangle \subset \mathbb{F}_2^n$ has dimension $m+1$ and generalized weight equal $m+1$, contradicting the minimality of $d_{m+1} = m+2$. $\square$

Let $1 \leq r_1 < \ldots < r_k \leq n$ be a sequence of integers, $N = \{1, 2, \ldots, n\}$, $I := \{i_1, \ldots, i_m\} \subset N$ and $I^c = N \backslash I$. We denoted by $\mathbb{F}^n(r_1, \ldots, r_k)$ the set of all the flags $D_{r_1} \subset \ldots \subset D_{r_k}$ formed by subspaces of $\mathbb{F}_2^n$ such that $\dim\left(D_{r_j}\right) = r_j$. We define the inclusions

$$i_I : \mathbb{F}_2^m \to \mathbb{F}_2^n,$$
$$\widehat{i}_I : A_{m-1}(2) \to \mathbb{F}^n(1, \ldots, m)$$
and
$$\widehat{\widehat{i}}_I : A_{n-m-1}(2) \to \mathbb{F}^n(m+1, \ldots, n-1)$$

respectively as

$$i_I(x_1, \ldots, x_m) = \left(0, \ldots, 0, (x_1)_{i_1}, 0, \ldots, 0, (x_m)_{i_m}, 0, \ldots, 0\right),$$
$$\widehat{i}_I(D_1 \subset \ldots \subset D_m) = i_I(D_1) \subset \ldots \subset i_I(D_m)$$
and
$$\widehat{\widehat{i}}_I(D_1 \subset \ldots \subset D_{n-m-1}) = i_{I^c}(D_1) \oplus i_I(\mathbb{F}_2^m) \subset \ldots \subset i_{I^c}(D_{n-m-1}) \oplus i_I(\mathbb{F}_2^m).$$

Given chamber systems Λ_1, Λ_2 over I_1, I_2, the *direct product* $\Lambda_1 \times \Lambda_2$ is a chamber system over the disjoint union $I_1 \cup I_2$. Its chambers are the pairs (α_1, α_2), with $\alpha_i \in \Lambda_i$, and (α_1, α_2) is said to be i-adjacent to (β_1, β_2) for $i \in I_t$ ($t = 1$ or 2) if $\alpha_j = \beta_j$ for $j \neq t$ and $\alpha_t \overset{i}{-} \beta_t$ in Λ_t.

Notice now that the direct product

$$\widehat{i}_I(\Delta_2(1, 2, \ldots, m-1)) \times \widehat{\widehat{i}}_I(\Delta_2(2, 3, \ldots, n-m))$$

is isomorphic to the set $\Delta_2 I$. So, if we place $\Delta = \Delta_2(1, 2, \ldots, m-1)$ and $\Delta' = \Delta_2(2, 3, \ldots, n-m)$, we have the *coproduct*

$$\coprod_{k=1}^{\binom{n}{m}} \Delta \times \Delta' = \bigcup_I \widehat{i}_I(\Delta) \times \widehat{\widehat{i}}_I(\Delta')$$

and as a particular case of Theorem 4.1 we have the following:

Corollary 4.3 *The set* $\Delta_2\left(1,2,...,\widehat{m+1},...,n\right)$ *is isomorphic to the coproduct*

$$\coprod_{k=1}^{\binom{n}{m}} \Delta\times\Delta',$$

where the codes of dimension m *and* $n-m-1$ *in each product* $\Delta\times\Delta'$ *are identified with the codes* $\left\langle\left\{e_{i_j}\right\}_{j=1}^{m}\right\rangle$ *and* $\left\langle\left\{v_{j_2}^{j_1}\right\}\right\rangle$, $j_1, j_2 \in I^c$. *Consequently* $\Delta_2\left(1,2,\ldots,\widehat{m+1},\ldots,n\right)$ *has exactly* $n!/2$ *chambers.*

5 Final Remark

We have characterized the connected components of the union of chambers of chain type in $\mathbb{F}_2^n$, $\bigcup\limits_{(d_1,\ldots,d_{n-1})}\Delta_2\left(d_1,\ldots,d_{n-1}\right)$, and determined the cardinality of each such connected component. Those results are summarized in the table below.

$\Delta_2\left(1,2,\ldots,n-1\right)$	$n!$	Proposition 4.3
$\Delta_2\left(2,3,\ldots,n\right)$	$n!/2$	Corollary 4.2
$\Delta_2\left\{i_1,\ldots,i_m\right\}$	$m!\left(n-m\right)!/2$	Theorem 4.1
$\Delta_2\left(1,2,\ldots,\widehat{m+1},\ldots,n\right)$	$n!/2$	Corollary 4.3

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