A semilinear heat equation with singular nonlinearity

M. Loayza* †

Instituto de Matemática, Universidade Estadual de Campinas,
Caixa postal 6065, 13081-970, Campinas, SP, Brazil

Abstract

We are interested in the parabolic equation $u_t - \Delta u = f(x,u)$ in a bounded domain of $\mathbb{R}^N$ with Dirichlet boundary condition and $f : \Omega \times [0,\infty) \to [0,\infty)$ a Carathéodory function. We study the existence of solution, life span and analyze the behavior of the global(when the time $t \to \infty$) solution with respect to the solution of the elliptic corresponding problem $-\Delta u = f(x,u)$ with the Dirichlet boundary condition. A typical example where the results are applied is when $f(x,s) = a(x)s^q + b(x)s^p$ with $0 < q < 1 < p$ and $a \in L^\alpha(\Omega)$, $b \in L^\beta(\Omega)$ with $\alpha, \beta > 1$ and $\alpha, \beta > N/2$.

Key words: Heat equation, life span, stationary solution

1 Introduction

Let Ω be a bounded domain in R^N, $N \geq 1$, with smooth boundary $\partial\Omega$, $T > 0$ and $f : \Omega \times [0,\infty) \to [0,\infty)$ is a Carathéodory function. In this work we consider the following nonlinear heat equation

$$\left\{\begin{array}{rll} u_t - \Delta u & = f(x,u) & \text{in } \Omega \times (0,T) \\ u & = 0 & \text{in } \partial\Omega \times (0,T) \\ u(0) & = u_0 & \text{in } \Omega \end{array}\right. \tag{1.1}$$

where $u_0 \in L^\infty(\Omega)$ and $u_0 \geq 0$ a.e in Ω. We consider only nonnegative solutions for (1.1).

The equation (1.1) appear in many areas of applications, e. g. in media flows and combustion theory and has been considered by different authors, see for example [8],[9],[10].

A elliptic version of (1.1) is the equation

$$\left\{\begin{array}{rll} -\Delta u & = f(x,u) & \text{in } \Omega \\ u & = 0 & \text{in } \partial\Omega \end{array}\right. \tag{1.2}$$

The existence of solutions of (1.2) has been extensively investigated, see [1],[5],[6],[12] for a survey.

The purpose of the present paper is to study the existence of nonnegative solutions of the initial value problem (1.1) and the relation between the global(in time) solution of (1.1) and its stationary elliptic problem (1.2) when the nonlinearity $f(x,s)$ can be concave for s small and convex for s sufficiently enough. The some sort of concavity and some sort of convexity of $f(x,s)$ will be required to hold only on open subsets Ω_1 and Ω_2 of Ω(cf. hypothesis (H_2) and (H_4)).

A typical example to which our result apply is when the term nonlinear is of the form

$$f(x,s) = \lambda a(x)s^q + b(x)s^p \text{ for } x \text{ a.e in } \Omega, \text{ for all } s \geq 0 \tag{1.3}$$

Here $\lambda > 0$ is a parameter, p, q satisfy $0 < q < 1 < p$, $a \in L^\alpha(\Omega), b \in L^\beta(\Omega)$, $\alpha, \beta \geq 1$.

Results for the equations (1.1) and (1.2) with f given by (1.3) are knows. For the equation (1.1) with $a = 0$ or $b = 0$ see [9], [10], [14] and for the case $a = b = 1$ see [8]. For the equation (1.2) in the case $a = b = 1$ see [1] and for a most general case [12].

*Supported by FAPESP
†E-mail addresses:migloa@ime.unicamp.br

Throughout the paper, $(S(t))_{t\geq 0}$ is the heat semigroup, i.e, $S(t)=e^{t\Delta}$. A function u will be called solution of (1.1) in the interval $[0,T)$, with $T\leq\infty$ if for all $T'<T$ we have that $u\in L^{\infty}((0,T')\times\Omega)$ and satisfies the equation

$$u(t)=S(t)u_0+\int_0^t S(t-\sigma)f(x,u(\sigma))d\sigma \tag{1.4}$$

for all $t\in[0,T)$. Here $f:\Omega\times[0,\infty)\to[0,\infty)$ is a Carathéodory function satisfying the following conditions:

(H_1) $t\mapsto f(\cdot,t)\in C([0,\infty),L^{\gamma}(\Omega))$ with $\gamma>1$, $\gamma>N/2$ and for every $M>0$ there exists a noincreasing sequence $\{f_n\}$ of functions $t\mapsto f_n(\cdot,t)$ and a sequence $\{a_n\}$ of functions $t\mapsto a_n(\cdot,t)$ in $C([0,\infty),L^{\gamma}(\Omega))$ such that $||f_n-f||_{C([0,M],L^{\gamma}(\Omega))}\to 0$ when $n\to\infty$ and for x a.e. in Ω we have

$$|f_n(x,s)-f_n(x,t)|\leq a_n(x,s_\theta)|s-t| \tag{1.5}$$

for all $s,t\geq 0$ and s_θ is some element of the interval $[s,t]$.

(H_2) There exists a nonempty open set $\Omega_1\subset\Omega$, $t_0>0$ and a continuous and concave function $g:[0,t_0)\to[0,\infty)$, such that

(i) $f(x,s)\geq g(s)$ a.e $x\in\Omega_1$ and $0\leq s\leq t_0$,

(ii) For every $M>0$, there exist $L>0$ such that

$$g(s)-g(t)\leq\frac{L}{t}(s-t)$$

for $0<t\leq s\leq M$.

(iii) $\displaystyle\int_0^{t_0}\frac{d\sigma}{g(\sigma)}<\infty$

(H_3) For $N\geq 3$(respect. N=1,2) there exist $0<q<1<p<2^*-1$,(respect. $0<q<1<p$) $a\in L^{\alpha}(\Omega)$ with $\alpha>(2^*/(q+1))'$(respect $\alpha>1$) and $a\geq 0$ a.e. in Ω, $b\in L^{\beta}(\Omega)$ with $\beta>(2^*/(p+1))'$(respect. $\beta>1$) and $b\geq 0$ a.e. in Ω such that

$$f(x,s)\leq a(x)s^q+b(x)s^p$$

a.e. $x\in\Omega$ and all $s\geq 0$. Here $2^*=(2N)/(N-2)$ for $N\geq 3$.

(H_4) There exist $0<q<1<p$, $\epsilon_2>0$ and nonnegative functions $\tilde{a}\in L^{\tilde{\alpha}}(\Omega_2),\tilde{b}\in L^{\tilde{\beta}}(\Omega_2)$ with $\tilde{\alpha},\tilde{\beta}>N/2$, $\tilde{\alpha},\tilde{\beta}>1$ defined on a nonempty open set $\Omega_2\subset\Omega$ such that $\tilde{b}\geq\epsilon_2$ for x a.e. in Ω_2, meas$(\Omega_2\cap\{\tilde{a}>0\}\cap\{\tilde{b}>0\})>0$ and

$$f(x,s)\geq\tilde{a}s^q+\tilde{b}s^p \text{ for all } x \text{ a.e. in } \Omega_2, s\geq 0.$$

Here we denote $\{a>0\}$ the set of all $x\in\Omega$ such that $a(x)>0$.

(H_5) There exist a nonempty open set $\Omega_3\subset\Omega$, $t_1>0$ and a continuous convex function $h:[0,\infty)\to[0,\infty)$, such that

(i) $f(x,s)\geq h(s)$ for x a.e in Ω_3 and $s\geq 0$.

(ii) $\liminf_{s\to\infty}\frac{h(s)}{s}>\lambda_1(\Omega_3)$, where $\lambda_1(\Omega_3)$ is the principal eigenvalue associated to principal eigenfunction ψ_1 of $-\Delta$ in $H_0^1(\Omega_3)$ such that $\int_{\Omega_3}\psi(x)dx=1$.

(iii) $\displaystyle\int_{t_1}^{\infty}\frac{d\sigma}{h(\sigma)}<\infty$.

We make some observations on equation (1.4).

Remark 1.1 *(i) Note that the definition make sense in $L^\gamma(\Omega)$ since $t \mapsto f(\cdot,t) \in C([0,\infty), L^\gamma(\Omega))$.*

(ii) Since $\gamma > 1$ and $f \in L^\infty((0,T), L^\gamma(\Omega))$ for all $T < \infty$ we have by maximal regularity that $u(t) - S(t)u_0 \in W^{1,r}((0,T), L^\gamma(\Omega)) \cap L^r((0,T), W^{2,\gamma}(\Omega) \cap W_0^{1,\gamma}(\Omega))$ for $1 < r < \infty$ and u satisfies the equation (1.1) for a.e. $t \in (0,T)$.

(iv) Since $\gamma > N/2$ and $\gamma > 1$, then it follows from (ii) and Sobolev's embedding that $u - S(t)u_0 \in W^{1,r}((0,T), H^{-1}(\Omega)) \cap L^r((0,T), H_0^1(\Omega))$ for $1 < r < \infty$.

The difficulty to show the existence of a solution for (1.1) arises because we are supposing that f can be "singular" in the sense that can be concave(H_2 condition) in the origin and therefore, it will not be of Lipschitz. In addition, we do not completely have the aid of the maximum principle. In order to show the existence of a solution of (1.1) we used with some modifications the methods of [9] and [8].

Before enunciating our first result we defined the following.

Definition 1.2 *A solution u of (1.1) defined in $[0,T)$ is a maximal solution of (1.1) in $[0,T)$ if given any other solution v of (1.1) defined in $[0,T)$, we have $v \le u$.*

On the existence of a solution we have the following result.

Theorem 1.3 *Let $u_0 \in L^\infty(\Omega)$ be a nonnegative function. If we assume (H_1) and (H_2) then there exists a positive function u defined on a maximal time interval $[0, T_{\max})$, $u \in L^\infty((0,T)\times\Omega)$ for all $T < T_{\max}$ maximal solution of (1.1).*

Moreover, we have the blow up alternative: either $T_{\max} = \infty$(global solution) or else $T_{\max} < \infty$ and $\lim_{t\to T_{\max}} ||u(t)||_{L^\infty} = \infty$(blow-up solution).

Here are some comments on the hypotheses $(H_1), ..., (H_4)$.

In the hypothesis (H_1) a continuity of f is a well know condition, because we want to obtain a solution u through approach of solutions u_n of (1.1) with f_n instead f. The nonincreasing condition of f_n is required to have the limit of the functions u_n as $n \to \infty$.

A situation where (H_1) is valid is when $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is continuous. Indeed, let $M > 0$ be and $(\rho_n)_{n\ge1}$ be a sequence of mollifiers. We define $h_n = \rho_n * \overline{f}$, where $\overline{f}$ is a continuous extension of f to $\mathbb{R}^N \times \mathbb{R}$. Since $h_n \to \overline{f}$ uniformly on compact sets, there exist a sequence (r_n) such that $||h_{r_n} - f||_{C(\overline{\Omega}\times[0,M])} \le 1/[2n(n+1)]$. Set $g_n = h_{r_n} + 1/n$ we have that $g_{n+1} \le g_n$ in $\overline{\Omega} \times [0,M]$ and set $f_n(x,t) = g_n(x,t)$ for $(x,t) \in \overline{\Omega} \times [0,M]$ and $f_n(x,t) = g_n(x,M)$ for $(x,t) \in \overline{\Omega} \times [T,\infty)$ we obtain a sequence $\{f_n\}$ nonincreasing and $f_n \to f$ in $C([0,M], L^\gamma(\Omega))$. In order to obtain a_n we consider $|\frac{\partial f_n}{\partial s}(x,s)|$ in $\Omega \times [0,M]$ and $|\frac{\partial f_n}{\partial s}(x,M)|$ in $\Omega \times [M,\infty)$.

Another situation where (H_1) is valid, is when $f(x,s) = a(x)h(s)$ with $a \in L^\gamma(\Omega)$ and $h : [0,\infty) \to [0,\infty)$ is continuous. In this case, $f_n(x,s) = a(x)h_n$ where h_n can be obtained using the same argument of the previous paragraph.

The condition (H_2) is necessary for the existence of a positive solution leaving $u(0) = 0$. (H_3) is a superior limitation of f, we used it to obtain estimates of energy associated to (1.1). Finally, (H_4) and (H_5) are used to obtain blow-up solutions in finite time.

With respect the global solutions, we have the following result.

Theorem 1.4 (Global solutions) *Assume $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$, (H_1) and (H_2). Let u be the maximal positive solution of (1.1) defined in the maximal interval $[0, T_{\max})$. If f satisfies (H_3) and $f(x,\cdot)$ is nondecreasing for x a.e. in Ω, then we can find explicitly constant $\eta > 0$ such that*

$$||a||_{L^\alpha}^{p-1} ||b||_{L^\beta}^{1-q} < \eta \tag{1.6}$$

and there exist $\delta > 0$ such that if $||u_0||_{H_0^1} \le \delta$ with

$$\int_\Omega \nabla u_0 \nabla \varphi - \int_\Omega f(x,u_0)\varphi \le 0 \tag{1.7}$$

for all $\varphi \in H_0^1(\Omega)$ with $\varphi \ge 0$, then $T_{\max} = \infty$.

Moreover, there exist a function $w \in H_0^1(\Omega)$ such that $u(t) \to w$ in $L^2(\Omega)$ as $t \to \infty$, where w is a solution of (1.2) in the following sense: for all $\varphi \in H_0^1(\Omega)$

$$\int_\Omega \nabla w \nabla \varphi dx = \int_\Omega f(x, w)\varphi dx. \tag{1.8}$$

Remark 1.5 *The values of η and δ are given by (4.28) and (4.29) respectively.*

To our example of application (1.3) with $u_0 = 0$ we have.

Corollary 1.6 *Assume f given by (1.3) with $\alpha > N/2$, $\beta > (\frac{2^*}{p+1})'$, $\alpha, \beta > 1$ and $u_0 = 0$. Let u be the maximal positive solution of (1.1) in a maximal interval $[0, T_{\max})$.*

Moreover, if $0 < q < 1 < p < 2^* - 1$, there exists $\epsilon_1 > 0$ and nonempty open set $\Omega_1 \subset \mathbb{R}^N$ such that $a(x) \geq \epsilon_1$ for x a.e. in Ω_1, then there exist a constant $\eta > 0$ such that for all

$$\lambda < \eta ||a||_{L^\alpha}^{-1} ||b||_{L^\beta}^{-(1-q)/(p-1)}$$

u is global and $u(t)$ converge to w in $L^2(\Omega)$ as $t \to \infty$, where w is the solution of (1.2) in the sense of (1.8).

Concerning the blow-up solution we have the following.

Theorem 1.7 (Blow-up solutions) *Assume $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, (H_1) and (H_2). Let u be the maximal positive solution of (1.1) defined in the maximal interval $[0, T_{\max})$.*

(i) If f satisfies (H_4), $u_0 \neq 0$ satisfies (1.7) and $\lambda_1(\tilde{m}, \Omega_2)$ is the principal eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ for the weight $\tilde{m} = \tilde{a}^{(p-1)/(p-q)} \tilde{b}^{(1-q)/(p-q)}$, then it is possible find explicitly a constant c=c(p,q) such that $T_{\max} < \infty$, if

$$\lambda_1(\tilde{m}, \Omega_2) < c(p, q).$$

(ii) If f satisfies (H_5), then there exist $\eta > 0$ such that if $\int_{\Omega_3} u_0 \psi_1 dx > \eta$, then $T_{\max} < \infty$.

Remark 1.8 *The value of $c = c(p, q)$ is given by (4.22).*

To our example of application (1.3) with $u_0 = 0$ we have.

Corollary 1.9 *Assume f given by (1.3) with $\alpha, \beta > N/2$, $\alpha, \beta > 1$, $u_0 = 0$ and u the maximal positive solution of (1.1) defined in a maximal interval $[0, T_{\max})$.*

If there exist $\epsilon_2 > 0$ and nonempty open set $\Omega_2 \subset \mathbb{R}^N$ such that $b(x) \geq \epsilon_2$ for x a.e. in Ω_2 and meas$(\Omega_2 \cap \{a > 0\} \cap \{b > 0\}) > 0$, then there exists $\eta' = \eta'(p, q, \Omega_2) > 0$ such that for $\lambda > \eta'$, $T_{\max} < \infty$.

We make to observe the existing relation between the conditions $(H_1), ..., (H_5)$ used here and the conditions used in [12] for the show the existence of a solution of (1.2). From [12], we have that if f is sublinear in 0, superlinear in ∞ on nonempty subdomains $\Omega_1, \Omega_2 \subset \Omega$ respectively, that is,

$$\begin{aligned} \liminf_{t \to 0} \frac{f(x,s)}{s} &> \lambda_1(\Omega_1) \text{ uniformly for } x \in \Omega_1, \\ \liminf_{t \to \infty} \frac{f(x,s)}{s} &> \lambda_1(\Omega_2) \text{ uniformly for } x \in \Omega_2 \end{aligned} \tag{1.9}$$

(compare with (H_2) and (H_5)), satisfy a standard subcritical growth condition, a weaker form of the classical condition of Ambrosetti-Rabinowitz [2] and the bound from above (H_3) condition then there exist $\eta = \eta(p, q, N) > 0$ such that if

$$||a_0||_{L^{\sigma_q}}^{p-1} ||b_0||_{L^{\sigma_p}}^{1-q} \leq \eta \tag{1.10}$$

with $\sigma_q = (\frac{2^*}{q+1})'$, $\sigma_p = (\frac{2^*}{p+1})'$, then the problem (1.2) has at less two solution. The solutions of (1.2) are understood in the sense of (1.8). They also showed that if there exists $0 \le q < 1 < p$ and nonnegative function $\tilde{a}, \tilde{b}$ such that

$$f(x,s) \ge \tilde{a}s^q + \tilde{b}s^p \text{ for a.e } x \in \Omega \text{ and all } s \ge 0 \tag{1.11}$$

with $\tilde{m} = \tilde{a}^{(p-1)/(p-q)}\tilde{b}^{(1-q)(p-q)} \neq 0$ on Ω and $\tilde{m} \in L^r(\Omega)$ for some $r > N/2$(compare with (H_4)), then it is possible find explicitly a constant $c = c(p,q) > 0$ such that (1.2) has no positive solution if $\lambda_1(\tilde{m},\Omega) < c(p,q)$, where $\lambda_1(\tilde{m},\Omega)$ is the principal eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ for the weight $\tilde{m}$.

With respect to uniqueness to the problem (1.1) with f given by (1.3) we have the following result.

Theorem 1.10 *Assume $a \in L^\alpha(\Omega), b \in L^\beta(\Omega)$, $a \neq 0$, $\alpha, \beta > 1$, $\alpha > N/(q+1)$ and $\beta > N/2$. Let u be the solution of (1.3) defined on a maximal time interval $[0, T_{\max})$*

(i) If $u_0 \neq 0$, then the solution is unique.

(ii) If $u_0 = 0$, then the set solutions of (1.1) consists of

- *(a) the trivial solution $u = 0$,*
- *(b) a solution u such that $u(x,t) > 0$ for any $t \in (0, T_{\max})$ and $x \in \Omega$,*
- *(c) a monoparametric family $\{u_\mu\}_{\mu>0}$ defined on the maximal interval $[0, T_{\max}+\mu)$, where $u_\mu(t) = u((t-\mu)_+)$, u is the solution obtained in (b) and $z_+ = \max\{z, 0\}$.*

We make a commentary of our results. The Theorem 1.3 has been proved in [8] for $f = f(u)$. We used some modifications in the arguments used by them.

One of the main results of [8] for $a = b = 1$ is the following: if $u_0 = 0$, then there exist λ^* such that for $0 < \lambda < \lambda^*$, then $T_{\max} = \infty$ and the solution $u_\lambda(t)$ converge to u_λ in $L^\infty(\Omega)$ when $t \to \infty$, where u_λ is a minimal solution of (1.2). If $\lambda > \lambda^*$, then $T_{\max} < \infty$. These results too are valid [4] if $f(x,s) = f(s)$ with $f \in C^1$ is convex plus a grown condition in the infinite. Our result(Corollary 1.6 and 1.9) show that part of these conclusions remain valid.

The paper is organized as follows. In section 2 we state the comparison principle for (1.1) for several situations of f. In section 3 we show the Theorem 1.3. The Theorem 1.4 is show in section 4. The Theorem 1.7 and its corollary are prove in the section 5. The section 6 is dedicated to show the Theorem 1.10.

2 Comparison principle

We start by studying a comparison result for the equation (1.1).

We say that a function $v \in L^2_{loc}((0,T), H^1(\Omega)) \cap W^{1,2}_{loc}((0,T), H^{-1}(\Omega)) \cap C([0,T], L^2(\Omega)) \cap L^\infty((0,T) \times \Omega)$ is a supersolution of (1.1) if

$$\begin{cases} v_t - \Delta v & \ge f(x,u) & \text{in } (0,T) \times \Omega \\ v & \ge 0 & \text{on } (0,T) \times \partial\Omega \\ v(0) & \ge u_0 & \text{in } \Omega. \end{cases} \tag{2.12}$$

subsolutions are defined analogously, with reversed inequalities in (2.12).

In the following proposition we establish a comparison result for the equation (1.3).

Proposition 2.1 *Let $a \in L^\alpha(\Omega)$, $b \in L^\beta(\Omega)$, $\alpha, \beta \ge 1$, $\alpha > N/(q+1)$ and $\beta > N/2$. If u is a supersolution of (1.1) with $u(0) \ge \gamma d_\Omega$ for some $\gamma > 0$ and v is a subsolution of (1.1) both defined on the interval $[0,T], T > 0$, then $u(t) \ge v(t)$ for all $t \in [0,T]$.*

Proof. Since $u_t - \Delta u \ge 0$ and $u(0) \ge \gamma d_\Omega$, there exists $\eta > 0$ such that $u(t) \ge \eta d_\Omega$ for all $t \in [0,T]$.

Multiplying by $w = (v-u)^+ \in H_0^1(\Omega)$ the difference of the inequalities satisfied by u and v, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 dx + \int_\Omega |\nabla w|^2 dx \le \underbrace{\int_\Omega a(v^q - u^q)w dx}_{I_1} + \underbrace{\int_\Omega b(v^p - u^p)w dx}_{I_2} \tag{2.13}$$

Let $\alpha_1, \alpha_2 \geq 1$ be defined by $1/(\alpha_1) = (1+q)/2 - 1/\alpha$ and $1/\alpha_2 = 1 - q/2$. Using Hölder's, Hardy's, Gagliardo-Niremberg's and Young's inequalities, we have

$$\begin{aligned} I_1 &= \int_\Omega a(v^q - u^q)w dx \\ &\leq q \int_{\{v>u\}} a u_\theta^{q-1} w^2 dx; \qquad u_\theta = \theta u + (1-\theta)v \text{ for some } \theta \in (0,1) \\ &\leq \frac{q}{\gamma^{1-q}} ||a||_{L^\alpha} ||w^{1+q}||_{L^{\alpha_1}} || \frac{w^{1-q}}{d_\Omega^{1-q}} ||_{L^{\alpha_2}} \\ &\leq C ||a||_{L^\alpha} ||w||^{q+1}_{\alpha_1(q+1)} ||\nabla w||^{1-q}_{L^2} \\ &\leq C ||a||_{L^\alpha} ||w||^{(q+1)(1-\theta_1)}_{L^2} ||\nabla w||^{\theta_1(q+1)+(1-q)}_{L^2} \\ &\leq \epsilon ||\nabla w||^2_{L^2} + C ||a||^{\Theta_1}_{L^\alpha} ||w||^2_{L^2} \end{aligned} \tag{2.14}$$

where $\theta_1 = N/(\alpha(q+1))$ and $\Theta_1 = 2/[(q+1)(1-\theta_1)]$.

Analogously, we have

$$\begin{aligned} I_2 &= \int_\Omega b(v^p - u^p) dx \\ &\leq p M^p \int_{\{v>u\}} b w^2 dx \\ &\leq p M^p ||b||_{L^\beta} ||w||^2_{L^{2\beta'}} \\ &\leq C ||b||_{L^\beta} ||\nabla w||^{2\theta_2}_{L^2} ||w||^{2(1-\theta_2)}_{L^2} \\ &\leq \epsilon ||\nabla w||^2_{L^2} + C ||b||^{\Theta_2}_{L^\beta} ||w||^2_{L^2} \end{aligned} \tag{2.15}$$

where $M = \max\{||u||_{L^\infty((0,T)\times\Omega)}, ||v||_{L^\infty((0,T)\times\Omega)}\}$, $\theta_2 = N/(2\beta)$ and $\Theta_2 = (2\beta)/(2\beta - N)$.

From (2.13), (2.14), (2.15) doing $0 < \epsilon < 1/2$ we obtain

$$\frac{1}{2}\frac{d}{dt} \int_\Omega w^2 dx \leq C(||a||^{\Theta_1}_{L^\alpha} + ||b||^{\Theta_2}_{L^\beta}) \int_\Omega w^2 dx$$

from which the result follows. $\square$

In the following result we analyze the case where $u_0 = 0$.

Proposition 2.2 *Let $a \in L^\alpha(\Omega)$, $b \in L^\beta(\Omega)$, $\alpha, \beta \geq 1$. If u is a supersolution positive of (1.1) with $u(0) = 0$ and v is a subsoluion of (1.1) defined on some interval $[0,T], T > 0$ with $u, v \in L^\infty((0,T) \times \Omega) \cap C([0,T], L^2(\Omega)) \cap L^2((0,T), H^1(\Omega)) \cap W^{1,2}((0,T), H^{-1}(\Omega))$ then, $u(t) \geq v(t)$ for all $t \in [0,T]$.*

Proof. Since $u_t - \Delta u \geq 0$, we see that $u(t) \geq S(t-\sigma)u(\sigma)$ for all $0 \leq s \leq t \leq T$ and since u is positive, it follows from the strong maximum principle that $u(t) \geq \delta(t) d_\Omega$ for all $t \in (0,T]$ with $\delta(t) > 0$. By proposition 2.1 we conclude that $u(t+\sigma) \geq v(t)$ for all $t \in [0, T-\sigma]$. Fixing $t \in [0,T]$, the result follows letting $\sigma \to 0$. $\square$

Remark 2.3 *It is possible to observe from the above proof that in the case f satisfies the following condition: there exist $a \in C([0,\infty), L^\gamma(\Omega))$ such that for all $0 \leq s, t \leq M (M > 0)$,*

$$|f(x,s) - f(x,t)| \leq a(x, s_\theta)|s - t|$$

and $s_\theta \in [s,t]$. Then, the maximum principle is hold without the condition $u(0) \geq \gamma d_\Omega$ for some $\gamma > 0$. Indeed, with the same notation of the proof of the previous theorem, coming in the same way that (2.13) and (2.14) we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 &\leq \int_\Omega [f(x,v) - f(x,v)] w dx \\ &\leq \int_\Omega a(x, u_\theta) w^2 dx, u_\theta = \theta u + (1-\theta)v \text{ for some } \theta \in [0,1] \\ &\leq ||a||_{C([0,M], L^\gamma)} ||w||^2_{L^{2\gamma'}} \\ &\leq \epsilon ||\nabla w||^2_{L^2} + C ||a||^{\Theta_2}_{C([0,M],L^\gamma)} ||w||^2_{L^2}. \end{aligned}$$

Therefore, choosing $0 < \epsilon < 1$, we obtain

$$\frac{d}{dt}\int_\Omega w^2 \leq C||a||^{\Theta_2}_{C([0,M],L^\gamma)}||w||^2_{L^2}$$

of which follows the result.

Another situation where the maximum principle is holds is the following.

Remark 2.4 *Assume that $f : [0,\infty) \to [0,\infty)$ is continuous and satisfies that for every $M > 0$, there exists $L > 0$ such that*

$$f(y) - f(x) \leq \frac{L}{x}(y - x)$$

for all $0 < x \leq y \leq M$.

In this case the maximum principle is valid as it was shown in [8]. In order to see this we used the notation of the demonstration of the proposition 2.1. Then,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 &\leq \int_\Omega [f(v) - f(u)]w \\
&\leq L_M \int_{\{v>u\}} \frac{w^2}{u} \\
&\leq \int_\Omega \frac{w^2}{d^2_\Omega} + C(\epsilon)\int_\Omega w^2
\end{aligned}$$

since $(\gamma d_\Omega)^{-1} \leq \epsilon d_\Omega^{-2} + C(\epsilon)$. Choosing $\epsilon > 0$ sufficiently small we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 \leq C\int_\Omega w^2.$$

3 proof of Theorem 1.3.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^N$ be a domain. If $1 \leq r \leq s$ and $u_0 \in L^r(\Omega)$, then $S(t)u_0 \in L^s(\Omega)$ and*

$$||S(t)u_0||_{L^s(\Omega)} \leq t^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{s})}||u_0||_{L^r}.$$

For the proof see [7].

We will divide the proof in some steps.

Step 1. Let $M = ||u_0||_{L^\infty} + 1 > 0$ be and suppose that $f \in C([0,\infty), L^\gamma(\Omega))$ and there exists a function $a \in C([0,\infty), L^\gamma(\Omega))$ such that

$$|f(x,s) - f(x,t)| \leq a(x,s_\theta)|s - t| \tag{3.16}$$

where $s_\theta \in [s,t]$, $s,t \geq 0$ and x a.e. in Ω.

For $T > 0$, let $E = L^\infty((0,T), L^\infty(\Omega))$ be and consider the set

$$B = \{u \in E; u(t) \geq 0 \text{ and } ||u(t)||_{L^\infty} \leq M \text{ for all } t \in (0,T)\}$$

and the application $\phi : B \to E$ defined by

$$\phi u(t) = S(t)u_0 + \int_0^t S(t-\sigma)f(x,u(\sigma))d\sigma.$$

It is clear that B is a space of Banach with the metric induced by E. We will show that $\phi : B \to B$ and it is a strict contraction if T is small enough.

Since $u_0 \geq 0$ and $f \geq 0$, by the positivity of $S(t)$ we have that $\phi u(t) \geq 0$. By the Lemma 3.1 we have that

$$\begin{aligned} ||\phi u(t)||_{L^\infty} &\leq ||u_0||_{L^\infty} + \int_0^t (t-\sigma)^{-\frac{N}{2\gamma}} ||f(\cdot, u(\sigma))||_{L^\gamma} d\sigma \\ &\leq ||u_0||_{L^\infty} + \int_0^t (t-\sigma)^{-\frac{N}{2\gamma}} [||a(\cdot, u(\sigma))||_{L^\gamma} ||u(\sigma)||_{L^\infty} + ||f(0)||_{L^\gamma}] d\sigma \\ &\leq ||u_0||_{L^\infty} + \frac{T^{1-\frac{N}{2\gamma}}}{1-\frac{N}{2\gamma}} [||a||_{C((0,M),L^\gamma)} M + ||f(0)||_{L^\gamma}] \end{aligned} \tag{3.17}$$

Similarly, one shows that for $u, v \in B$,

$$||\phi u(t) - \phi v(t)||_{L^\infty} \leq \frac{T^{1-\frac{N}{2\gamma}}}{1-\frac{N}{2\gamma}} ||a||_{C((0,M),L^\gamma)} ||u-v||_{L^\infty((0,T),L^\infty(\Omega))} \tag{3.18}$$

It follows from the estimates (3.17) and (3.18) that if T is small enough (depending on M), then $\phi : B \to B$ is a strict contraction. Thus ϕ has a unique fixed point in B.

Using (3.16) we can to show a uniqueness and thus the solution can be extended to maximal interval $[0, T_{\max})$. On the other hand, since T depend only of $||u_0||_{L^\infty}$ the blow-up occurs in standard way.

Step 2. By the assumption $(H1)$ we have that for $n \geq 1$, f_n satisfies (3.16). Let u^n be the solution of (1.1) defined on the maximal interval $[0, T^n_{\max})$ obtained in the step 1, that is, $u_n \in L^\infty((0,T) \times \Omega)$ for all $T < T^n_{\max}$ and satisfies

$$\begin{cases} (u^n)_t - \Delta u^n &= f_n(x, u^n) \quad (x,t) \in \Omega \times (0, T^n_{\max}) \\ u(x,t) &= 0 \qquad\qquad (x,t) \in \partial\Omega \times (0, T^n_{\max}) \\ u(x,0) &= u_0 \qquad\qquad x \in \Omega. \end{cases} \tag{3.19}$$

Since $f_{n+1} \leq f_n$ and the condition (3.16) is valid, we have by the maximal principle(remark 2.3) that $u_{n+1}(t) \leq u_n(t)$ for all $t \in [0, T)$ with $T = \min\{T^n_{\max}, T^{n+1}_{\max}\}$. By the blow-up alternative we conclude that $T^n_{\max} \leq T^{n+1}_{\max}$. Therefore, there exists $\lim_{n\to\infty} u_n(t) = u(t) \geq 0$ for all $t \in [0,T]$ with $T < T^1_{\max}$.

On the other hand, u^n satisfies the equation

$$u^n(t) = S(t)u_0 + \int_0^t S(t-\sigma) f_n(\cdot, u^n(\sigma)) d\sigma. \tag{3.20}$$

Let $M = ||u^1||_{L^\infty((0,T)\times\Omega)}$ be. Since $||u^n||_{L^\infty((0,T)\times\Omega)} \leq M$, we obtain of the convergence $f_n \to f$ in $C([0,M)], L^\gamma(\Omega))$ when $n \to \infty$ and the Lemma 3.1 that there exist a constant $C > 0$ such that $||S(t - \sigma) f_n(\cdot, u^n(\sigma))||_{L^\infty} \leq C(t-\sigma)^{1-\frac{N}{2\gamma}} \in L^1(0,t)$. In addition, by continuity of $f \in C([0,M], L^\gamma(\Omega))$ and the triangular inequality of $f_n(\cdot, u_n(\sigma))$, $f(\cdot, u^n(\sigma))$ and $f(\cdot, u^n(\sigma))$ we have that $S(t-\sigma) f_n(\cdot, u^n(\sigma)) \to f(\cdot, u(\sigma))$ for σ a.e. in $(0,t)$. Using the dominated convergence theorem we conclude of (3.20) doing $n \to \infty$ that u is the nonnegative solution of (1.1).

Step 3. Existence of a maximal solution. Let u the nonnegative defined in the interval $[0,T]$ and obtained by the step 2. We claim that u is the maximal solution in [0,T]. Indeed, if v is any other solution of (1.1) defined on $[0,T]$ then v is a subsolution of (3.19) in $[0,T]$, since that $f \leq f_n$ for each $n \geq 1$. By the remark 2.3, $u^n \geq v$ for all n. Letting $n \to \infty$ we have that $u \geq v$.

Uniqueness of the maximal solution implies that the solution can be extended to a maximal interval $[0, T_{max})$ and the blow-up alternative follows.

Step 4. Existence of a nonnegative solution for $u_0 = 0$. We used the ideas of [8]. For $t \in [0, t_0]$ we define

$$h(t) = \int_0^t \frac{d\sigma}{g(\sigma)}.$$

Then, $h(0) = 0$ and h is increasing in $[0, t_0]$. Let $v(t) = w(t)S(t)1_{\Omega_1}$ for all $t \in [0, h(t_0)]$ and $w(t) = h^{-1}(t)$. We have that

$$v_t - \Delta v = w'(t)S(t)1_{\Omega_1} = g(w(t))S(t)1_{\Omega_1}$$

Since that $S(t)1_{\Omega_1} \leq 1$ and $w(t) \leq t_0$ by the concavity of the function g and (i) we have that $g(w(t))S(t)1_{\Omega_1} \leq g(w(t)S(t)1_{\Omega_1}) = g(v)$. Thus, v is the subsolution of

$$\begin{cases} u_t - \Delta u & = g(u) \quad x \in \Omega_1, t > 0 \\ u(t,x) & = 0 \qquad x \in \partial\Omega_1, t > 0 \\ u(x,0) & = 0 \qquad x \in \Omega_1. \end{cases}$$

Moreover, if $\lambda_1(\Omega_1)$ is the principal eigenvalue and ψ_1 is the corresponding eigenfunction of $-\Delta$ in Ω_1, then there exists a constant $C > 0$ such that

$$v(t) = w(t)S(t)1_{\Omega_1} \geq Cw(t)S(t)\psi_1 = Cw(t)e^{-\lambda_1(\Omega_1)t} > 0 \tag{3.21}$$

for all $t \in (0, h(t_0)]$.

On the other hand, let $\tilde{u} = u|_{\Omega_1}$ be the restriction of the function u to Ω_1. By the Remark 1.1, we have that $\tilde{u} \in L^\infty((0,T) \times \Omega_1) \cap L^r((0,T), H^1(\Omega_1)) \cap W^{1,r}((0,T), H^{-1}(\Omega_1)), r < \infty$ where $T < \min\{T_{\max}, h(t_0)\}$ and by (H_2) we have that $\tilde{u}$ satisfies

$$\begin{cases} u_t - \Delta u & \geq g(u) \quad (x,t) \in \Omega_1 \times (0,T) \\ u(t,x) & \geq 0 \qquad (x,t) \in \partial\Omega_1 \times (0,T) \\ u(x,0) & = 0 \qquad x \in \Omega_1. \end{cases}$$

Then, by the maximum principle(remark 2.4) and (3.21) we have that $\tilde{u}(t) \geq v(t) > 0$. Thus $u(t) \neq 0$ for all $t \in [0,T]$ and x a.e. in Ω_1.

Let $t \in (0, T_{\max})$ and $\tau \in (0,t)$ be with $\tau < T$. Since that $u(t) \geq S(t-\tau)u(\tau)$ and $u(\tau) \neq 0$, by the strong maximum principle we have that $u(t) > 0$ for $t > \tau$. Let $\tau \to 0$ we have that $u(t) > 0$ for $t > 0$. □

Remark 3.2 *(i) In the proof we consider the sequence approximation $(f_n)_{n\geq 1}$ and we obtain the maximal solution u_n. It is clearly to observe that this solution does not depend of the choice of the sequence (f_n).*

(ii) Let u be the solution obtained in the Theorem 1.3 and defined on a maximal interval $[0, T_{\max})$. If $w(t) = u(t) - S(t)u_0 = \int_0^t S(t-\sigma)f(\cdot, u(\sigma))d\sigma$ for all $t \in [0, T_{\max})$, then $w \in C([0, T_{\max}), L^\infty(\Omega))$. This is consequence of the following lemma.

Lemma 3.3 *Assume that $f : [\tau, T] \to L^s(\Omega)$ is measurable for some $1 \leq s \leq \infty$. If $1 \leq r \leq \infty$, $\theta \geq 0$ and*

(i) for each $t \in (\tau, T)$, $S(t - \cdot)f(\cdot) \in D((-\Delta^\theta))$

(ii) for each $t \in (\tau, T), (-\Delta)^\theta S(t - \cdot)f(\cdot) \in L^1((\tau, T), L^r(\Omega))$,

(iii) if $\tau \leq t_0 < t < T$, then $\lim_{t\to t_o} || \int_{t_0}^t (-\Delta)^\theta S(t-\sigma)f(\sigma)d\sigma||_{L^r} = 0$,

then the function $w : [\tau, T] \to L^r(\Omega)$ defined by

$$w(t) = \int_\tau^t (-\Delta)^\theta S(t-\sigma)f(\sigma)d\sigma$$

is continuous in $[\tau, T]$.

For the proof, see [15](Lema 2.1). For more information on $(-\Delta)^\theta$ see [13].

Proof of (ii) of remark 3.2. Let $M \geq ||u||_{L^\infty((0,T)\times\Omega)}$ be with $T \in (0, T_{\max})$. For $0 \leq t_0 < t < T$ it follows

$$\begin{aligned} ||w(t)||_{L^\infty} &\leq \int_{t_0}^t (t-\sigma)^{-\frac{N}{2\gamma}} ||f(\cdot, u(\sigma))||_{L^\gamma} \\ &\leq C||f||_{C([0,M],L^\gamma)}(t-t_0)^{1-\frac{N}{2\gamma}} \to 0, \text{ if } t \to t_0. \end{aligned}$$

4 Proof of the Theorem 1.4.

We start this section enunciating some preliminary results.

Lemma 4.1 *Let $A, B \geq 0$ and $0 \leq q < 1 < p$. Then there exists a constant $c = c(p,q) > 0$ such that*

$$As^q + Bs^p \geq cA^{(p-1)/(p-q)}B^{(1-q)/(p-q)}s$$

for all $s \geq 0$.

It is followed directly of Young's inequality. The value of c is

$$c = (\max\{\frac{p-1}{p-q}, \frac{1-q}{p-q}\})^{-1}. \tag{4.22}$$

Lemma 4.2 *Assume that $0 \leq q < 1 < p$, $A, B > 0$, and consider the function $f(t) = t^2 - At^{q+1} - Bt^{p+1}$ for $t \geq 0$. Then $\max_{t\geq 0}\{f(t)\}$ is positive if and only if*

$$A^{p-1}B^{1-q} < (p-1)^{p-1}(1-q)^{1-q}(p-q)^{q-p}.$$

For the proof see [12].

Lemma 4.3 (A singular Gronwall inequality) *Let $T > 0$, $A \geq 0$, $r, s \in [0,1]$ and let f be a nonnegative function with $f \in L^p(0,T)$ for some $p > 1$ such that $p' \max\{r,s\} < 1$. Consider a nonnegative function $\varphi \in L^\infty(0,T)$ such that*

$$\varphi(t) \leq At^{-r} + \int_0^t (t-\sigma)^{-s} f(\sigma)\varphi(\sigma)d\sigma \text{ for almost all } t \in [0,T].$$

Then there exists C, depending only on T, r, s, p and $||f||_{L^p}$ such that

$$\varphi(t) \leq ACt^{-r}$$

for almost all $t \in [0,T]$.

For the proof, see e.g. [7].

For the equation (1.1) we defined the associated energy

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(u) \tag{4.23}$$

where $G(u) = \int_0^u f(x,\sigma)d\sigma$. We have the following lemma.

Lemma 4.4 *Assume $u_0 \in L^\infty(\Omega) \cap H_0^1(\Omega)$. If u is the solution of (1.1) defined on the maximal interval $[0, T_{\max})$, then $u \in C([0, T_{\max}), H_0^1(\Omega))$.*

Proof. Since $S(t)u_0 \in C([0,\infty), H_0^1(\Omega))$, it suffices show that $w(t) = \int_0^t S(t-\sigma)f(\cdot,u(\sigma))d\sigma$ is continuous at $[0,T_{\max})$. Let $M \geq ||u(t,x)||_{L^\infty((0,T)\times\Omega)}$ be with $T \in (0,T_{\max})$. For $0 \leq t_0 < t < T < T_{\max}$ we have when $\alpha < 2$,

$$\begin{aligned} ||w(t)||_{H_0^1} &= ||\textstyle\int_{t_0}^t S(t-\sigma)f(\cdot,u(\sigma))d\sigma||_{H_0^1} \\ &\leq C\textstyle\int_{t_0}^t (t-\sigma)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{\alpha}-\frac{1}{2})}||f(\cdot,u(\sigma))||_{L^\gamma}ds \\ &\leq C||f||_{L^\infty(0,T),L^\gamma)}(t-t_0)^{1-\Theta} \end{aligned}$$

where $\Theta = (1/2 + N/2)(1/\alpha - 1/2) < 1$. By analogy when $\alpha \geq 2$ we have

$$||w(t)||_{H_0^1} \leq C||f||_{L^\infty(0,T),L^\gamma)}(t-t_0)^{1/2}.$$

Hence, w is continuous in $[0,T]$. Since T is arbitrary we have that w is continuous in $[0,T_{\max})$. $\square$

Proof of the Theorem 1.4. *Proof of (i).* We only analyze the case $N\geq 3$, because our arguments can easily be be adapted by standard modifications to the cases $N=1$ or $N=2$. We consider some steps.

Step 1. We affirmed that if u_0 satisfies (1.7) then the solution u is nondecreasing. Indeed, Let $M\geq ||u_0||_{L^\infty}$ be and let u_n be the sequence of solutions of (1.1) corresponding to f_n, a_n given by the $(H1)$ hypothesis and defined in $[0,T^n_{\max})$, then $0\leq \Delta u_0+f(x,u_0)\leq \Delta u_0+f_n(x,u_0)$ a.e in Ω. Thus, u_0 is a subsolution of the equation

$$\begin{cases} (u_n)_t-\Delta u_n &= f_n(x,u_n) \quad t>0, x\in\Omega\\ u_n &= 0 \qquad\qquad\quad t>0, x\in\partial\Omega\\ u_n(0) &= u_0, x\in\Omega \end{cases}$$

Therefore, by the remark 2.3 we have that $u_n(\tau)\geq u_0$ for all $\tau\in[0,T^n_{\max})$. Again, by the remark 2.3, we have that $u_n(t+\tau)\geq u_n(t)$ for all $t\in[0,T^n_{\max}-\tau)$. Since u_n is nonincreasing, doing $n\to\infty$, we have that $u(t+\tau)\geq u(t)$ for all $t\in[0,T^1_{\max}-\tau)$. Thus we concluded that $u_t\geq 0$.

By the remark 1.1 (ii), we have that $u_t-\Delta u=f(x,u)$ in $L^\gamma(\Omega)$. Since that $u(t)\in W^{2,\gamma}\hookrightarrow L^{\gamma'}(\Omega)$, $u(t)\in H^1_0(\Omega)$ and $u_t\geq 0$ we obtain that $\int_\Omega|\nabla u|^2\leq\int_\Omega f(x,u)udx$ and by (H_3) we concluded that

$$\int_\Omega|\nabla u|^2\leq\int_\Omega a(x)u^{q+1}+b(x)u^{p+1}. \tag{4.24}$$

Step 2. We show now that

$$\sup_{0<t<T_{\max}}||u(t)||_{H^1_0}<\infty. \tag{4.25}$$

It follows from (4.24) that

$$E(u(t))\leq\frac{1}{2}[\int_\Omega a(x)u^{q+1}+b(x)u^{p+1}]$$

and by (H_3) we have that

$$E(u(t))\geq\frac{1}{2}\int_\Omega|\nabla u|^2-\frac{1}{q+1}\int_\Omega a(x)u^{q+1}-\frac{1}{p+1}\int_\Omega b(x)u^{p+1}.$$

Therefore,

$$\frac{1}{2}\int_\Omega|\nabla u|^2-\frac{q+2}{2(q+1)}\int_\Omega a(x)u^{q+1}-\frac{p+2}{2(p+1)}\int_\Omega b(x)u^{p+1}\leq 0$$

and by Holders inequality

$$\frac{1}{2}||u||_{H^1_0}-C_1||a||_{L^\alpha}||u||^{q+1}_{L^{2^*}}-C_2||b||_{L^\beta}||u||^{p+1}_{L^{2^*}}\leq 0$$

where $C_1=\frac{q+2}{2(q+1)}|\Omega|^{\frac{1}{\alpha'}-\frac{q+1}{2^*}}$, $C_2=\frac{p+2}{2(p+1)}|\Omega|^{\frac{1}{\beta'}-\frac{p+1}{2^*}}$. By Sobolev's inequality we have that

$$\frac{1}{2}||u||_{H^1_0}-C_1'||a||_{L^\alpha}||u||^{q+1}_{H^1_0}-C_2'||b||_{L^\beta}||u||^{p+1}_{H^1_0}\leq 0 \tag{4.26}$$

where $C_1'=C_1/S^{(q+1)/2}, C_2'=C_2/S^{(p+1)/2}$ and S is the best Sobolev constant, which is independent of Ω, i.e.

$$S=\inf\{\int_\Omega|\nabla u|^2; u\in H^1_0(\Omega) \text{ and } \int_\Omega|u|^{2^*}=1\}.$$

We consider now the function $f(s)=s^2-2C_1'||a||_{L^\alpha}t^{q+1}-2C_2'||b||_{L^\beta}t^{p+1}$. By the Lemma 4.2 we have that if

$$||a||^{p-1}_{L^\alpha}||b||^{1-q}_{L^\beta}<\eta \tag{4.27}$$

with

$$\eta = \frac{(p-1)^{p-1}(1-q)^{1-q}(p-q)^{q-p}}{2^{p-q}C_1'^{p-1}C_2'^{1-q}} \tag{4.28}$$

then $\max_{s\geq 0}\{f(s)\}$ is positive. Let $\delta > 0$ be the smaller root of the equation $f(s) = 0$, that is,

$$\delta^{1-q} - 2C_1'||a||_{L^\alpha} - 2C_2'||b||_{L^\beta}\delta^{p-q} = 0. \tag{4.29}$$

We choose u_0 such that $||u_0||_{H_0^1} < \delta$. Since that $u \in C([0, T_{\max}, H_0^1(\Omega))$ and (4.26) we conclude that $||u(t)||_{H_0^1}$ is trapped in the interval $[0, s_0]$. Thus we obtain that (4.25) is valid.

Step 3. We show that $T_{\max} = \infty$. We use the follow argument. Let $s = 2^*/(p-1)$ be. Since that $\beta > (\frac{2^*}{p+1})'$ it is possible to observe that $\frac{N}{2}(\frac{1}{\beta} + \frac{1}{s}) < 1$. Hence,

$$\begin{aligned} ||u(t)||_{L^\infty} &\leq ||u_0||_{L^\infty} + \int_0^t (t-\sigma)^{-\frac{N}{2\alpha}}||a||_{L^\alpha}||u^q||_{L^\infty}d\sigma + \\ &\qquad \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{1}{s})}||b||_{L^\beta}||u||^p_{L^{sp}}d\sigma. \end{aligned} \tag{4.30}$$

Since, $sp > 2^*$, we have that

$$||u||_{L^{sp}} \leq ||u||_{L^{2^*}}^{\frac{p-1}{p}}||u||_{L^\infty}^{\frac{1}{p}}. \tag{4.31}$$

It follows from (4.25), (4.30) and (4.31) that

$$\begin{aligned} ||u(t)||_{L^\infty} &\leq ||u_0||_{L^\infty} + \int_0^t (t-\sigma)^{-\frac{N}{2\alpha}}||a||_{L^\alpha}||1+u||_{L^\infty}d\sigma + \\ &\qquad \int_0^t (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{1}{s})}||b||_{L^\beta}||u||^{p-1}_{L^{2^*}}||u||_{L^\infty}d\sigma \\ &\leq ||u_0||_{L^\infty} + C_1 t^{1-\frac{N}{2\alpha}} + C_2\int_0^t [(t-s)^{-\frac{N}{2\alpha}} + (t-\sigma)^{-\frac{N}{2}(\frac{1}{\beta}+\frac{1}{s})}]||u(\sigma)||_{L^\infty}d\sigma \end{aligned}$$

for all $t \in (0, T_{\max})$. Therefore, if $T_{\max} < \infty$, then it follows the generalized Gronwall inequality(Lemma 4.3) that

$$\sup_{0\leq t<T_{\max}}\{||u(t)||_{L^\infty}\} < \infty.$$

Impossible.

Step 4. Asymptotic behavior. We used the ideas of [4]. Let $L = \sup_{t\geq 0}\{||u(t)||_{H_0^1}\}$ be finite by the step 2. Multiplying the equation (1.1) by the principal eigenvector φ_1 of $-\Delta$ in $H_0^1(\Omega)$ and integrating

$$\int_\Omega u_t\varphi_1 - \int_\Omega \Delta u\varphi_1 = \int_\Omega f(x,u)\varphi_1. \tag{4.32}$$

Since $-\Delta u \in L^\gamma(\Omega) \hookrightarrow H^{-1}(\Omega)$ we have that

$$\begin{aligned} \int_\Omega -\Delta u\varphi_1 &=< -\Delta u, \varphi_1 >_{H^{-1},H_0^1} \\ &= \int_\Omega \nabla u \nabla\varphi_1 \\ &\leq L||\nabla\varphi_1||_{L^2}. \end{aligned} \tag{4.33}$$

On the other hand, by Poincare's inequality

$$\int_\Omega u(t)\varphi_1 \leq \frac{1}{\lambda_1(\Omega)}||\nabla u(t)||_{L^2}||\varphi_1||_{L^2} \leq \frac{L}{\lambda_1(\Omega)}||\varphi_1||_{L^2} \tag{4.34}$$

Since $u_t \geq 0$ and $f(x,.)$ is nondecreasing, integrating (4.32) in $(t, t+1)$ and using (4.33), (4.34) we have that

$$\begin{aligned} \int_\Omega f(x,u(t))\varphi_1 &\leq \int_t^{t+1}\int_\Omega f(x,u)\varphi \\ &\leq \frac{L}{\lambda_1}||\varphi_1||_{L^2} + L||\nabla\varphi||_{L^2} \\ &\leq \frac{L}{\lambda_1}(1+\lambda_1)||\varphi_1||_{H^1}. \end{aligned} \tag{4.35}$$

Since $|\Omega| < \infty$ and by (4.35) we have that

$$\sup_{t\geq 0}\int_\Omega u(t,x)dx \leq \frac{L}{\lambda_1}|\Omega|^{1/2} \tag{4.36}$$

and

$$\sup_{t\geq 0}\int_\Omega f(x,u)\varphi_1 dx \leq \frac{L}{\lambda_1}(1+\lambda_1)||\varphi_1||_{H^1}. \tag{4.37}$$

Since $u_t \geq 0$ by monotone convergence, it follows from (4.36) and (4.37) that $u(t)$ has a limit w in $L^1(\Omega)$ and that $f(x,u(t))$ converges to $f(x,w)$ in $L^1(\Omega, d_\Omega dx)$ as $t \to \infty$.

Let ζ be with $\zeta \in C(\overline{\Omega})$ and $\zeta|_{\partial\Omega} = 0$. Multiplying (1.1) by ζ and integrating on Ω and $(t, t+1)$ it follows

$$[\int_\Omega u\zeta]|_t^{t+1} - \int_t^{t+1}\int_\Omega u\Delta\zeta = \int_t^{t+1}\int_\Omega f(x,u)\zeta.$$

Letting $t \to \infty$ we find

$$\int_\Omega w(-\Delta\zeta) = \int_\Omega f(x,w)\zeta. \tag{4.38}$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta|_{\partial\Omega} = 0$.

From (4.25) we have that there exist a sequence (t_n) with $t_n \to \infty$ when $n \to \infty$ such that $u(t_n) \rightharpoonup z \in H_0^1(\Omega)$(weak convergence). By the compact immersion of $H_0^1(\Omega) \to L^2(\Omega)$(Rellich's theorem) and after extracting possibly a subsequence we have that $u(t_n) \to z$ in $L^2(\Omega)$ when $n \to \infty$. Therefore, we conclude that $w = z \in H_0^1(\Omega)$.

Taking $\zeta \in C_0^\infty(\Omega)$ in (4.38) and integrating by parts we obtain

$$\int_\Omega \nabla w\nabla\zeta = \int_\Omega f(x,w)\zeta \tag{4.39}$$

and by density we have that (4.39) is valid for all $\zeta \in H_0^1(\Omega)$.

5 Proof of the Theorem 1.7

proof of (i). We assumed that Ω_2 is smooth enough. Let ψ_1 be the positive eigenfunction associated to the principal Dirichlet eigenvalue $\lambda_1(\tilde{m}, \Omega_2)$ of $-\Delta$ on $H_0^1(\Omega)$ for the weigh $\tilde{m} = \tilde{a}^{(p-1)/(p-q)}\tilde{b}^{(1-q)/(p-q)}$. Since $\tilde{\alpha}, \tilde{\beta} > N/2$ and $meas\{\Omega_2 \cap \{a > 0\} \cap \{b > 0\}\} > 0$, we have that $\tilde{m} \in L^r(\Omega)$ for some $r > N/2$ and $\tilde{m} \neq 0$. By regularity theory we have that $\psi_1 \in C^1(\Omega_2 \cup \partial\Omega_2) \cap H^2(\Omega)$ and $\frac{\partial\psi}{\partial\nu} \leq 0$ on $\partial\Omega_2$ where ν denotes the unit exterior normal to Ω_2.

Multiplying (1.1) by ψ_1, integrating on Ω_2, using (H_4) and Lemma 4.1 we obtain

$$\frac{d}{dt}\int_{\Omega_2} u\psi_1 - \int_{\Omega_2}\Delta u\psi_1 \geq c(p,q)\int_{\Omega_2}\tilde{m}u\psi. \tag{5.40}$$

Since u_0 satisfies (1.7) we have that u is nondecreasing(step 1 of the proof of (i)). Therefore, if $\lambda_1(\tilde{m}, \Omega_2) < c(p,q)$ and $u_0 \neq 0$, then we have from (5.40)

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_2} u\psi_1 &\geq [c(p,q) - \lambda_1(\tilde{m}, \Omega_2)]\int_{\Omega_2}\tilde{m}u\psi \\ &\geq [c(p,q) - \lambda_1(\tilde{m}, \Omega_2)]\int_{\Omega_2}\tilde{m}u_0\psi. \end{aligned}$$

Thus, we have that

$$\int_{\Omega_2} u(t)\psi_1 \geq C[c(p,q) - \lambda_1(\tilde{m},\Omega_2)]t \tag{5.41}$$

for all $t \in [0, T_{\max})$, if $\lambda_1(\tilde{m},\Omega_2) < c(p,q)$.

On the other hand, if ϕ_1 is the principal eigenfunction associated to the principal eigenvalue $\lambda_1(\Omega_2)$ of $-\Delta$ on $H_0^1(\Omega_2)$, such that $\int_{\Omega_2}\phi_1 = 1$. Then multiplying (1.1) for ϕ_1, integrating in Ω_2, using Jensen's inequality and the fact that $b \geq \epsilon_2$ on Ω_2 we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\Omega_2} u\phi_1 + \lambda_1(\Omega_2)\int_{\Omega_2} u\phi_1 &\geq \int_{\Omega_2} bu^p\phi_1 \\ &\geq \epsilon_2(\int_{\Omega_2} u\phi)^p. \end{aligned} \tag{5.42}$$

From (5.42) we know that if there exist $t_0 \in [0, T_{\max})$ such that

$$\int_{\Omega_2} u(t_0)\phi_1 > (\frac{\lambda_1(\Omega_2)}{\epsilon_2})^{1/p-1} \tag{5.43}$$

then u blow-up in finite time. Therefore, if $T_{\max} = \infty$, then from (5.41) we have that (5.43) is valid for some $t_0 > 0$. Impossible.

Proof of (ii). Assume that $T_{\max} = \infty$. Since that ψ_1 is the principal eigenfunction of $-\Delta u$ in $H_0^1(\Omega_3)$ we have that $\frac{\partial\psi_1}{\partial\nu} < 0$ on $\partial\Omega_3$. Multiplying the equation (1.1) by ψ_1, integrating on Ω_3 and using (i) of (H_5) we obtain

$$\frac{d}{dt}\int_{\Omega_3} u\psi_1 + \lambda_1(\Omega_3)\int_{\Omega_3} u\psi_1 \geq \int_{\Omega_3} h(u)\psi_1. \tag{5.44}$$

Again by (ii) of (H_5), there exist $\eta, \kappa > 0$ such that

$$h(s) \geq \lambda_1(\Omega_3) + \kappa h(s), \text{ for all } s \geq \eta. \tag{5.45}$$

From (5.44), (5.45) and since that ψ_1 is normalized, that is, $\int_{\Omega_3}\psi_1 = 1$, we obtain using Jensen's inequality that

$$\frac{d}{dt}\int_{\Omega_3} u\psi_1 \geq \kappa h(\int_{\Omega_3} u\psi_1). \tag{5.46}$$

Thus, from (iii) of (H_5) and (5.46) we have that $\int_{\Omega_3} u\psi_1$ blow up in finite time, if $\int_{\Omega_3} u_0\psi_1 \geq \eta$. Impossible.

Proof of the Corollary 1.9. Since u is positive there exist $\tau > 0$ such that $u(\tau) > 0$. Let $v(t) = u(t+\tau)$ for all $t \in [0, T_{\max} - \tau)$. Since that u is nondecreasing(step 1 of the proof of Theorem 1.4) we have that $v(0) = u(\tau)$ satisfies (1.7) and therefore, the result is followed of the Theorem 1.4.

Remark 5.1 *In (5.40), (5.42) and (5.44) we used the following inequality*

$$\int_\Omega (-\Delta u)\phi \leq \lambda_1(\Omega)\int_\Omega u\phi$$

where $u \in W^{2,\gamma}(\Omega)$, $\gamma > N/2$ and ϕ_1 is the principal eigenfunction associated to principal eigenvalue $\lambda_1(\Omega)$ of $-\Delta$ in $H_0^1(\Omega)$. This is clear if $u \in C_0^\infty(\Omega)$, since $\phi = 0$ on $\partial\Omega$ and $\frac{\partial\phi}{\partial\nu} < 0$ by Green's identities. For the general case we used a density argument.

6 Proof of the theorem 1.10.

For the proof we consider some steps.

Step 1. Uniqueness of the positive solution for $u_0 = 0$. Let u the maximal solution and v any other solution positive with $v(0) = 0$. Since that u is maximal, we have that $u \geq v$. On the other hand, since v is a positive solution by the maximum principle (Proposition 2.2) it follow that $v \geq u$. Thus, $u = v$.

Step 2. Non-uniqueness for $u_0 = 0$. It is clear that for every $\mu > 0$, $u_\mu(t) = u((t-\mu)+)$, $t \in [0, T_{\max} + \mu)$, is a solution of (1.1). We show that if v be an other nontrivial solution of (1.3) with $v(0) = 0$ defined on the maximal interval $[0, \tilde{T}_{\max})$ and different of the positive solution u, then v belongs to the family $\{u_\mu\}_{\mu>0}$.

Since v is nontrivial and different of a positive solution

$$\tau = \inf\{t; v(x,t) > 0 \text{ for some } x \in \Omega\}$$

verifies $\tau \in (0, \tilde{T}_{\max})$. Then $v(\tau) = 0$ and since that $v(t) \neq 0$ for $t \in (\tau, \tilde{T}_{\max})$ it follows that $v(t) > 0$ for $t \in (\tau, \tilde{T}_{\max})$. Defining $\tilde{u}(t) = v(t+\tau)$ for $t \in [0, \tilde{T}_{\max} - \tau)$ we concluded by the maximum principle(Proposition 2.2) and the uniqueness of positive solution that $\tilde{u} = u$ and $\tilde{T}_{\max} = T_{\max} + \tau$. Therefore, $v = u_\tau$.

Step 3. Uniqueness for $u(0) = u_0 \neq 0$. Let u be a maximal solution of (1.3) obtained in the Theorem 1.3 and let v be another solution of (1.1) such that $v(0) = u_0$. We assume that both solutions are defined in the same interval $[0, T]$, $T > 0$. Thus $u(t) \geq v(t)$ for all $t \in [0, T]$.

Let $M > \max\{||u||_{L^\infty((0,T)\times\Omega)}, ||v||_{L^\infty((0,T)\times\Omega)}\}$ be. Since $u^p - v^p \leq pM^{p-1}(u-v)$ it follows that

$$\begin{aligned}(u-v)_t - \Delta(u-v) &= a(u^q - v^q) + b(u^p - v^p) \\ &\leq a(u-v)^q + pbM^{p-1}(u-v)\end{aligned}$$

and $(u-v)(0) = 0$. Let us consider now z the positive solution of the equation

$$\begin{cases} u_t - \Delta u &= au^q + pM^{p-1}bu \quad \text{in } (0, T_1) \times \Omega \\ u &= 0 \quad \text{on } (0, T_1) \times \partial\Omega \\ u(0) &= 0 \quad \text{in } \Omega. \end{cases} \tag{6.47}$$

Then, by the proposition 2.2 we have that $z \geq u - v$ in $[0, T_2)$ with $T_2 = \min\{T, T_1\}$.

On the other hand, since $u_0 \neq 0$, exist $T_3 \in (0, T_2]$ such that $u(t) \geq z(t)$ in $[0, T_3]$(Remark 3.2 (ii)). Thus, if $w(t) = z(t) - u(t) + v(t) \geq 0$ for all $t \in [0, T_2)$, then $w(0) = 0$ and

$$\begin{aligned} w_t - \Delta w &\geq a(z^q - u^q + v^q) + pbM^{p-1}(z - u + v) \\ &\geq aw^q + pbM^{p-1}w \end{aligned}$$

where the last inequality is obtained from following inequality

$$(x+\alpha)^q - (y+\alpha)^q \leq x^q - y^q$$

for $0 \leq y \leq x$, $\alpha \geq 0$, doing $x = v$, $y = z - u + v$, $\alpha = u - v$. Therefore, w is a supersolution of (6.47) in $[0, T_2]$. Suppose that $w = 0$, this is $u = z + v$, in some interval $[0, \tau)$ with $\tau < T_3$ sufficiently small, then

$$\begin{aligned} a(z+v)^q + b(z+v)^p &= u_t - \Delta u \\ &= z_t - \Delta z + v_t - \Delta v \\ &= a(z^q + v^q) + b(pM^{p-1}z + v^p). \end{aligned}$$

Thus $a[(z+v)^q - (z^q + v^q)] + b[(z+v)^p - v^p - pM^{p-1}z] = 0$, which is absurd, because as z, v is positive we have $(z+v)^q < z^q + v^q$ and for some $\theta \in (0,1)$, $(z+v)^p - v^p - pM^{p-1}z = pz[(\theta z + v)^{p-1} - M^{p-1}] < 0$, if τ is choosing such that $||z||_{L^\infty((0,\tau)\times\Omega)} + ||v||_{L^\infty((0,\tau)\times\Omega)} < M$. Therefore, for all $t \in (0, T_3)$ $w(t) \neq 0$. But since $w \geq 0$ we deduce from (6.47) that $w(t) > 0$ for all $t \in (0, T_2)$. By the Step 1, we have that $w = z$ in $[0, T_3)$, this is $u = v$ in $[0, T_3)$. □

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