FOLD, TRANSCRITICAL AND PITCHFORK SINGULARITIES FOR TIME-REVERSIBLE SYSTEMS

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In this paper singularly perturbed reversible vector fields defined in R^2 without normal hyperbolicity conditions are discussed. The main results give conditions for the existence of infinitely many periodic orbits and heteroclinic cycles converging to singular orbits with respect to to the Hausdorff distance. Besides we classify the slow manifolds via exhibition of three normal forms on neighborhoods of non normally hyperbolic points.

1. Introduction

Consider the singularly perturbed ordinary differential equations on the plane given by

$$\begin{cases} x' = f(x, y, \varepsilon) \\ y' = \varepsilon g(x, y, \varepsilon) \end{cases} x = x(\tau), \ y = y(\tau), \ 0 \le \varepsilon \ll 1, \tag{1}$$

and $f,\, g \in C^\infty$.

The main trick in the geometric singular perturbation is to consider the family (1) in addition to the family

$$\begin{cases} \varepsilon \dot{x} = f(x, y, \varepsilon) \\ \dot{y} = g(x, y, \varepsilon) \end{cases} x = x(t), \ y = y(t) \tag{2}$$

obtained after the time rescaling $t = \varepsilon\tau$. Equation (1) is called the *fast system* and (2) the *slow system.*

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We call $\mathcal{S} = \{(x, y) \in R^2 | f(x, y, 0) = 0\}$ the slow manifold. We observe that for $\varepsilon = 0$, $\mathcal{S}$ is a set of all singular points of (1), but the equation (2) defines a dynamical system on $\mathcal{S}$ called *reduced problem*.

Combining results on the dynamics of these two limiting problems one obtains information on the dynamics for small values of ε.

We say that $(x_0, y_0) \in \mathcal{S}$ is *normally hyperbolic* if $\dfrac{\partial f}{\partial x}(x_0, y_0) \neq 0$. Let $p, q \in \mathcal{S}$ be normally hyperbolic points. A *singular orbit* consists of 3 pieces of smooth curves: an orbit of the reduced problem in the unstable manifold $W_1^u(p)$, an orbit of the reduced problem in the stable manifold $W_2^s(q)$ and a heteroclinic orbit of the fast problem connecting the two previous pieces.

The main question in GSP-theory is to exhibit conditions under which a singular orbit can be approached by regular orbits for $\varepsilon \downarrow 0$, with respect to to the Hausdorff distance. The usual approach consists is to consider singular orbits along which the vector field X_0 is normally hyperbolic. We refer [1], [2] and [3] for related problems. A general question is what remains of this picture when the Normal Hyperbolicity assumption is dropped.

To fix our notation we suppose that the non normally hyperbolicity occurs on $(0, 0)$.

In our approach we assume that the system is time-reversible with respect to the linear involution $R(x, y) = (-x, y)$. It means that

(a) $X(R(x, y)) = -R(X(x, y))$.
(b) The phase portrait of X_ε is symmetric with respect to $\{x = 0\}$.

Here we observe that the assumption that the involution R is linear is not restrictive because we can linearize it using montgomery-bochner theorem. For general properties of time-reversible systems we recommend the survey [4].

Let X_ε be a reversible singularly perturbed vector field and Γ be a singular orbit. Summarizing, in what follows we give a rough all-over description of the main results of the paper:

(1) Assume that X_ε is of type $(2; 1)$. We give conditions on X_ε under which Γ is approximated by regular orbits (see Theorem A). Moreover we exhibit conditions, under which X_ε possesses a periodic orbit or heteroclinic cycle Γ_ε converging to Γ.
(2) The main results contained in [5,6] are revisited. In these papers the planar systems presenting singular perturbation problems are classified via exhibition of three normal forms on neighborhoods of non-normally hyperbolic points and the dynamics near these points

is analyzed. By means of the techniques developed in this paper, we prove variations of these results for the reversible case.

In section 2 we prove Theorem A and discuss the existence of infinitely many periodic orbits or heteroclinic cycles converging to singular orbits. Section 3 is devoted to present Theorem B, which discusses the planar classification of Krupa-Szmolyan mentioned above.

2. Singularly Perturbed Reversible Vector Fields on the plane

Here we examine the solutions of reversible singular perturbation problems expressed by X_ϵ as in (1).

Definition 2.1. We say that a R-reversible vector field X_ϵ satisfies the QG-condition if:

(1) $(x, y) \in R^2, \varepsilon \geq 0$ and $f, g \in C^\infty$;
(2) The slow manifold $\mathcal{S} = \{(x, y)\,|\,f(x, y, 0) = 0\}$ is a regular curve;
(3) $0 \in \mathcal{S} \pitchfork Fix(R)$ is a non normally hyperbolic point;
(4) p and q are R-symmetric points on $\mathcal{S}$ and Γ is a singular orbit passing through 0 that is composed of 3 pieces: an orbit of the reduced problem in the unstable manifold $W_1^u(p)$, an orbit of the reduced problem in the stable manifold $W_2^s(q)$, and a heteroclinic orbit of the fast problem that conects them;
(5) 0 is the unique non normally hyperbolic point on Γ.

Theorem A *Suppose that* X_ε *given by* (1) *satisfies the QG-condition. There exists a neighborhood* $U \subset R^2$ *of* 0 *as in (3), such that if* $\Gamma \subset U$ *then for each* $\varepsilon > 0$ *there exists an orbit* Γ_ε *of* X_ε *that approaches* Γ *as* $\varepsilon \downarrow 0$*, with respect to the Hausdorff distance.*

Proof. Consider the auxiliary vector field $X_\varepsilon^*(x, y, \varepsilon) = (f(x, y, \varepsilon), \varepsilon g(x, y, \varepsilon), 0)$ on R^3. Take $r = (0, y_0)$ on $\Gamma \cap Fix(R)$. Such a point exists because Γ connects R-symmetric invariant manifolds on $\mathcal{S}$. Let $l = \{(x, y, \varepsilon) : x = 0,\ y = y_0,\ 0 \leq \varepsilon < \varepsilon_0\}$, where ε_0 is some sufficiently small positive number. Thus l is a segment of a curve transverse to $Fix(R)$ at r.

Denote by C_l the saturate of l, that is the closure of the union of segments of orbits of X_ε^* through the points of l and taken between the first intersection of this curve with $Fix(R)$ in negative time $\left(C_l^-\right)$ and in positive time $\left(C_l^+\right)$. The orbits on C_l are the orbits of X_ε. C_l crosses $Fix(R)$

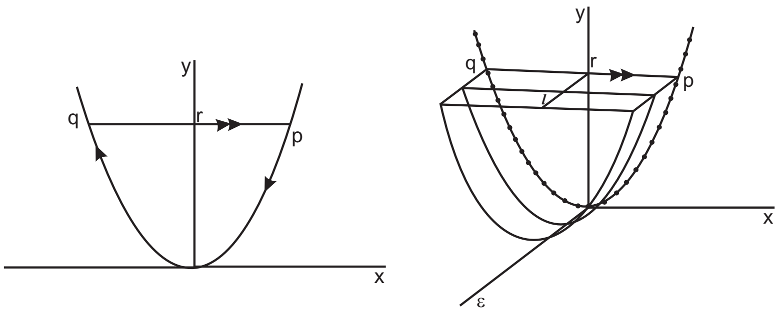

Figure 1. Singular Orbit Γ and Center Manifold C_l

transversally because the singular orbit Γ does. Finally, the reversibility of X_ε implies that $C_l^- \cap Fix\,(R) = C_l^+ \cap Fix\,(R)$. To complete the proof it is enough to choose Γ_ε to be an orbit contained on C_l. □

We observe that if p and q, as in the QG-condition, are hyperbolic singular points of the slow system then the family obtained in Theorem A is composed of heteroclinic cycles. Besides if $\frac{\partial f}{\partial y}(0,0,0)\,.\frac{\partial g}{\partial x}(0,0,0) < 0$, Γ is a singular orbit of X_0 passing through $(0,0)$ and connecting R-symmetric points on the slow manifold then there exists a sequence of periodic orbits Γ_ε, of X_ε, that approaches Γ for $\varepsilon \downarrow 0$. In fact, if X_ε is R-reversible then we have

$$f\,(x,y,\varepsilon) = \varphi\left(x^2,y,\varepsilon\right)$$
$$g\,(x,y,\varepsilon) = x\psi\left(x^2,y,\varepsilon\right)$$

for some smooth functions φ and ψ on R^2. The singular points for $\varepsilon > 0$ are given by the equations:

$$x\psi\left(x^2,y,\varepsilon\right) = 0$$
$$\varphi\left(x^2,y,\varepsilon\right) = 0$$

By the implicit function theorem there is a smooth function $y = y(\varepsilon)$ provided that $\frac{\partial f}{\partial y}(0,0,0) \neq 0$. Moreover

(1) $\dfrac{\partial f}{\partial x}(0,y(\varepsilon),\varepsilon) = \dfrac{\partial g}{\partial y}(0,y(\varepsilon),\varepsilon) = 0;$

(2) $\dfrac{\partial f}{\partial y}(0,y(\varepsilon),\varepsilon)\,.\dfrac{\partial g}{\partial x}(0,y(\varepsilon),\varepsilon) < 0;$

(3) $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial g}{\partial x}$ are continuous functions.

Finally, the reversibility condition ensures that the singularity $(0, y(\varepsilon), \varepsilon)$ is a center for X_ε.

3. Fold, Transcritical and Pitchfork Singularities

Consider the singularly perturbed ordinary differential equations on the plane given by (1).

Assume that $(0,0) \in \mathcal{S}$ is a non-normally hyperbolic singular point. a generic classification of slow manifolds in [5,6] is done by the exhibition of three normal forms:

1- Fold Singularity $X_\varepsilon : \begin{cases} x' = -y + x^2 \\ y' = \varepsilon g(x,y) \end{cases}, \ g(0,0) \neq 0.$

2- Transcritical Singularity $X_\varepsilon : \begin{cases} x' = -y^2 + x^2 \\ y' = \varepsilon g(x,y) \end{cases}, \ g(0,0) \neq 0.$

3- Pitchfork Singularity $X_\varepsilon : \begin{cases} x' = xy - x^3 \\ y' = \varepsilon g(x,y) \end{cases}, \ g(0,0) \neq 0.$

These singularities have been analyzed by analytic methods, asymptotic expansions, non-standard analysis and blowup techniques(see for instance [5,6]).

If $\mathcal{S}_0 \subset \mathcal{S}$ is a normally hyperbolic submanifold, then it persists for sufficiently small $\varepsilon > 0$ as a nearby locally invariant slow manifold $\mathcal{S}_\varepsilon$, for $\varepsilon \downarrow 0$. In [5,6], the case where $\mathcal{S}_0$ contains non normally hyperbolic points was studied. The main question is to understand the behavior of $\mathcal{S}_\varepsilon$ at these points. This analysis is made by the inspection of the transition maps defined on transversal sections to slow manifolds at points where the normal hyperbolicity occurs.

Theorem 3.1. *(see [5,6])*

a) Assume that $(0,0)$ is a fold singularity and $g(0,0) < 0$. Let $\Sigma_i, i = 1,2$, be transversal sections to $\mathcal{S}$ at $p_i = (x_i, y_i)$, with $x_1 < 0$ and $x_2 > 0$. Consider $\pi : \Sigma_1 \to \Sigma_2$ the transition map and $K \subset S$ a normal hyperbolic part of the slow manifold with $p_1 \in K$. Then K_ε, invariant manifold of X_ε converging to K, crosses transversally Σ_2 for $\varepsilon \downarrow 0$. Moreover the map π is a contraction with constant $O\left(e^{-\frac{c}{\varepsilon}}\right)$ for some constant $c > 0$. (see illustration on figure 2-(a)).

b) Assume that $(0,0)$ is a transcritical singularity and $g(0,0) > 0$. Then X_ε is topologically equivalent to $X(x,y) = (-y^2 + x^2 + \lambda\varepsilon, \varepsilon)$. Moreover let Σ_1 be transversal to $y = x$ at $p_1 = (x_1, x_1)$, with $x_1 < 0$ and Σ_2 transversal to $y = -x$ at $p_2 = (x_2, -x_2)$ if $\lambda > 1$, $x_2 < 0$ and to $y = 0$ at $p_2' = (x_2', 0)$ if $\lambda < 1$ and $x_2' > 0$. Consider $\pi : \Sigma_1 \to \Sigma_2$ the transition map and $K \subset S$ a *normally hyperbolic part of the slow manifold with* $p_1 \in K$. *Then as before* K_ε *crosses* Σ_2 *transversally for* $\varepsilon \downarrow 0$. *(see figure 2-(b)).*

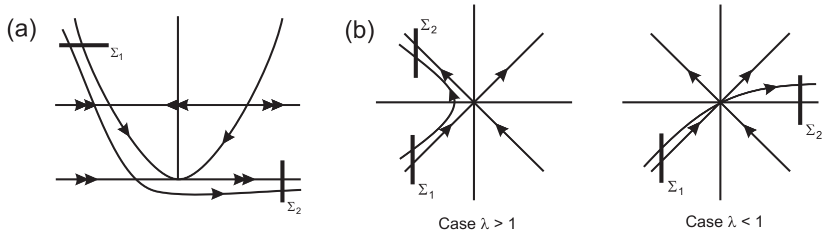

Figure 2. Transition Maps.

We recall that the cases treated are not reversible with respect to $R(x, y) = (-x, y)$.

Extending Slow Manifolds with Reversibility:

In our approach we treat the case $g\,(0, 0) = 0$.

Definition 3.1. $(0, 0)$ is a simple reversible singularity if it is either a fold or a transcritical singularity and $\frac{\partial g}{\partial x}\,(0, 0) \neq 0$.

If $\frac{\partial g}{\partial x}\,(0, 0) < 0$ the dynamics is not interesting (see figure 3-(A)) its analysis will be omitted. The next result is a immediate consequence of the Theorem A.

Theorem B *Let* X_ε *be given by* $X_\varepsilon(x, y) = (f(x, y), \varepsilon g\,(x, y))$. *Assume further that* $(0, 0)$ *is a simple reversible singularity with* $\frac{\partial g}{\partial x}\,(0, 0) > 0$. *Let* Σ_1 *be a transversal section to* $\mathcal{S}$ *at* $p = (a, b)$ *with* $a < 0$, $b > 0$ *and* $\Sigma_2 = R(\Sigma_1)$. *Then either the* $\varpi-$ *limit set of* p *is* $\{(0, 0)\}$ *or the orbit of* X_ε *through* p, $\mathcal{O}(p)$, *satisfies* $\mathcal{O}(p) \cap \Sigma_2 = \{R\,(p)\}$.

We observe that the pitchfork singularity is a degenerate singularity. Its generic canonical form is $X(x, y) = (xy - x^3, \pm\varepsilon)$. It is not $R-$reversible. We consider the generic reversible pitchfork singularity. It is represented by the following normal form $X(x, y) = (-x^2y + x^4, \varepsilon x)$, and all possible fast and slow dynamics are illustrated in figures 3-(E) and 3-(F).

Figure 3 describes the dynamics of some situations that occur for reversible singular perturbation problems. They are:

(A) $f(x, y) = -y + x^2$ and $\frac{\partial g}{\partial x}(0, 0) < 0$.
(B) $f(x, y) = -y + x^2$ and $\frac{\partial g}{\partial x}(0, 0) > 0$.
(C) $f(x, y) = -y^2 + x^2$ and $\frac{\partial g}{\partial x}(0, 0) > 0$.

(D) $f(x,y) = -y^2 + x^2$ and $\frac{\partial g}{\partial x}(0,0) < 0$.
(E) $f(x,y) = -x^2(y - x^2)$ and $\frac{\partial g}{\partial x}(0,0) > 0$.
(F) $f(x,y) = -x^2(y - x^2)$ and $\frac{\partial g}{\partial x}(0,0) < 0$.

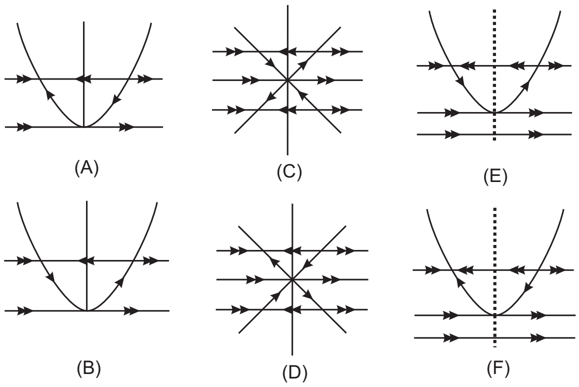

Figure 3. Fast and slow dynamics.

In what follows we give an example of a reversible singularly perturbed vector field $X_{\varepsilon,\lambda}$ possessing a 1-parameter family of singular orbits Γ_λ and a 2-parameter family of regular orbits $\Gamma_{\varepsilon,\lambda}$ having the following properties: (1) If $\lambda < 0$, then $\Gamma_{\varepsilon,\lambda}$ are periodic solutions and $\Gamma_{\varepsilon,\lambda} \to \Gamma_\lambda$ as $\varepsilon \to 0$. (2) If $\lambda > 0$, then $\Gamma_{\varepsilon,\lambda}$ are homoclinic solutions and $\Gamma_{\varepsilon,\lambda} \to \Gamma_\lambda$ as $\varepsilon \to 0$.

Consider the following family of time-reversible singularly perturbed systems $X(x,y) = (-y^3 + x^2 + \lambda y, \varepsilon x)$.

This family is time-reversible with respect to the involution $\phi(x,y) = (-x,y)$. The slow system and the fast system are given, respectively, by

$$\begin{cases} 0 = -y^3 + x^2 + \lambda y \\ \dot{y} = x \end{cases} \qquad \begin{cases} \dot{x} = -y^3 + x^2 + \lambda y \\ \dot{y} = 0. \end{cases} \tag{3}$$

We show how the dynamics of $X_{\varepsilon,\lambda}$ varies in figure 4.

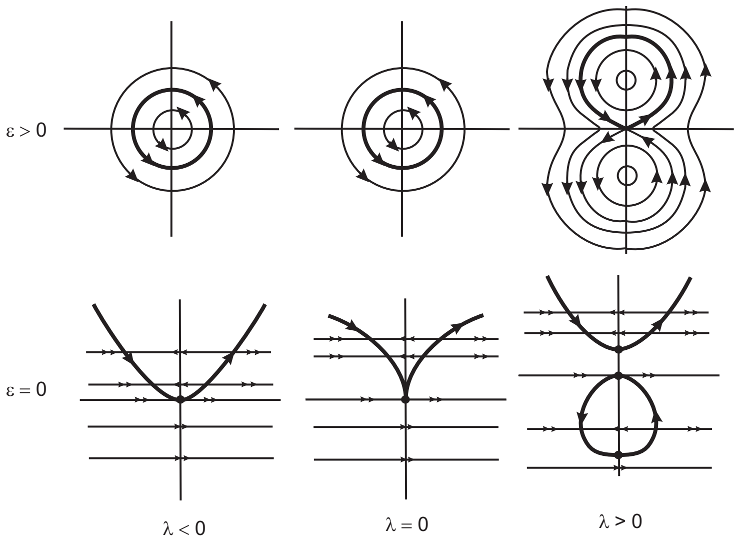

Figure 4. Bifurcation

References

1. F.Dumortier, R. Roussarie. *Canard Cycles and Center Manifolds*, Memoirs of the A.M.S, V. 121, 1996.
2. N.Fenichel. Geometric Singular Perturbation Theory for Ordinary Differential Equations, *J.D.E* **31**, 53-98, 1979.
3. P.Szmolyan. Transversal Heteroclinic and Homoclinic Orbits in Singular Perturbation Problems , *J.D.E* **92**, 252-281, 1991.
4. J.S.W. Lamb and J.A.G. Roberts. Time-reversal symmetry in dynamical systems: a survey. *Phys. D*, **112**(1-2):1–39, 1998. Time-reversal symmetry in dynamical systems (Coventry, 1996).
5. M.Krupa and P.Szmolyan. Extending geometric singular perturbation theory to nonhyperbolic points: fold and Canard points in two dimensions, *SIAM J. Math. Anal.* **33**, 2, 286-314, 2001.
6. M.Krupa and P.Szmolyan. Extending slow manifolds near transcritical and pitchfork singularities, *Nonlinearity* **14**, 1473-1491, 2001.