COMPLETENESS CONDITIONS IN CERTAIN WEYL COMPLEXES, COMBINATORICS AND PARSIMONY

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Abstract. We describe the combinatorial relationship between different completeness theorems in the Buchsbaum-Rota resolution of Weyl modules. In particular this gives a completely elementary, combinatorial description of the Buchsbaum-Rota construction that shows that it is a resolution.

1. Introduction

The context of this work is the Buchsbaum-Rota program of constructing resolutions of Weyl modules in a characteristic free setting, using Letter-Place methods. The study of these resolutions have several motivations: the representation theory of $GL(n)$ [2], where the Giambelli and Jacobi-Trudy identities appear in the guise of Euler-Poincaré characteristic of the resolution; the sudy of determinantal ideals [3, 4, 9], and the invariant theory of $GL(n)$, all in general characteristic [7].

In [6], Buchsbaum and Rota constructed a characteristic-free resolution over the Schur algebra of 3-rowed Weyl modules with at most one triple overlap. The fact that the Buchsbaum-Rota complex is a resolution is proven through a highly indirect proof, using the fundamental exact sequence.

The series of papers [10, 11, 12] and the present one is an attempt to understand this resolution in a more elementary, combinatorial way. The usefulness of this understanding lies in the fact that the general program is to construct resolutions for all Weyl modules using Letter-Place methods; a deep combinatorial understanding of the known resolutions is a neccessary step for this program to succeed.

In [10, 11], a basis for the syzygies of the Buchsbaum-Rota resolution was constructed under certain conditions; a fundamental element of the construction is the division of the basis of each term in the resolution into two complementary subsets, the *essential* and *non-essential* elements. This division satisfies two conditions: the *completeness condition*, that roughly says that there are "enough" essential elements, and the *rank condition*, that says that there are *just* enough of them.

Then in [12], it was realized that a strong form of the completeness theorem would produce a splitting contracting homotopy for the Buchsbaum-Rota complex, (not assuming that it is a resolution) thus giving a more elementary proof of the fact that the complex is a resolution.

The main objective of this paper is to describe the combinatorial identities relating the different "Completeness theorems" produced by the author in the study of the Buchsbaum-Rota resolutions of 3-rowed Weyl modules ([10, 11, 12]); this combinatorial relationship in particular will give a homotopy for these complexes that does not depend on $d^2 = 0$; in fact it does give in addition a neat way of organizing the computation that implies that $d^2 = 0$, and therefore gives a completely elementary proof of the fact that the Buchsbaum-Rota construction is a complex and, by the homotopy constructed in [12] assuming $d^2 = 0$, a resolution; in short, [10, 12] and this paper gives a proof of the Buchsbaum-Rota resolution that uses exclusivley the combinatorics of the divided powers and the polarization operators applied to bistandard bitableaux.

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For a more precise description of these completeness theorems, see section 3. In this paper we show that certain terms (the "essential correcting terms" M_α and M'_α of section 3) in the completeness theorems, which look very different in the strong completeness theorem of [12] compared to the terms in the weak completeness theorem of [10], are in fact equal. As we have said, this equality will show that the Buchsbaum-Rota construction is a resolution without assuming anything at all.

The paper is organized as follows: in section 2 we give the minimum background neccessary on Letter-Place and generalized bar complex construction of [6] to carry out the computations. Then in section 3 we describe briefly the Weak and Strong Completeness Theorems proved in [10] and [12], respectively. In that section we also hope that the examples given shed some light into the heuristics of how to find the essential/non-essential partition and its relation to the "parsimony principle" mentioned in [7]. And finally in section 4 we show how these completeness theorems are related via combinatorial identites, in particular showing that $d^2 = 0$.

2. Notation and Preliminaries

In this section we give very brief introduction, in fact just setting the notation, to the Buchsbaum-Rota resolution of 3-rowed Weyl modules using Letter-Place. For more in-depth descriptions, the reader can look at [8] (Letter-Place), [7], [10].

Letter-Place is a symbolic method which generalizes the ordinary algebra of polynomials in a set of A variables with integer coefficients. The set A is the union of three disjoint subsets A^+, A^0 and A^-; whose elements are called positively, neutral and negatively signed. These variables satisfy the following rules:

1) Positively signed variables act like the divided powers. (Recall that the divided power functor is isomorphic to the symmetric powers in carahteristic zero but not in general carachteristic, [1])

$$a^{(i)}a^{(j)} = \binom{i+j}{i} a^{(i+j)},$$

$$(a^{(i)})^{(j)} = \frac{(ij)!}{j!(i!)^j} a^{(ij)},$$

and

$$(a+b)^i = \sum_{j+k=i} a^{(j)} b^{(k)}$$

These variables and their divided powers commute.

2) Neutral variables behave like ordinary polynomial variables; in particular they also commute.

3) Negatively signed variables satisfy the rules of ordinary exterior algebra: $ab = -ba$ and $a^2 = b^2 = 0$.

We take the following discussion of Letter-Place from [7]:

The main idea is to note not only the basis elements of a given tensor product, but also to keep track of their "places". Thus we have the positive alphabet, or basis of the underlying free module, *and* we also have a *'place alphabet'* of positive places. For example, an element $w \otimes w' \in D_p \otimes D_q$ would be written, in letter-place algebra, as $(w|1^{(p)})(w'|2^{(q)})$ to indicate that it is the tensor product of a basis element of degree p in the first factor, and one of degree q in the second. This is then written in double tableau form as

$$\left(\begin{array}{c|c} w & 1^{(p)} \\ w' & 2^{(q)} \end{array} \right).$$

In this paper, instead of denoting the place alphabet by $1, 2, 3, \ldots$, we denote it by $a, b, c \ldots$ (this is in order to be consistent with the notation in [5] and [6]).

Let us also use the symbol $(v|a^{(r)}b^{(s)})$ to denote $\sum v(r)\otimes v(s)$, where this stands for the diagonalization of the element $v\in D_{r+s}$ into its image under the diagonalization map in $D_r\otimes D_s$,. Then we also have double tableaux

$$\left(\begin{array}{c|c} w & a^{(p)}b^{(k)} \\ w' & b^{(q-k)} \end{array}\right),$$

which means the element $\sum w(p)\otimes w(k)w'\in D_p\otimes D_q$. Ordering the basis elements of the underlying free module and the place alphabet as well, we can now talk about 'standard' and 'double standard' tableaux. By the standard basis theorem ([8]), the set of double standard tableaux form a basis for $D_p\otimes D_q$. In a similar fashion, the Letter-Place language is used in $D_{p_1}\otimes D_{p_2}\otimes\cdots\otimes D_{p_n}$ where the 'places' run from 1 to n., and also with mixed products of divided and exterior powers, and negative and positive places. In this work, we will work mainly with three factors, so that our place alphabet will be $\{a,b,c\}$.

We will be working mainly on the set of standard tableux

$$\left\{\left(\begin{array}{c|ccc} W & a^{(\eta)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right),\ \text{where } a<b<c \text{ are positive places}\right\}.$$

In this set, the polarization operators are given by

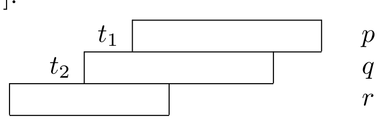
$$D_{ba}^{(k)}\left(\begin{array}{c|ccc} W & a^{(\eta)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)=\binom{k+\sigma_1}{k}\left(\begin{array}{c|ccc} W & a^{(\eta-k)} & b^{(k+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right),$$

$$D_{cb}^{(k)}\left(\begin{array}{c|ccc} W & a^{(\eta)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)=$$

$$=\sum_{i=0}^{k}\binom{i+\rho_1}{\rho_1}\binom{k-i+\rho_2}{\rho_2}\left(\begin{array}{c|ccc} W & a^{(\eta)} & b^{(\sigma_1-i)} & c^{(i+\rho_1)} \\ W' & b^{(\sigma_2-k+i)} & c^{(k-i+\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right),$$

$$D_{ca}^{(k)}\left(\begin{array}{c|ccc} W & a^{(\eta)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)=\binom{k+\rho_1}{\rho_1}\left(\begin{array}{c|ccc} W & a^{(\eta-k)} & b^{(\sigma_1)} & c^{(k+\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right).$$

Let us describe the Buchsbaum-Rota resolution of 3-rowed Weyl modules using Letter-Place and the *differential bar complex* of [7]:



where the number of triple overlaps is at most 1, i.e., $r-t_1-t_2\leq 1$. Here, from [6], we have a resolution

$$\cdots\longrightarrow P_k\xrightarrow{d_k}\cdots\longrightarrow P_3\xrightarrow{d_3}P_2\xrightarrow{d_2}P_1\xrightarrow{d_1}P_0\longrightarrow K_{\lambda/\mu}$$

modeled on a subquotient of the differential bar complex as follows: consider free bar module $\mathrm{Bar}(Super(L|\{a,b,c\}),A(Z_{ba},Z_{cb},Z_{ca}),\{x,y\})$, where $Super(L|\{a,b,c\})$ is the letter-place algebra with the places a,b,c we have been working with, x and y are two separators. The algebra $A(Z_{ba},Z_{cb},Z_{ca})$ is the associative noncommutative algebra generated by the variables Z_{ba},Z_{cb},Z_{ca}, subject to the commutation relations $Z_{ca}Z_{cb}=Z_{cb}Z_{ca}$ and $Z_{ca}Z_{ba}=Z_{ba}Z_{ca}$. The algebra $A(Z_{ba},Z_{cb},Z_{ca})$ acts on the module $Super(L|\{a,b,c\})$ by letting Z_{ba}, Z_{cb} and Z_{ca} act like the polarization operators D_{ba},D_{cb} and D_{ca}.

Let us impose now the relations

$$Z_{cb}^{(\alpha)}Z_{ba}^{(\beta)}=\sum_{k=0}^{\alpha}Z_{ba}^{(\beta-\alpha+k)}xZ_{cb}^{(k)}Z_{ca}^{(\alpha-k)}$$
$$Z_{ca}x=xZ_{ca}$$
$$Z_{cb}x=xZ_{cb}$$

The module P_k is freely spanned by all elements of the form

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}x\left(\begin{array}{c|ccc} W & a^{(\pi)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right),$$

where all the integers α_i and β_j are positive, $\beta_1 \geq t_2+1$ and $\alpha_1 > t_1+\sum_j \beta_j$, $\pi = p+\sum_i \alpha_i$, $\sigma_1+\sigma_2 = q+\sum_j \beta_j - \sum_i \alpha_i$, $\rho_1+\rho_2+\rho_3 = r-\sum_j \beta_j$ and $\lambda+\mu = k$.

A remark on notation. Sometimes, such elements -especially linear combinations of them- do not fit in a single line. In such cases what should be a single line splits into several lines in order to fit the printed page.

The boundary operator is $\partial_x+\partial_y$, given by place polarization by taking away the separators. Let us do an example to describe how this boundary operator works:

$$d_4\left(Z_{cb}^{(\beta_1)}yZ_{cb}^{(\beta_2)}yZ_{ba}^{(\alpha_1)}xZ_{ba}^{(\alpha_2)}x\left(\begin{array}{c|ccc} W & a^{(p+\alpha_1+\alpha_2)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$

$$\binom{\alpha_2+\sigma_1}{\alpha_2}Z_{cb}^{(\beta_1)}yZ_{cb}^{(\beta_2)}yZ_{ba}^{(\alpha_1)}x\left(\begin{array}{c|ccc} W & a^{(p+\alpha_1)} & b^{(\alpha_2+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)-$$

$$\binom{\alpha_1+\alpha_2}{\alpha_1}Z_{cb}^{(\beta_1)}yZ_{cb}^{(\beta_2)}yZ_{ba}^{(\alpha_1+\alpha_2)}x\left(\begin{array}{c|ccc} W & a^{(p+\alpha_1+\alpha_2)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)+$$

$$\sum_{k=0}^{\beta_2}\sum_{j=0}^{k}\binom{\beta_2-j}{\beta_2-k}\binom{\rho_1+\beta_2-j}{\rho_1}\binom{\rho_2+j}{\rho_2}Z_{cb}^{(\beta_1)}yZ_{ba}^{(\alpha_1-\beta_2+k)}xZ_{ba}^{(\alpha_2)}x$$

$$\left(\begin{array}{c|ccc} W & a^{(p+\alpha_1+\alpha_2-\beta_2+k)} & b^{(\sigma_1-k+j)} & c^{(\rho_1+\beta_2-j)} \\ W' & b^{(\sigma_2-j)} & c^{(\rho_2+j)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)-$$

$$\binom{\beta_1+\beta_2}{\beta_1}Z_{cb}^{(\beta_1+\beta_2)}yZ_{ba}^{(\alpha_1)}xZ_{ba}^{(\alpha_2)}x\left(\begin{array}{c|ccc} W & a^{(p+\alpha_1+\alpha_2)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right).$$

3. Completeness conditions

The main tool in [10, 12] is the division of the canonical basis of bistandard bitableaux of each module P_i in the complex in two complementary subsets, the "essential elements" and the "non-essential elements", so that $P_i = E_{i-1} \oplus N_i$, $P_i \supset E_{i-1} = span(\text{essential elements})$, $P_i \supset N_i = span(\text{non-essential elements})$ in P_i. the essential elements are those which

- Have at least one $Z^{(\cdot)}_{ba}$ variable in front, and no b in the first row.
- Have only $Z^{(\cdot)}_{cb}$ variables in front, no c in the first row, and $\sigma \leq t_1$.

And $\mathcal{N}_i$ is the complement of the $\mathcal{E}_{i-1}$ in the canonical basis of P_i, that is, non-essential elements are those which

- Have at least one $Z^{(\cdot)}_{ba}$ variable, and b appears in the first row.
- Have only $Z^{(\cdot)}_{cb}$ variables, and either c appears in the first row, or $\sigma \geq t_1$.

For example in the first level a basis for P_1 is given by bistandard bitableaux of the form

$$Z^{(\alpha)}_{ba} x \left(\begin{array}{c|lll} W & a^{(p+\alpha)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\alpha > t_1$, $\sigma_1 + \sigma_2 = q - \alpha$ and $\rho_1 + \rho_2 + \rho_3 = r$, and

$$Z^{(\beta)}_{cb} y \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\beta \geq t_2 + 1$, $\sigma_1 + \sigma_2 = q + \beta$ and $\rho_1 + \rho_2 + \rho_3 = r - \beta$.

The essential elements $\mathcal{E}_0$ are the elements of the form

$$Z^{(\alpha)}_{ba} x \left(\begin{array}{c|ll} W & a^{(p+\alpha)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right),$$

where $\alpha \geq t_1 + 1$ and $\rho_1 + \rho_2 + \rho_3 = r$, plus elements of the form

$$Z^{(\beta)}_{cb} y \left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma+\beta)} \\ W' & b^{(q-\sigma)} & c^{(\rho_1)} \\ W'' & c^{(\rho_2)} & \end{array} \right),$$

where $\beta \geq t_2 + 1, q - p \leq \sigma \leq t_1$ and $\rho_1 + \rho_2 = r - \beta$.

And the non-essential elements $\mathcal{N}_1$ are given by the complement of $\mathcal{E}_0$.

These partition satisfies the following:

•**Weak Completeness Condition:** [10]: Given a non-essential basis element $T_\alpha \in \mathcal{N}_{i+1}$, there exits an explicit $M_\alpha \in E_i$ such that $d_{i+1}(T_\alpha) = d_{i+1}(M_\alpha)$.

•**Strong Completeness Condition** [12]: Given a non-essential basis element $T_\alpha \in \mathcal{N}_{i+1}$, there exists an explicit $C_\alpha \in E_{i+1}$ such that $d_{i+2}(C_\alpha) = T_\alpha - M'_\alpha$, where $M'_\alpha \in E_i$.

Note that assuming $d^2 = 0$ the Strong Completeness Condition implies the Weak Completeness Condition. The terms M_α and M'_α are called the *essential correcting terms*, and the main Theorem states that they are equal and its consequences. Many computations in this theory are done modulo essential correcting terms (e.g. the main construction of [12])

The Weak Completeness Theorem (i.e. showing that the Weak Completeness Condition hold for the resolution at hand) is used in constructing a basis for the syzygies by setting $z_\alpha = d_{i+1}(\epsilon_\alpha)$; the set of $z_\alpha \in P_i$ where ϵ_α ranges in the essential basis elements of P_{i+1} forms a basis for the syzygies ([10]).

The Strong Completeness Theorem is then used for the construction of the homotopy: define $s_{i+1} : P_{i+1} \to P_{i+2}$ by $s_{i+1}(T_\alpha) = 0$ if T_α is essential; if T_α is non-essential then $s_{i+1}(T_\alpha) = C_\alpha$ where C_α is as in the Strong Completeness Condition above. The proof that s_i forms a splitting contracting homotopy reduces to showing that $s_i d_{i+1}(\epsilon) = \epsilon$ for essential elements ϵ, which is the main computation of [12], and the non-essential case then follows from the fact that $d^2 = 0$ and the construction.

Let's do an example to ilustrate how the Weak and Strong Completeness Condition works in practice:

Example. Given the following non-essential basis element

$$Z_{cb}^{(\beta)} y \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\sigma_1 > t_1$, $\beta \geq t_2 + 1$, $\rho_1 + \rho_2 + \rho_3 = r - \beta$ and $\rho_1 \geq 0$, we have that

$$d_1 \left(Z_{cb}^{(\beta)} y \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right) \right) =$$

$$d_1 \left(\sum_{i=0}^{\beta} \binom{\beta + \rho_1 - i}{\rho_1} \binom{\rho_2 + i}{i} Z_{ba}^{(\sigma_1+i)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma_1+i)} & c^{(\beta+\rho_1-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_2+i)} \\ W'' & c^{(\rho_3)} & \end{array} \right) \right) =$$

$$\sum_{i=0}^{\beta} \binom{\beta + \rho_1 - i}{\rho_1} \binom{\rho_2 + i}{i} \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+i)} & c^{(\beta+\rho_1-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_2+i)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right).$$

Note that two different elements give the same element under the boundary map. Sometimes we can get an element either by taking away exponentials from b's and putting it in c's or taking away exponential from a's and putting it in b's. The "principle of parsimony" states that we should try to get a given element with the least amount of work possible; thus in this case the essential elements are chosen as the ones that take exponential from a and put it into b; that is for this hind of bitableuax the essential element are the ones of the form

$$Z_{ba}^{(\alpha)} x \left(\begin{array}{c|ll} W & a^{(p+\alpha)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right),$$

where $\alpha \geq t_1 + 1$ and $\rho_1 + \rho_2 + \rho_3 = r$. Then the example just provided shows that the elements of the form

$$Z_{cb}^{(\beta)} y \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\sigma_1 > t_1$, $\beta \geq t_2 + 1$, $\rho_1 + \rho_2 + \rho_3 = r - \beta$ and $\rho_1 \geq 0$, are "redundant" for the boundary map, that is, non-essential. In this case,

$$M_\alpha = \sum_{i=0}^{\beta} \binom{\rho_1 + \beta - i}{\rho_1} \binom{\rho_2 + i}{\rho_2} Z_{ba}^{(\sigma_1+i)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma_1+i)} & c^{(\rho_1+\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right).$$

Let us use the same kind of bitableaux to illustrate the Strong Completeness Condition. For that non-essential element we also have that

$$d_2\left(Z_{cb}^{(\beta)} y Z_{ba}^{(\sigma_1+\beta)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right) \right) = Z_{cb}^{(\beta)} y \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right) -$$

$$\sum_{i=0}^{\beta} \binom{\rho_1+\beta-i}{\rho_1} \binom{\rho_2+i}{\rho_2} Z_{ba}^{(\sigma_1+i)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma_1+i)} & c^{(\rho_1+\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right).$$

The previous equation is taking this game one step further, that is the non-essential element of the form

$$Z_{cb}^{(\beta)} y \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\sigma_1 > t_1$, $\beta \geq t_2+1$, $\rho_1+\rho_2+\rho_3 = r-\beta$ and $\rho_1 \geq 0$, can be reached by the boundary map applied to an essential level one level higher modulo essential element. In this case,

$$M'_\alpha = \sum_{i=0}^{\beta} \binom{\rho_1+\beta-i}{\rho_1} \binom{\rho_2+i}{\rho_2} Z_{ba}^{(\sigma_1+i)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma_1+i)} & c^{(\rho_1+\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right).$$

Note that in this easy example there is nothing to prove, that is, $M_\alpha = M'_\alpha$ directly. In the general case they look quite different but Theorem 1 shows their equality.

4. Combinatorial equivalence of completeness conditions

Let us recall the completeness conditions of the previous section:

•**Weak Completeness Condition:** Given a non-essential basis element $T_\alpha \in \mathcal{N}_{i+1}$, there exits an explicit $M_\alpha \in E_i$ such that $d_{i+1}(T_\alpha) = d_{i+1}(M_\alpha)$.

•**Strong Completeness Condition:** Given a non-essential basis element $T_\alpha \in \mathcal{N}_{i+1}$, there exists an explicit $C \in E_{i+1}$ such that $d_{i+2}(C) = T_\alpha - M'_\alpha$, where $M'_\alpha \in E_i$.

We have the following theorem, that relates the Weak and Strong Completeness Conditions in a combinatorial way.

Theorem 1. *For all α parametrizing non-essential elements T_α, the linear combination of essential elements M_α of the Weak Completeness Condition is equal to the linear combination of essential elements M'_α of the Strong Completeness Condition.*

Before we prove this theorem, let us notice the following

Corollary. *The Buchsbaum-Rota sequence satisfies $d^2 = 0$.*

Let us show the corollary: we call it a corollary because Theorem 1 is the last step in a series of *purely computational facts* that imply that $d^2 = 0$ and then that the Buchsbaum-Rota sequence is a resolution. The merit of this way of arriving at this is that both the completeness condition plus Theorem 1 rely purely on elementary combinatorial computations instead of, for example, the homological considerations of the fundamental exact sequence of [7].

Let us show how the corrollary follows from these results:

(1) The Weak Completeness Condition implies that it suffices to show that $d^2 = 0$ for essential basis elements.
(2) $d^2 = 0$ for linear combinations of essential elements that are in the image of the homotopy $s_{i+1} : P_{i+1} \to P_{i+2}$: indeed, the image of s_{i+1} is spanned by $\epsilon_\alpha = s_{i+1}(T_\alpha)$, $T_\alpha \in P_{i+1}$ non-essential basis element. But $d_{i+2}(\epsilon_\alpha) = T_\alpha - M'_\alpha$ as above. Then theorem 1 says that $d_{i+2}(\epsilon_\alpha) = T_\alpha - M_\alpha$ where M_α is of the Weak Completeness Condition, which satisfies $d_{i+1}(T_\alpha) = d_{i+1}(M_\alpha)$. Therefore for such elements $d^2 = 0$.
(3) The homotopy condition applied to essential elements reads $s_i d_{i+1}(\epsilon) = \epsilon$ when $\epsilon \in E_i$. This is shown by computation in [12] without assuming $d^2 = 0$ (but $d^2 = 0$ is needed to show the homotopy condition $sd + ds = 1$ for non-essential elements). This implies that the image of s_i spans the essential space E_i.

□

Proof of Theorem 1

We will do the prove in a pedagogical sequence, first we will show that $M_\alpha = M'_\alpha$ (corresponding to the non-essential element T_α) for the level one and then for the general level.

Level one. In this level we will have three cases, corresponding to the different ways in which an element can be non-essential.

Case 1.For non-essential elements of the form

$$Z_{ba}^{(\alpha)} x \left(\begin{array}{c|lll} W & a^{(p+\alpha)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\alpha > t_1$, $\sigma_1 > 0$ and $\rho_1 + \rho_2 + \rho_3 = r$, the corresponding

$$M_\alpha = M'_\alpha = \binom{\alpha + \sigma_1}{\alpha} Z_{ba}^{(\alpha+\sigma_1)} x \left(\begin{array}{c|ll} W & a^{(p+\alpha+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right).$$

Case 2.For non-essential elements of the form

$$Z_{cb}^{(\beta)} y \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\sigma_1 \geq t_1$, $\beta \geq t_2 + 1$, $\rho_1 \geq 0$ and $\rho_1 + \rho_2 + \rho_3 = r - \beta$, the corresponding

$$M_\alpha = \sum_{i=0}^{\beta} \binom{\beta - i + \rho_2}{\rho_2} \binom{i + \rho_1}{i} Z_{ba}^{(\sigma_1+\beta-i)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma_1+\beta-i)} & c^{(\rho_1+i)} \\ W' & b^{(q-\sigma_1-\beta+i)} & c^{(\beta-i+\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right),$$

and the corresponding

$$M'_\alpha = \sum_{j=0}^{\beta} \binom{\rho_1 + \beta - j}{\beta - j} \binom{\rho_2 + j}{\rho_2} Z_{ba}^{(\sigma_1+j)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma_1+j)} & c^{(\rho_1+\beta-j)} \\ W' & b^{(q-\sigma_1-j)} & c^{(\rho_2+j)} \\ W'' & c^{(\rho_3)} & \end{array} \right).$$

If we put $k = \beta - j$ in M'_α we get that $M'_\alpha = M_\alpha$.

Case 3.For non-essential elements of the form

$$Z_{cb}^{(\beta)} y \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $q-p \le \sigma_1 \le t_1$, $\beta \ge t_2+1$, $\rho_1 > 0$, $t_1 = \sigma_1 + 1 + \sigma$ and $\rho_1+\rho_2+\rho_3 = r-\beta$. This case splits in two different cases.

3.1) If $\rho_1 > \sigma+1$ then the corresponding

$$M'_\alpha = \sum_{k=0}^{\sigma+1} (-1)^k \binom{\rho_2+k}{\rho_2} \binom{\beta+\rho_1-k}{\beta} Z_{cb}^{(\beta+\rho_1-k)} y \left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array} \right) -$$

$$\sum_{l=1}^{\rho_1-\sigma-1} \sum_{n=0}^{\beta} (-1)^{\sigma+1} \binom{\rho_2+\sigma+1+l}{\rho_2} \binom{\sigma+l}{\sigma+1} \binom{\beta+\rho_1-\sigma-1-n-l}{\beta-n} \binom{\rho_2+\sigma+1+l+n}{\rho_2+\sigma+1+l}$$

$$Z_{ba}^{(t_1+l+n)} x \left(\begin{array}{c|ll} W & a^{(p+t_1+l+n)} & c^{(\beta+\rho_1-\sigma-1-n-l)} \\ W' & b^{(q-t_1-n)} & c^{(\rho_2+\sigma+1+n+l)} \\ W'' & c^{(\rho_3)} & \end{array} \right),$$

and the corresponding

$$M_\alpha = \sum_{i=0}^{\sigma+1} (-1)^i \binom{\rho_2+i}{\rho_2} \binom{\beta+\rho_1-i}{\rho_1-i} Z_{cb}^{(\beta+\rho_1-i)} y \left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_2+i)} \\ W'' & c^{(\rho_3)} & \end{array} \right) +$$

$$\sum_{i=\sigma+2}^{\rho_1} \sum_{j=0}^{\beta+\rho_1-i} (-1)^i \binom{\rho_2+i}{\rho_2} \binom{\beta+\rho_1-i}{\rho_1-i} \binom{\beta+\rho_1+\rho_2-j}{\rho_2+i}$$

$$Z_{ba}^{(\sigma_1+\beta+\rho_1-j)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma_1+\beta+\rho_1-j)} & c^{(j)} \\ W' & b^{(q-\sigma_1-\beta-\rho_1+j)} & c^{(\beta+\rho_1+\rho_2-j)} \\ W'' & c^{(\rho_3)} & \end{array} \right).$$

Now, let $M = A_1 + A_2$ where

$$A_1 = \sum_{i=0}^{\sigma+1} (-1)^i \binom{\rho_2+i}{\rho_2} \binom{\beta+\rho_1-i}{\rho_1-i} Z_{cb}^{(\beta+\rho_1-i)} y \left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_2+i)} \\ W'' & c^{(\rho_3)} & \end{array} \right),$$

and

$$A_2 = \sum_{i=\sigma+2}^{\rho_1} \sum_{j=0}^{\beta+\rho_1-i} (-1)^i \binom{\rho_2+i}{\rho_2} \binom{\beta+\rho_1-i}{\rho_1-i} \binom{\beta+\rho_1+\rho_2-j}{\rho_2+i}$$

$$Z_{ba}^{(\sigma_1+\beta+\rho_1-j)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma_1+\beta+\rho_1-j)} & c^{(j)} \\ W' & b^{(q-\sigma_1-\beta-\rho_1+j)} & c^{(\beta+\rho_1+\rho_2-j)} \\ W'' & c^{(\rho_3)} & \end{array} \right).$$

Claim: the following combinatorial identities hold

1) $B = A_1$
2) $-D = A_2$

The first identity is clear, we only need to show the second one. In order to prove the second identity we have to consider two cases.

3.1.1) If in the expression for D we take $n+l=t$, where $1 \le t \le \rho_1-\sigma-1$, then

$$(-1)^{\sigma+1}\sum_{l=1}^{t}\binom{\rho_2+\sigma+1+n+l}{\rho_2+\sigma+1+l}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}\binom{\beta+\rho_1-\sigma-1-n-l}{\beta-n}=$$

$$(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{l=1}^{t}\binom{\sigma+1+t}{\sigma+1+l}\binom{\sigma+l}{\sigma+1}\binom{\beta+\rho_1-\sigma-1-t}{\beta-n}=X$$

If we put $p=n$, $m=\beta$ and $q=\beta+\rho_1-\sigma-1-t+n$ in the following formula

$$\binom{q-p}{m-p}=\sum_{i=0}^{p}(-1)^i\binom{q-i}{m}\binom{p}{i},$$

we get that

$$\binom{\beta+\rho_1-\sigma-1-t}{\beta-n}=\sum_{i=0}^{n}(-1)^i\binom{\beta+\rho_1-\sigma-1-t+n-i}{\beta}\binom{n}{i}.$$

So

$$X=(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{l=1}^{t}\sum_{i=0}^{t-l}(-1)^i\binom{\sigma+1+t}{t-l}\binom{\sigma+l}{\sigma+1}\binom{t-l}{i}\binom{\beta+\rho_1-\sigma-1-l-i}{\beta}$$

Let $l+i=z$, where $1\le z\le t$

$$X=(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{z=1}^{t}\sum_{l=1}^{z}(-1)^{z-l}\binom{\sigma+1+t}{t-l}\binom{\sigma+l}{\sigma+1}\binom{t-l}{t-z}\binom{\beta+\rho_1-\sigma-1-z}{\beta}=$$

$$(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{z=1}^{t}(-1)^z\sum_{l=1}^{z}(-1)^l\binom{\sigma+1+t}{t-z}\binom{\sigma+l}{\sigma+1}\binom{\sigma+1+z}{z-l}\binom{\beta+\rho_1-\sigma-1-z}{\beta}=$$

$$(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{z=1}^{t}(-1)^z\binom{\sigma+1+t}{t-z}\left[\sum_{l=1}^{z}(-1)^l\binom{\sigma+1+z}{z-l}\binom{\sigma+l}{\sigma+1}\right]\binom{\beta+\rho_1-\sigma-1-z}{\beta}=$$

Set $v=l-1$, then

$$(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{z=1}^{t}(-1)^{z+1}\binom{\sigma+1+t}{t-z}\left[\sum_{v=0}^{z-1}(-1)^v\binom{\sigma+1+z}{z-1-v}\binom{\sigma+v+1}{\sigma+1}\right]\binom{\beta+\rho_1-\sigma-1-z}{\beta}=$$

$$(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{z=1}^{t}(-1)^{z+1}\binom{\sigma+1+t}{t-z}\binom{z-1}{z-1}\binom{\beta+\rho_1-\sigma-1-z}{\beta}=$$

$$(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{z=1}^{t}(-1)^{z+1}\binom{\sigma+1+t}{t-z}\binom{\beta+\rho_1-\sigma-1-z}{\beta}$$

Now, if in the expression for A_2 we let $j=\beta+\rho_1-\sigma-1-t$ then we get

$$\sum_{i=\sigma+2}^{\sigma+t+1}(-1)^i\binom{\rho_2+\sigma+1+t}{\rho_2}\binom{\sigma+1+t}{i}\binom{\beta+\rho_1-i}{\beta}.$$

Set $p = i - \sigma - 1$

$$(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{p=1}^{t}(-1)^p\binom{\sigma+1+t}{p+\sigma+1}\binom{\beta+\rho_1-\sigma-1-z}{\beta}.$$

3.1.2) If in the expression for D we take $n+l=t$, where $\rho_1-\sigma \leq t \leq \beta+\rho_1-\sigma-1$ then $t=\rho_1-\sigma+\alpha$, where $0 \leq \alpha \leq \beta-1$. Thus $l+n=\rho_1-\sigma+\alpha$ and

$$(-1)^{\sigma+1}\binom{\rho_1+\rho_2+\alpha+1}{\rho_2}\sum_{n=\alpha+1}^{\beta}\binom{\rho_1+\alpha+1}{\rho_1+\alpha+1-n}\binom{\rho_1+\alpha-n}{\sigma+1}\binom{\beta-1-\alpha}{\beta-n}=$$

$$(-1)^{\sigma+1}\binom{\rho_1+\rho_2+\alpha+1}{\rho_2}\sum_{n=\alpha+1}^{\beta}\binom{\rho_1+\alpha+1}{n}\binom{\rho_1+\alpha-n}{\sigma+1}\binom{\beta-1-\alpha}{\beta-n}.$$

Set $i = n - \alpha - 1$

$$(-1)^{\sigma+1}\binom{\rho_1+\rho_2+\alpha+1}{\rho_2}\sum_{i=0}^{\beta-\alpha-1}\binom{\rho_1+\alpha+1}{i+\alpha+1}\binom{\rho_1-1-i}{\sigma+1}\binom{\beta-1-\alpha}{i}.$$

Now, if in the expression for A_2 we let $j=\beta+\rho_1-\sigma-1-t=\beta-1-\alpha$ we get

$$\binom{\rho_1+\rho_2+\alpha+1}{\rho_2}\sum_{i=\sigma+2}^{\rho_1}(-1)^i\binom{\rho_1+\alpha+1}{i}\binom{\beta+\rho_1-i}{\beta}=$$

$$\binom{\rho_1+\rho_2+\alpha+1}{\rho_2}\sum_{i=\sigma+2}^{\rho_1}\sum_{k=0}^{\beta-\alpha-1}(-1)^i\binom{\rho_1+\alpha+1}{i}\binom{\rho_1+\alpha+1-i}{\beta-k}\binom{\beta-\alpha-1}{k}=$$

$$\binom{\rho_1+\rho_2+\alpha+1}{\rho_2}\sum_{k=0}^{\beta-\alpha-1}\binom{\rho_1+\alpha+1}{\beta-k}\left[\sum_{i=\sigma+2}^{\rho_1}(-1)^i\binom{\rho_1+\alpha+1-\beta+k}{\rho_1+\alpha+1-i-\beta+k}\right]\binom{\beta-\alpha-1}{k}=$$

$$\binom{\rho_1+\rho_2+\alpha+1}{\rho_2}\sum_{k=0}^{\beta-\alpha-1}\binom{\rho_1+\alpha+1}{\beta-k}\left[\sum_{i=\sigma+2}^{\rho_1}(-1)^i\binom{\rho_1+\alpha+1-\beta+k}{i}\right]\binom{\beta-\alpha-1}{k}=$$

$$-\binom{\rho_1+\rho_2+\alpha+1}{\rho_2}\sum_{k=0}^{\beta-\alpha-1}\binom{\rho_1+\alpha+1}{\beta-k}\left[\sum_{i=0}^{\sigma+1}(-1)^i\binom{\rho_1+\alpha+1-\beta+k}{i}\right]\binom{\beta-\alpha-1}{k}=$$

$$-(-1)^{\sigma+1}\binom{\rho_1+\rho_2+\alpha+1}{\rho_2}\sum_{k=0}^{\beta-\alpha-1}\binom{\rho_1+\alpha+1}{\beta-k}\binom{\rho_1+\alpha-\beta+k}{\sigma+1}\binom{\beta-\alpha-1}{k}=Y$$

If we set $p=\beta-\alpha-1-k$ in Y we get

$$-(-1)^{\sigma+1}\binom{\rho_1+\rho_2+\alpha+1}{\rho_2}\sum_{p=0}^{\beta-\alpha-1}\binom{\rho_1+\alpha+1}{p+\alpha+1}\binom{\rho_1-1-p}{\sigma+1}\binom{\beta-\alpha-1}{p}.$$

3.2) If $\rho_1 \leq \sigma+1$ then the corresponding

$$M'_\alpha=\sum_{k=0}^{\rho_1-1}\sum_{j=0}^{\rho_1-k}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\rho_2+k+j}{\rho_2+k}Z_{cb}^{(\beta)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta+k+j)} & c^{(\rho_1-k-j)} \\ W' & b^{(q-\sigma_1-k-j)} & c^{(\rho_2+k+j)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)-$$

$$\sum_{k=0}^{\rho_1-1}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\beta+\rho_1-k}{\beta}Z_{cb}^{(\beta+\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)=A-B$$

where in the first term $k+j \geq 1$, and the corresponding

$$M_\alpha = \sum_{i=0}^{\rho_1} (-1)^i \binom{\rho_2+i}{\rho_2} \binom{\beta+\rho_1-i}{\beta} Z_{cb}^{(\beta+\rho_1-i)} y \left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_2+i)} \\ W'' & c^{(\rho_3)} & \end{array} \right).$$

To prove that $M'_\alpha = M_\alpha$ we note that:

-If in the expression for A we take $k+j=t$, where $0 < t \leq \rho_1 - 1$ then

$$\sum_{k=0}^{t} \binom{\rho_2+t}{\rho_2+k} \binom{\rho_2+k}{\rho_2} = \binom{\rho_2+t}{\rho_2} \sum_{k=0}^{t} (-1)^k \binom{t}{k},$$

which is equal to 0 if t is different to 0.

-If in the expression for A we take $k+j=\rho_1$ then

$$\sum_{k=0}^{\rho_1-1} (-1)^k \binom{\rho_2+\rho_1}{\rho_2+k} \binom{\rho_2+k}{\rho_2} = \binom{\rho_2+\rho_1}{\rho_2} \sum_{k=0}^{\rho_1-1} (-1)^k \binom{\rho_1}{k} = -(-1)^{\rho_1} \binom{\rho_2+\rho_1}{\rho_2}.$$

General level. Let us prove the equality in general now. We will have four cases corresponding to the different ways in which an element can be non-essential.

Case 1. For non-essential elements of the form

$$Z_{ba}^{(\alpha_1)} x \ldots x Z_{ba}^{(\alpha_i)} x \left(\begin{array}{c|lll} W & a^{(p+|\alpha|)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\alpha_1 > t_1$, $\alpha_j > 0 for j = 2, ..., i$, $\sigma_1 > 0$ and $\rho_1+\rho_2+\rho_3 = r$, the corresponding

$$M_\alpha = M'_\alpha = \binom{\alpha_i+\sigma_1}{\alpha_i} Z_{ba}^{(\alpha_1)} x \ldots x Z_{ba}^{(\alpha_i+\sigma_1)} x \left(\begin{array}{c|ll} W & a^{(p+|\alpha|+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right) +$$

$$\sum_{m=i-1}^{1} (-1)^{i-m} \binom{\alpha_m+\alpha_{m+1}}{\alpha_m} Z_{ba}^{(\alpha_1)} x \ldots x Z_{ba}^{(\alpha_{m-1})} x Z_{ba}^{(\alpha_m+\alpha_{m+1})} x Z_{ba}^{(\alpha_{m+2})} x \ldots x Z_{ba}^{(\alpha_i)} x Z_{ba}^{(\sigma_1)} x$$
$$\left(\begin{array}{c|ll} W & a^{(p+|\alpha|+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right).$$

Case 2. For non-essential elements of the form

$$Z_{cb}^{(\beta_1)} y \ldots y Z_{cb}^{(\beta_\lambda)} y Z_{ba}^{(\alpha_1)} x \ldots x Z_{ba}^{(\alpha_\mu)} x \left(\begin{array}{c|lll} W & a^{(p+|\alpha|)} & b^{(\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\rho > 0$, $\lambda - \mu = i$, $\beta_1 \geq t_2 + 1$ and $\beta_j > 0$ for $j = 2, ..., \lambda$, $\alpha_1 > t_1 + |\beta|$ and $\alpha_j > 0$ for $j = 2, ..., \mu$, $|\beta| = \sum \beta_j$, $|\alpha| = \sum \alpha_j$, and $\rho_1 + \rho_2 + \rho_3 = r - |\beta|$, the corresponding

$$M_\alpha = M'_\alpha = Z_{cb}^{(\beta_1)} y ... y Z_{cb}^{(\beta_\lambda)} y Z_{ba}^{(\alpha_1)} x ... x Z_{ba}^{(\alpha_\mu)} x \left(\begin{array}{c|lll} W & a^{(p+|\alpha|)} & b^{(\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right) +$$

$$\binom{\alpha_\mu+\rho}{\alpha_\mu} Z_{cb}^{(\beta_1)} y ... y Z_{cb}^{(\beta_\lambda)} y Z_{ba}^{(\alpha_1)} x ... x Z_{ba}^{(\alpha_{\mu-1})} x Z_{ba}^{(\alpha_\mu+\rho)} x \left(\begin{array}{c|ll} W & a^{(p+|\alpha|+\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right) +$$

$$\sum_{k=\mu-1}^{1} (-1)^{\mu-k} \binom{\alpha_k+\alpha_{k+1}}{\alpha_k} Z_{cb}^{(\beta_1)} y ... y Z_{cb}^{(\beta_\lambda)} y Z_{ba}^{(\alpha_1)} x ... x Z_{ba}^{(\alpha_k+\alpha_{k+1})} x ... x Z_{ba}^{(\alpha_\mu)} x Z_{ba}^{(\rho)} x$$

$$\left(\begin{array}{c|ll} W & a^{(p+|\alpha|+\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right) +$$

$$\sum_{l=\lambda-1}^{1} (-1)^{\mu+\lambda-l} \binom{\beta_l+\beta_{l+1}}{\beta_l} Z_{cb}^{(\beta_1)} y ... y Z_{cb}^{(\beta_l+\beta_{l+1})} y ... y Z_{cb}^{(\beta_\lambda)} y Z_{ba}^{(\alpha_1)} x ... x Z_{ba}^{(\alpha_\mu)} x Z_{ba}^{(\rho)} x$$

$$\left(\begin{array}{c|ll} W & a^{(p+|\alpha|+\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right) +$$

$$(-1)^\mu \sum_{k_1=0}^{\beta_\lambda} \sum_{i=2}^{\mu} \sum_{k_i=0}^{k_{i-1}} \sum_{m=0}^{k_\mu} M_{\lambda,\mu,k} \binom{\beta_\lambda+\rho_1-k_\mu}{\rho_1} \binom{\beta_\lambda+\rho_1-m}{k_\mu-m} \binom{\rho_2+m}{\rho_2}$$

$$Z_{cb}^{(\beta_1)} y ... y Z_{cb}^{(\beta_{\lambda-1})} y Z_{ba}^{(\alpha_1-\beta_\lambda+k_1)} x Z_{ba}^{(\alpha_2-k_1+k_2)} x ... x Z_{ba}^{(\alpha_\mu-k_{\mu-1}+k_\mu)} x Z_{ba}^{(\rho-k_\mu+m)} x$$

$$\left(\begin{array}{c|ll} W & a^{(p+|\alpha|-\beta_\lambda+\rho+m)} & c^{(\beta_\lambda+\rho_1-m)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho-m)} & c^{(\rho_2+m)} \\ W'' & c^{(\rho_3)} & \end{array} \right),$$

where

$$M_{\lambda,\mu,k} = \binom{\beta_\lambda-k_\mu}{\beta_\lambda-k_1, k_1-k_2, ..., k_{\mu-1}-k_\mu} = \frac{(\beta_\lambda-k_\mu)!}{(\beta_\lambda-k_1)!(k_1-k_2)!...(k_{\mu-1}-k_\mu)!}$$

is the monomial coefficient.

Case 3. For non-essential elements of the form

$$Z_{cb}^{(\beta_1)} y ... y Z_{cb}^{(\beta_i)} y \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\sigma_1 > t_1$, $\beta_1 \geq t_2+1$, $\beta_j > 0$ for $j = 2, ..., i$, $t_1 = \sigma_1 + 1 + \sigma$ and $\rho_1 \geq 0$, the corresponding

$$M_\alpha = M'_\alpha = \sum_{j=0}^{\beta_i} \binom{\beta_i+\rho_1-j}{\rho_1} \binom{\rho_2+j}{\rho_2} Z_{cb}^{(\beta_1)} y ... y Z_{cb}^{(\beta_{i-1})} y Z_{ba}^{(\sigma_1+|\beta|-\beta_i+j)} x$$

$$\left(\begin{array}{c|ll} W & a^{(p+\sigma_1+|\beta|-\beta_i+j)} & c^{(\beta_i+\rho_1-j)} \\ W' & b^{(q-\sigma_1-j)} & c^{(\rho_2+j)} \\ W'' & c^{(\rho_3)} & \end{array} \right) +$$

$$\sum_{l=i-1}^{1}(-1)^{i-l}\binom{\beta_l+\beta_{l+1}}{\beta_l}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_l+\beta_{l+1})}y...yZ_{cb}^{(\beta_i)}yZ_{ba}^{(\sigma_1+|\beta|)}x\left(\begin{array}{c|cc} W & a^{(p+\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right).$$

Case 4. For non-essential elements of the form

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|ccc} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right),$$

where $q-p\le\sigma_1\le t_1,\ \rho_1>0,\ \beta_1\ge t_2+1,\ \beta_j>0$ for $j=2,...,i,\ t_1=\sigma_1+1+\sigma$ and $\rho_1+\rho_2+\rho_3=r-|\beta|$. Here we have to consider two cases.

4.1) If $\rho_1>\sigma+1$ then the corresponding

$$M'_\alpha=\sum_{k=0}^{\sigma+1}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\beta_i+\rho_1-k}{\beta_i}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{cb}^{(\beta_i+\rho_1-k)}y\left(\begin{array}{c|cc} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)+$$

$$\sum_{k=0}^{\sigma+1}\sum_{m=i-1}^{1}(-1)^{k+i-m}\binom{\rho_2+k}{\rho_2}\binom{\beta_m+\beta_{m+1}}{\beta_m}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-k)}y$$

$$\left(\begin{array}{c|cc} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-$$

$$\sum_{l=1}^{\rho_1-\sigma-1}\sum_{n=0}^{\beta_i}(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+l+n}{\rho_2+\sigma+1+l}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}\binom{\beta_i+\rho_1-\sigma-1-n-l}{\beta_i-n}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{ba}^{(t_1+|\beta|-\beta_i+l+n)}x\left(\begin{array}{c|cc} W & a^{(p+t_1+|\beta|-\beta_i+l+n)} & c^{(\beta_i+\rho_1-\sigma-1-n-l)} \\ W' & b^{(q-t_1-l-n)} & c^{(\rho_2+\sigma+1+l+n)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-$$

$$\sum_{l=1}^{\rho_1-\sigma-1}\sum_{m=i-1}^{1}(-1)^{\sigma+1+i-m}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}\binom{\beta_m+\beta_{m+1}}{\beta_m}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}yZ_{ba}^{(t_1+|\beta|+l)}x\left(\begin{array}{c|cc} W & a^{(p+t_1+|\beta|+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} \\ W'' & c^{(\rho_3)} & \end{array}\right)=B+C+D-E-F$$

and the corresponding

$$M_\alpha=\sum_{k=0}^{\sigma+1}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\beta_i+\rho_1-k}{\rho_1-k}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}y...yZ_{cb}^{(\beta_i+\rho_1-k)}y$$

$$\left(\begin{array}{c|cc} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-\sum_{k=\sigma+2}^{\rho_1-1}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\beta_i+\rho_1-k}{\rho_1-k}\binom{\beta_{i-1}+\beta_i+\rho_1-k}{\beta_i+\rho_1-k}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-2})}yZ_{cb}^{(\beta_{i-1}+\beta_i+\rho_1-k)}yZ_{ba}^{(\sigma_1+|\beta|+\rho_1)}x$$

$$\left(\begin{array}{c|c} W & a^{(p+\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)}c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} \end{array}\right)-(-1)^{\rho_1}\binom{\rho_2+\rho_1}{\rho_1}\binom{\beta_{i-1}+\beta_i}{\beta_i}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-2})}yZ_{cb}^{(\beta_{i-1}+\beta_i)}yZ_{ba}^{(\sigma_1+|\beta|+\rho_1)}x\left(\begin{array}{c|l} W & a^{(p+\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-\rho_1)}c^{(\rho_2+\rho_1)} \\ W'' & c^{(\rho_3)} \end{array}\right)+$$

$$\sum_{k=\sigma+2}^{\rho_1-1}\sum_{l=i-2}^{1}(-1)^{k+i-l}\binom{\rho_2+k}{\rho_2}\binom{\beta_i+\rho_1-k}{\rho_1-k}\binom{\beta_l+\beta_{l+1}}{\beta_l}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_l+\beta_{l+1})}y...yZ_{cb}^{(\beta_{i-1})}yZ_{cb}^{(\beta_i+\rho_1-k)}yZ_{ba}^{(\sigma_1+|\beta|+\rho_1)}x\left(\begin{array}{c|l} W & a^{(p+\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)}c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} \end{array}\right)+$$

$$\sum_{l=i-2}^{1}(-1)^{\rho_1+i-l}\binom{\rho_2+\rho_1}{\rho_2}\binom{\beta_l+\beta_{l+1}}{\beta_l}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_l+\beta_{l+1})}y...yZ_{cb}^{(\beta_{i-1})}yZ_{cb}^{(\beta_i)}yZ_{ba}^{(\sigma_1+|\beta|+\rho_1)}x$$

$$\left(\begin{array}{c|l} W & a^{(p+\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-\rho_1)}c^{(\rho_2+\rho_1)} \\ W'' & c^{(\rho_3)} \end{array}\right)+\sum_{k=\sigma+2}^{\rho_1}\sum_{j=0}^{\beta_i+\rho_1-k}(-1)^k\binom{\rho-2+k}{\rho_2}\binom{\beta_i+\rho_1-k}{\rho_1-k}\binom{\rho_2+k+j}{\rho_2+k}$$

$$Z_{cb}^{()}\beta_1 y...yZ_{cb}^{()}\beta_{i-1}yZ_{ba}^{(\sigma_1+|\beta|-\beta_i+k+j)}x\left(\begin{array}{c|ll} W & a^{(p+\sigma_1+|\beta|-\beta_i+k+j)} & c^{(\beta_i+\rho_1-k-j)} \\ W' & b^{(q-\sigma_1-k-j)} & c^{(\rho_2+k+j)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-$$

$$\sum_{m=1}^{i-1}\sum_{\delta=0}^{\sigma+1}(-1)^{i-1-m+\delta}\binom{\beta_m+\beta_{m+1}}{\beta_m}\binom{\rho_2+\delta}{\delta}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-\delta)}y$$

$$\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-\delta)} & c^{(\rho_2+\delta)} \\ W'' & c^{(\rho_3)} & \end{array}\right)+\sum_{m=1}^{i-2}\sum_{\delta=\sigma+2}^{\rho_1-1}(-1)^{i-1-m+\delta}\binom{\beta_m+\beta_{m+1}}{\beta_m}\binom{\rho_2+\delta}{\rho_2}\binom{\beta_i+\rho_1-\delta}{\rho_1-\delta}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i+\rho_1-\delta)}yZ_{ba}^{(\sigma_1+|\beta|+\rho_1)}x\left(\begin{array}{c|l} W & a^{(p+\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-\delta)}c^{(\rho_2+\delta)} \\ W'' & c^{(\rho_3)} \end{array}\right)+$$

$$\sum_{\delta=\sigma+2}^{\rho_1-1}(-1)^{\delta}\binom{\beta_{i-1}+\beta_i}{\beta_i}\binom{\rho-2+\delta}{\delta}\binom{\beta_{i-1}+\beta_i+\rho_1-\delta}{\rho_1-\delta}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-2})}yZ_{cb}^{(\beta_{i-1}+\beta_i+\rho_1-\delta)}yZ_{ba}^{(\sigma_1+|\beta|+\rho_1)}x$$

$$\left(\begin{array}{c|l} W & a^{(p+\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-\delta)}c^{(\rho_2+\delta)} \\ W'' & c^{(\rho_3)} \end{array}\right)+\sum_{m=1}^{i-1}\sum_{\delta=\sigma+2}^{\rho_1-1}\sum_{l=i-1}^{m+2}(-1)^{\delta-1-m-l}\binom{\beta_m+\beta_{m+1}}{\beta_m}\binom{\rho_2+\delta}{\delta}\binom{\beta_l+\beta_{l+1}}{\beta_l}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_l+\beta_{l+1})}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-\delta)}yZ_{ba}^{(\sigma_1+|\beta|+\rho_1)}x\left(\begin{array}{c|l} W & a^{(p+\sigma_1+|\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-\delta)}c^{(\rho_2+\delta)} \\ W'' & c^{(\rho_3)} \end{array}\right)+$$

$$\sum_{m=1}^{i-1}\sum_{\delta=\sigma+2}^{\rho_1-1}(-1)^{\delta-1-m-(m+3)}\binom{\beta_m+\beta_{m+1}}{\beta_m}\binom{\rho_2+\delta}{\delta}\binom{\beta_m+\beta_{m+1}+\beta_{m+2}}{\beta_m+\beta_{m+1}}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1}+\beta_{m+2})}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-\delta)}yZ_{ba}^{(\sigma_1+|\beta|+\rho_1)}x\left(\begin{array}{c|l} W & a^{(p+\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-\delta)}c^{(\rho_2+\delta)} \\ W'' & c^{(\rho_3)} \end{array}\right)+$$

$$\sum_{m=1}^{i-1}\sum_{\delta=\sigma+2}^{\rho_1-1}(-1)^{\delta-1-m-(m+4)}\binom{\beta_m+\beta_{m+1}}{\beta_m}\binom{\rho_2+\delta}{\delta}\binom{\beta_{m-1}+\beta_m+\beta_{m+1}}{\beta_m+\beta_{m+1}}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{m-1}+\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-\delta)}yZ_{ba}^{(\sigma_1+|\beta|+\rho_1)}x\left(\begin{array}{c|l} W & a^{(p+\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-\delta)}c^{(\rho_2+\delta)} \\ W'' & c^{(\rho_3)} \end{array}\right)-$$

$$\sum_{m=1}^{i-1}\sum_{\delta=\sigma+2}^{\rho_1-1}\sum_{l=m-2}^{1}(-1)^{\delta-1-m-l}\binom{\beta_m+\beta_{m+1}}{\beta_m}\binom{\rho_2+\delta}{\delta}\binom{\beta_l+\beta_{l+1}}{\beta_l}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_l+\beta_{l+1})}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-\delta)}yZ_{ba}^{(\sigma_1+|\beta|+\rho_1)}x\left(\begin{array}{c|l} W & a^{(p+\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-\delta)}c^{(\rho_2+\delta)} \\ W'' & c^{(\rho_3)} \end{array}\right)-$$

$$\sum_{m=1}^{i-1}\sum_{\delta=\sigma+2}^{\rho_1-1}\sum_{j=0}^{\rho_1-\delta}(-1)^{i-1-m+\delta}\binom{\beta_m+\beta_{m+1}}{\beta_m}\binom{\rho_2+\delta}{\delta}\binom{\rho_2+\delta+j}{\rho_2+\delta}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}yZ_{ba}^{(\sigma_1+|\beta|+\delta+j)}x\left(\begin{array}{c|ll} W & a^{(p+\sigma_1+|\beta|+\delta+j)} & c^{(\rho_1-\delta-j)} \\ W' & b^{(q-\sigma_1-\delta-j)} & c^{(\rho_2+\delta+j)} \\ W'' & c^{(\rho_3)} & \end{array}\right)=$$

$$A_1-A_2-A_3+A_4+A_5+A_6-A_7+A_8+A_9+A_{10}+A_{11}+A_{12}-A_{13}-A_{14}$$

Claim:

$$\begin{array}{lll} 1)B=A_1 & 4)F=-A_{14}-A_3+A_5 & 7)-A_2+A_9=0 \\ 2)C=-A_7 & 5)A_{10}-A_{13}=0 & 8)A_4+A_8=0 \\ 3)E=A_6 & 6)A_{11}+A_{12}=0 & \end{array}$$

Proof of the claim: The identities 1), 2), 5) and 8) are clear.

4)

$$-A_{14}-A_3+A_5=-\sum_{m=1}^{i-1}\sum_{\delta=\sigma+2}^{\rho_1}\sum_{j=0}^{\rho_1-\delta}(-1)^{i-1-m+\delta}\binom{\beta_m+\beta_{m+1}}{\beta_m}\binom{\rho_2+\delta}{\rho_2}\binom{\rho_2+\delta+j}{\rho_2+\delta}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}yZ_{ba}^{(\sigma_1+|\beta|+\delta+j)}x\left(\begin{array}{c|ll} W & a^{(p+\sigma_1+|\beta|+\delta+j)} & c^{(\rho_1-\delta-j)} \\ W' & b^{(q-\sigma_1-\delta-j)} & c^{(\rho_2+\delta+j)} \\ W'' & c^{(\rho_3)} & \end{array}\right).$$

We know that $1\le l\le \rho_1-\sigma-1$ and $\sigma+2\le \delta+j\le \rho_1$. So we let $\delta+j=\sigma+1+l$ or $\delta=\sigma+1+l-j$. Thus

$$-\sum_{j=0}^{l-1}(-1)^{i-1-m+\sigma+1+l-j}\binom{\rho_2+\sigma+1+l}{\rho_2+\delta}\binom{\rho_2+\delta}{\rho_2}=-\sum_{j=0}^{l-1}(-1)^{i-m+\sigma+l-j}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+1+l}{j}=$$

$$-(-1)^{i-m+\sigma+l}\binom{\rho_2+\sigma+1+l}{\rho_2}\sum_{j=0}^{l-1}(-1)^j\binom{\sigma+1+l}{j}=-(-1)^{i-m+\sigma+l}\binom{\rho_2+\sigma+1+l}{\rho_2}(-1)^{l-1}\binom{\sigma+l}{l-1}=$$

$$-(-1)^{i-m+\sigma+1}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{l-1}$$

6)$A_{11}+A_{12}=0$ because we have

$$\binom{\beta_m+\beta_{m+1}+\beta_{m+2}}{\beta_m+\beta_{m+1}}\binom{\beta_m+\beta_{m+1}}{\beta_m}=\binom{\beta_m+\beta_{m+1}+\beta_{m+2}}{\beta_m}\binom{\beta_{m+1}+\beta_{m+2}}{\beta_{m+1}}$$

in A_{11} where $1 \leq m \leq i-2$ and

$$\binom{\beta_{m-1}+\beta_m+\beta_{m+1}}{\beta_{m-1}}\binom{\beta_m+\beta_{m+1}}{\beta_m}$$

in A_{12} where $2 \leq m \leq i-1$.

7)$-A_2+A_9=0$ because

$$\binom{\beta_{i-1}+\beta_i+\rho_1-\delta}{\beta_{i-1}+\beta_i}\binom{\beta_{i-1}+\beta_i}{\beta_{i-1}}=\binom{\beta_{i-1}+\beta_i+\rho_1-\delta}{\beta_{i-1}}\binom{\beta_i+\rho_1-\delta}{\beta_i}$$

In order to prove the identity (3) we have to consider two cases depending of ρ_1. Recall that $\rho_1>\sigma+1$.

Case 1. If in the expression for E we take $n+l=t$, where $1 \leq t \leq \rho_1-\sigma-1$, then from (3.1.1) we have that

$$(-1)^{\sigma+1}\sum_{l=1}^{t}\binom{\rho_2+\sigma+1+n+l}{\rho_2+\sigma+1+l}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}\binom{\beta_i+\rho_1-\sigma-1-n-l}{\beta_i-n}=$$

$$(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{z=1}^{t}(-1)^{z+1}\binom{\sigma+1+t}{t-z}\binom{\beta_i+\rho_1-\sigma-1-z}{\beta_i}$$

On ther other hand, let $u=k-\sigma-1$ then

$$A_6=\sum_{u=1}^{\rho_1-\sigma-1}\sum_{j=0}^{\beta_i+\rho_1-\sigma-1-u}(-1)^{u+\sigma+1}\binom{\rho_2+\sigma+1+u}{\rho_2}\binom{\rho_2+\sigma+1+u+j}{\rho_2+\sigma+1+u}\binom{\beta_i+\rho_1-\sigma-1-u}{\beta_i}$$

$$Z_{cb}^{(\beta)}y\dots Z_{cb}^{(\beta_{i-1})}yZ_{ba}^{(t_1+|\beta|-\beta_i+u+j)}x\left(\begin{array}{c|ll} W & a^{(p+t_1+|\beta|-\beta_i+u+j)} & c^{(\beta_i+\rho_1-\sigma-1-u-j)} \\ W' & b^{(q-t_1-u-j)} & c^{(\rho_2+\sigma+1+u+j)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

Now, we let $u+j=t$, where $1 \leq t \leq \rho_1-\sigma-1$, then

$$(-1)^{\sigma+1}\sum_{u=1}^{t}(-1)^u\binom{\rho_2+\sigma+1+t}{\rho_2+\sigma+1+u}\binom{\rho_2+\sigma+1+u}{\rho_2}\binom{\beta_i+\rho_1-\sigma-1-u}{\beta_i}=$$

$$(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+t}{\rho_2}\sum_{u=1}^{t}(-1)^u\binom{\sigma+1+t}{t-u}\binom{\beta_i+\rho_1-\sigma-1-u}{\beta_i}$$

Case 2. If in the expression for E we take $n+l=t$, where $\rho_1-\sigma \leq t \leq \beta_i+\rho_1-\sigma-1$ then $t=\rho_1-\sigma+\alpha$, where $0 \leq \alpha \leq \beta_i-1$. Thus $n+l=\rho_1-\sigma+\alpha$ and

$$(-1)^{\sigma+1}\binom{\rho_2+\rho_1+1+\alpha}{\rho_2}\sum_{n=\alpha+1}^{\beta_i}\binom{\rho_1+\alpha+1}{n}\binom{\rho_1+\alpha-n}{\sigma+1}\binom{\beta_i-1-\alpha}{\beta_i-n}$$

Set $j=n-\alpha-1$

$$(-1)^{\sigma+1}\binom{\rho_2+\rho_1+1+\alpha}{\rho_2}\sum_{j=0}^{\beta_i-\alpha-1}\binom{\rho_1+\alpha+1}{j+\alpha+1}\binom{\rho_1-1-j}{\sigma+1}\binom{\beta_i-1-\alpha}{\beta_i-\alpha-1-j}$$

Now, if in A_6 we let $k+j=\sigma+1+t=\rho_1+\alpha+1$ we get

$$\binom{\rho_2+\rho_1+\alpha+1}{\rho_2}\sum_{k=\sigma+2}^{\rho_1}(-1)^k\binom{\rho_1+\alpha+1}{k}\binom{\beta_i+\rho_1-k}{\beta_i}=$$

$$-(-1)^{\sigma+1}\binom{\rho_2+\rho_1+\alpha+1}{\rho_2}\sum_{j=0}^{\beta_i-\alpha-1}\binom{\rho_1+\alpha+1}{j+\alpha+1}\binom{\rho_1-1-j}{\sigma+1}\binom{\beta_i-\alpha-1}{k}$$

4.2) If $\rho_1\le\sigma+1$ then the corresponding

$$M'_\alpha=-\sum_{k=0}^{\rho_1-1}\sum_{j=0}^{\rho_1-k}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\rho_2+k+j}{\rho_2+k}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y$$

$$\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|+k+j)} & c^{(\rho_1-k-j)} \\ W' & b^{(q-\sigma_1-k-j)} & c^{(\rho_2+k+j)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)+\sum_{k=0}^{\rho_1-1}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\beta_i+\rho_1-k}{\beta_i}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{cb}^{(\beta_i+\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)+$$

$$\sum_{k=0}^{\rho_1-1}\sum_{m=i-1}^{1}(-1)^{k+i-m}\binom{\rho_2+k}{\rho_2}\binom{\beta_m+\beta_{m+1}}{\beta_m}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{m-1})}yZ_{cb}^{(\beta_m+\beta_{m+1})}yZ_{cb}^{(\beta_{m+2})}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-k)}y$$

$$\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

where in the first double sum $k+j\ge 1$, and the corresponding

$$M_\alpha=\sum_{k=0}^{\rho_1}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\beta_i+\rho_1-k}{\rho_1-k}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{cb}^{(\beta_i+\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-$$

$$\sum_{m=1}^{i-1}\sum_{\delta=0}^{\rho_1-1}(-1)^{i-1-m+\delta}\binom{\beta_m+\beta_{m+1}}{\beta_m}\binom{\rho_2+\delta}{\delta}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-\delta)}y$$

$$\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-\delta)} & c^{(\rho_2+\delta)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

The proof is exactly the same as in (3.2), and this finishes the proof of theorem 1. $\square$

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