REPRODUCTIVITY FOR A NEMATIC LIQUID CRYSTAL MODEL

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Abstract

In this paper we prove existence of weak solution with the reproductivity in time property, for a penalized PDE's system related to a nematic liquid crystal model.

This problem is relatively explicit when time-independent Dirichlet boundary conditions are imposed for the orientation of crystal molecules. For the time-dependent case, the verification of a maximum principle for weak reproductive solutions is fundamental in the argument.

Finally, the relation between reproductive and periodic in time solutions and their dependence respect to the regularity will be pointed out, being completely different the $2D$ and $3D$ cases.

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1 Introduction

In this work, a nematic liquid crystal model in a simplified Ericksen-Leslie version is considered; see for instance in [7] a more completed formulation of the problem.

This model can be seen as a variant of the Navier-Stokes problem(respect to the unknowns velocity-pressure $(\mathbf{u}, p)$) coupled with a convection-diffusion system for a new variable $\mathbf{d}$, a unit vectorial function modelling the orientation of the crystal molecules. On the other hand, it is usual to consider an approximation by Ginzburg-Landau penalization ([1]) for the constraint $|\mathbf{d}| = 1$ ($|\mathbf{d}| = |\mathbf{d}(t, x)|$ denotes the point-wise euclidean norm).

This penalized model (in which the constraint $|\mathbf{d}| = 1$ is relaxed by $|\mathbf{d}| \le 1$) was introduced by Lin in [5] and studied (from a mathematical point of view) by Lin and Liu in [6, 7] and by Coutand and Shkoller in [2]. The main difficulties of the model coming from the strongly nonlinear coupling between the orientation vector $\mathbf{d}$ and the velocity-pressure $(\mathbf{u}, p)$ and from the constraint $|\mathbf{d}| \le 1$, jointly with the well known difficulties for the Navier-Stokes problem (a nonlinear parabollic system with the free divergence constraint related to the pressure).

In all previous works, initial conditions and time-independent Dirichlet boundary conditions for $\mathbf{d}$ are considered. Now, we are interested in time reproductive solutions. In particular, we will see that this study is completely different for time-independent or time-dependent boundary conditions for $\mathbf{d}$.

We assume a (newtonian) fluid confined in an open bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3) with regular boundary $\partial\Omega$. In the penalized model the constraint $|\mathbf{d}| = 1$ is partially conserved to $|\mathbf{d}| \le 1$ as consequence of the maximum principle for the Ginzburg-Landau equation considering the penalization function

$$\mathbf{f}(\mathbf{d}) = \varepsilon^{-2}(|\mathbf{d}|^2 - 1)\mathbf{d}$$

where $\varepsilon > 0$ is the penalization parameter. There exists a potential function

$$F(\mathbf{d}) = \frac{1}{4\varepsilon^2}(|\mathbf{d}|^2 - 1)^2$$

such that $\mathbf{f}(\mathbf{d}) = \nabla_{\mathbf{d}}(F(\mathbf{d}))$ for each $\mathbf{d} \in \mathbb{R}^N$. Then, we consider the following PDE system in $(0, T) \times \Omega$:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = -\lambda\nabla \cdot (\nabla\mathbf{d} \odot \nabla\mathbf{d}), \quad \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + (\mathbf{u} \cdot \nabla)\mathbf{d} = \gamma(\Delta\mathbf{d} - f(\mathbf{d})), \quad |\mathbf{d}| \le 1, \end{cases} \tag{1}$$

The constants ν, λ and γ are positives, representing respectively, the fluid viscosity, an elasticity constant and a time relaxation. Here, the following tensorial notation is used:

$$(\nabla\mathbf{d} \odot \nabla\mathbf{d})_{ij} = \sum_{k=1}^{N} \partial_{x_i} d_k \partial_{x_j} d_k, \quad \forall\, i, j = 1, \ldots, N.$$

The problem (1) is completed with the (Dirichlet) boundary conditions

$$\mathbf{u}(x,t)=0, \qquad \mathbf{d}(x,t)=\mathbf{h}(x,t) \qquad \text{on } \partial\Omega\times(0,T) \tag{2}$$

(assuming as novelty a time-depending boundary data for $\mathbf{d}$ given by $\mathbf{h}:\partial\Omega\times(0,T)\mapsto\mathbb{R}^N$, in [6, 7] only a time-independent boundary data is considered) and the reproductivity conditions:

$$\mathbf{u}(x,0)=\mathbf{u}(x,T), \qquad \mathbf{d}(x,0)=\mathbf{d}(x,T) \qquad \text{in } \Omega \tag{3}$$

These reproductive conditions jointly with the constraint $|\mathbf{d}|\le 1$ are the main difficulties of the problem (1)-(3), while the time-dependent boundary conditions $\mathbf{d}=\mathbf{h}(x,t)$ produces some additional difficulties. In particular, an adequate lifting of this condition must be done.

It is important to remark that reproductive solution with the following boundary data independent of time $\mathbf{d}(x,t)_{|\partial\Omega\times(0,T)}=\mathbf{d}_0(x)$ has the trivial stationary (static) solution:

$$\begin{aligned}
&\mathbf{u}\equiv 0,\\
&\mathbf{d}\quad \text{solution of the elliptic problem:}\quad -\Delta\mathbf{d}+f(\mathbf{d})=0 \quad \text{in } \Omega, \quad \mathbf{d}_{|\partial\Omega}=\mathbf{d}_0,\\
&p\quad \text{such that}\quad \nabla p=-\lambda\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}).
\end{aligned}$$

Therefore, in this work will be fundamental assume time-dependent boundary data for $\mathbf{d}$.

The concept of reproductive solution in the Navier-Stokes context, appears for the first time in [4], see also [8].

The goal of this paper is to obtain existence of (global in time) weak solution of problem (1)–(3). We start defining a variational formulation, testing the maximum principle for $\mathbf{d}$ and the energy inequality. Afterwards, we introduce a Galerkin discretization of the problem, proving existence and uniqueness of approximate solution associated to arbitrary initial conditions. Then, a Leray-Schauder argument (by means of fixed point process about initial and final in time values of the solutions) allows us to obtain a reproductive Galerkin solution, which converges towards a continuous reproductive solution. Finally, some comments on the relation between reproductive and periodic solutions and their dependence with the regularity will be pointed out, being completely different the $2D$ and $3D$ cases.

In our opinion, an interesting open problem related with this work, is the asymptotic behaviour of reproductive solutions of (1)–(3) when $\varepsilon\to 0$. In the case of initial-boundary problem, existence of weak solution of the limit problem is obtained in [3].

2 Variational Formulation

For simplicity, we denote L^2 and H^1 instead of $L^2(\Omega)^N$ and $H^1(\Omega)^N$, $L^\infty(H^1)$ instead of $L^\infty(0,T;H^1(\Omega)^N)$, etc. Also, the scalar product in L^2 will be denoted by $(\cdot,\cdot)$, and $\langle\cdot,\cdot\rangle$ will denote some duality products.

Let us consider the following function spaces:

$$\begin{array}{rcl} H & = & \{\mathbf{u} \in L^2 : \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega, \mathbf{u}\cdot n = 0 \text{ on } \partial\Omega\} \\ V & = & \{\mathbf{u} \in H_0^1 : \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\} \end{array}$$

Without less of generality, we fix the constants $\nu = \lambda = \gamma = 1$.

Obviously, the following compatibility conditions will be imposed in this work for the boundary data function $\mathbf{h}$:

$$|\mathbf{h}| \le 1 \quad \text{on } \partial\Omega\times(0,T) \quad \text{and} \quad \mathbf{h}(0) = \mathbf{h}(T) \quad \text{on } \partial\Omega.$$

2.1 The variational problem for $(\mathbf{u}, \mathbf{d})$

Definición 1 *We say that* $(\boldsymbol{u}, \boldsymbol{d})$ *is a weak reproductive solution of* (1)–(3) *if*

$$\begin{array}{l} \boldsymbol{u} \in L^2(V)\cap L^\infty(H) \\ \boldsymbol{d} \in L^\infty(H^1), \quad \Delta\boldsymbol{d} \in L^2(L^2), \quad \boldsymbol{d}|_{\partial\Omega\times(0,T)} = \boldsymbol{h} \end{array}$$

verifying

$$\begin{array}{l} \langle\partial_t\boldsymbol{u}, \boldsymbol{v}\rangle + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}) + (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + (\nabla\boldsymbol{d}^t\Delta\boldsymbol{d}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in V\cap L^\infty, \\ \langle\partial_t\boldsymbol{d}, \boldsymbol{e}\rangle + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{d}, \boldsymbol{e}) + (f(\boldsymbol{d}), \boldsymbol{e}) - (\Delta\boldsymbol{d}, \boldsymbol{e}) = 0 \quad \forall\, \boldsymbol{e} \in L^3, \\ \boldsymbol{u}(0) = \boldsymbol{u}(T), \quad \boldsymbol{d}(0) = \boldsymbol{d}(T) \quad \textit{in } \Omega. \end{array}$$

□

In order to arrive at the previous variational formulation, the following equalities have been used:

$$\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}) = \nabla\left(\frac{|\nabla\mathbf{d}|^2}{2}\right) + \nabla\mathbf{d}^t\Delta\mathbf{d}$$

and

$$\left(\nabla\left(\frac{|\nabla\mathbf{d}|^2}{2}\right), \mathbf{v}\right) = 0 \quad \forall\,\mathbf{v} \in V.$$

Notice that the reproductivity conditions $\mathbf{u}(0) = \mathbf{u}(T)$ and $\mathbf{d}(0) = \mathbf{d}(T)$ have sense, because $\mathbf{u}$ and $\mathbf{d}$ are (at least weakly) continuous functions from $[0,T]$ onto some Banach spaces.

2.2 Weak Maximum Principle for d

An essential characteristic of the problem for $\mathbf{d}$ (given $\mathbf{u}$) is the following weak maximum principle:

Lemma 2.1 *Assume* $|\boldsymbol{h}(x,t)| \leq 1$ *a.e. on* $\partial\Omega\times(0,T)$. *Then, given* $\boldsymbol{u} \in L^2(0,T;V)\cap L^\infty(0,T;H)$, *any weak solution for the* $\boldsymbol{d}$*-problem, i.e.* $\boldsymbol{d} \in L^2(0,T;H^1)\cap L^\infty(0,T;L^2)$ *such that*

$$\langle \partial_t \boldsymbol{d}, \boldsymbol{e}\rangle + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{d}, \boldsymbol{e}) + (f(\boldsymbol{d}), \boldsymbol{e}) + (\nabla \boldsymbol{d}, \nabla \boldsymbol{e}) = 0, \quad \forall\, \boldsymbol{e} \in H^1_0, \tag{4}$$

$$\boldsymbol{d}_{|\partial\Omega\times(0,T)} = \boldsymbol{h} \quad \text{and} \quad \boldsymbol{d}(0) = \boldsymbol{d}(T), \tag{5}$$

verifies $|\boldsymbol{d}(x,t)| \leq 1$ *a.e. in* $\Omega\times(0,T)$.

□

Proof.

Let us define the functions:

$$\varphi(x,t) = (|\mathbf{d}(x,t)|^2 - 1)_+ \quad \text{and} \quad \psi = \varphi\mathbf{d},$$

where $z_+ = max(z,0)$ for each $z \in \mathbb{R}$. Hypothesis $|\mathbf{h}| \leq 1$ implies $\varphi = 0$ on $\partial\Omega\times(0,T)$. Taking ψ as test function in the variational formulation for $\mathbf{d}$:

$$\frac{1}{2}\int_\Omega \partial_t(|\mathbf{d}|^2)\varphi + \int_\Omega (\mathbf{u}\cdot\nabla)|\mathbf{d}|^2\varphi + \int_\Omega \nabla\mathbf{d} : \nabla(\varphi\mathbf{d}) + \int_\Omega \varphi\, f(\mathbf{d})\cdot\mathbf{d} = 0 \tag{6}$$

If we define $\Omega_+ = \{x \in \Omega : |\mathbf{d}(t)| > 1\}$, then $\varphi = 0$ in $\Omega\setminus\Omega_+$. Consequently, the first three terms of (6) can be written as follows:

$$\frac{1}{2}\int_\Omega \partial_t(|\mathbf{d}|^2)\varphi = \frac{1}{2}\int_{\Omega_+} \partial_t(|\mathbf{d}|^2 - 1)\varphi = \frac{1}{4}\frac{\mathbf{d}}{dt}\|\varphi\|^2_{L^2(\Omega)},$$

$$\int_\Omega (\mathbf{u}\cdot\nabla)|\mathbf{d}|^2\varphi = \int_\Omega (\mathbf{u}\cdot\nabla)(|\mathbf{d}|^2 - 1)\varphi = \int_{\Omega_+} (\mathbf{u}\cdot\nabla\varphi)\varphi = 0,$$

$$\begin{aligned}\int_\Omega \nabla\mathbf{d} : \nabla(\varphi\mathbf{d}) &= \frac{1}{2}\int_\Omega \nabla(|\mathbf{d}|^2)\cdot\nabla\varphi + \int_\Omega |\nabla\mathbf{d}|^2\varphi \\ &\geq \frac{1}{2}\int_{\Omega_+} \nabla(|\mathbf{d}|^2 - 1)\nabla\varphi = \frac{1}{2}\|\nabla\varphi\|^2_{L^2(\Omega)}.\end{aligned}$$

Taking into account that $f(\mathbf{d})\cdot\mathbf{d} > 0$ as $|\mathbf{d}| > 1$ (i.e. in Ω_+) in the last term of (6), we arrive at the differential inequality:

$$\frac{d}{dt}\|\varphi\|^2_{L^2} + 2\|\nabla\varphi\|^2_{L^2} \leq 0.$$

Integrating in $[0,T]$ one has

$$\|\varphi(T)\|_{L^2}^2 + 2\int_0^T \|\nabla\varphi\|_{L^2}^2 \le \|\varphi(0)\|_{L^2}^2$$

Since $\varphi(T) = \varphi(0)$, the previous inequality implies $\nabla\varphi = 0$ a.e. in $\Omega \times (0,T)$. Therefore $\varphi(\cdot,t) \equiv$ constant in Ω. But, since $\varphi(\cdot,t) = 0$ on $\partial\Omega$, we conclude $\varphi(x,t) = 0$ a.e. in $(x,t) \in \Omega \times (0,T)$, i.e. $|\mathbf{d}(x,t)| \le 1$ a.e. in $\Omega \times (0,T)$. □

Although in formulation of problem (1)–(3), the constraint $|\mathbf{d}| \le 1$ has been explicitly included, the previous Lemma say us that it is not necessary because this constraint can be obtained a posteriori.

In the sequel, we can consider the penalized function $\mathbf{f}$ as a bounded function. Indeed, from maximum principle one has $|\mathbf{d}| \le 1$ (recall that $|\mathbf{h}| \le 1$). Therefore, using the auxiliary function

$$\widetilde{\mathbf{f}}(\mathbf{d}) = \begin{cases} \dfrac{1}{\varepsilon^2}(|\mathbf{d}|^2 - 1)\mathbf{d} & \text{if} \quad |\mathbf{d}| \le 1 \\ 0 & \text{if} \quad |\mathbf{d}| > 1 \end{cases}$$

one can consider the problem (1) with $\widetilde{\mathbf{f}}$ instead of $\mathbf{f}$. Indeed, if $(\mathbf{u}, p, \mathbf{d})$ is a solution of (1) with $\widetilde{\mathbf{f}}$, in particular $|\mathbf{d}| \le 1$ (because the maximum principle is also verified, since $\widetilde{\mathbf{f}}(\mathbf{d}) \cdot \mathbf{d} > 0$ as $|\mathbf{d}| > 1$), then $(\mathbf{u}, p, \mathbf{d})$ is also a solution of (1) with $\mathbf{f}$. It is easy to verify the inverse implication.

Notice that, $|\widetilde{\mathbf{f}}(\mathbf{d})| \le 1/\varepsilon^2$ for each $\mathbf{d} \in \mathbb{R}^N$.

2.3 Variational Formulacion in $\mathbf{u}$ and $\widehat{\mathbf{d}}$

Since a time-dependent boundary data $\mathbf{h}(x,t)$ on $\partial\Omega \times (0,T)$ has been considered, an adequate lifting is necessary. Assuming $\mathbf{h} \in H^1(0,T; H^{1/2}(\partial\Omega)^N)$, if we define $\widetilde{\mathbf{d}}(t)$ as the weak solution of

$$\begin{cases} -\Delta\widetilde{\mathbf{d}} = 0 & \text{in } \Omega \\ \widetilde{\mathbf{d}}|_{\partial\Omega} = \mathbf{h}(t) & \text{on } \partial\Omega \end{cases}$$

then $\widetilde{\mathbf{d}} \in H^1(0,T; H^1(\Omega)^N)$ (notice that only weak regularity of Laplace-Dirichlet problems are used, therefore it is suffices regularity Lipschitz for the domain Ω). Moreover, since $\mathbf{h}(0) = \mathbf{h}(T)$ on $\partial\Omega$, then $\widetilde{\mathbf{d}}(0) = \widetilde{\mathbf{d}}(T)$ in Ω.

Therefore, if we define $\widehat{\mathbf{d}}(t) = \mathbf{d}(t) - \widetilde{\mathbf{d}}(t)$, then $\widehat{\mathbf{d}}(t) \in H_0^1(\Omega)^N$, $\Delta\widehat{\mathbf{d}} = \Delta d$ in $\Omega \times (0,T)$ and $\mathbf{d}(0) = \mathbf{d}(T)$ if and only if $\widehat{\mathbf{d}}(0) = \widehat{\mathbf{d}}(T)$.

Definición 2 *We say that* $(\boldsymbol{u}, \boldsymbol{d})$ *is a weak reproductive solution of* (1)–(3) *if* $\boldsymbol{d} = \widehat{\boldsymbol{d}} + \widetilde{\boldsymbol{d}}$ *with* $\widetilde{\boldsymbol{d}}$ *previously defined and for* $(\boldsymbol{u}, \widehat{\boldsymbol{d}})$ *such that*

$$\boldsymbol{u} \in L^2(V) \cap L^\infty(H),$$
$$\widehat{\boldsymbol{d}} \in L^\infty(H_0^1), \quad \Delta\widehat{\boldsymbol{d}} \in L^2(L^2),$$

verifying

$$\langle \partial_t \boldsymbol{u}, \boldsymbol{v}\rangle + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}) + (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\nabla \boldsymbol{d}^t \Delta \widehat{\boldsymbol{d}}, \boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in V \cap L^\infty,$$
$$\langle \partial_t \widehat{\boldsymbol{d}}, \boldsymbol{e}\rangle + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{d}, \boldsymbol{e}) + (f(\boldsymbol{d}), \boldsymbol{e}) - (\Delta \widehat{\boldsymbol{d}}, \boldsymbol{e}) = -(\partial_t \widetilde{\boldsymbol{d}}, \boldsymbol{e}) \quad \forall\, \boldsymbol{e} \in L^3,$$
$$\boldsymbol{u}(0) = \boldsymbol{u}(T), \quad \widehat{\boldsymbol{d}}(0) = \widehat{\boldsymbol{d}}(T) \quad in\ \Omega.$$

□

2.4 Energy Inequality

Taking $\mathbf{v} = \mathbf{u}$ and $\mathbf{e} = -\Delta\widehat{\mathbf{d}}$ as test functions in previous formulation, one has

$$\begin{cases} (\partial_t \mathbf{u}, \mathbf{u}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{u}) + (\nabla \mathbf{u}, \nabla \mathbf{u}) + (\nabla \mathbf{d}^t \Delta \widehat{\mathbf{d}}, \mathbf{u}) = 0 \\ -(\partial_t \widehat{\mathbf{d}}, \Delta\widehat{\mathbf{d}}) - ((\mathbf{u} \cdot \nabla)\mathbf{d}, \Delta\widehat{\mathbf{d}}) - (\mathbf{f}(\mathbf{d}), \Delta\widehat{\mathbf{d}}) - (\Delta\widehat{\mathbf{d}}, \Delta\widehat{\mathbf{d}}) = (\partial_t \widetilde{\mathbf{d}}, \Delta\widehat{\mathbf{d}}). \end{cases}$$

Adding up, taking into account that

$$((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{u}) = 0 \quad \text{and} \quad (\nabla \mathbf{d}^t \Delta \widehat{\mathbf{d}}, \mathbf{u}) - ((\mathbf{u} \cdot \nabla)\mathbf{d}, \Delta\widehat{\mathbf{d}}) = 0,$$

one arrives (at least formally) at the following energy equality:

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|_{L^2}^2 + \|\nabla\widehat{\mathbf{d}}\|_{L^2}^2\right) + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\Delta\widehat{\mathbf{d}}\|_{L^2}^2 = (\mathbf{f}(\mathbf{d}), \Delta\widehat{\mathbf{d}}) + (\partial_t \widetilde{\mathbf{d}}, \Delta\widehat{\mathbf{d}}). \quad (7)$$

Consequently, one has the energy inequality:

$$\frac{d}{dt}\left(\|\mathbf{u}\|_{L^2}^2 + \|\nabla\widehat{\mathbf{d}}\|_{L^2}^2\right) + 2\|\nabla \mathbf{u}\|_{L^2}^2 + \|\Delta\widehat{\mathbf{d}}\|_{L^2}^2 \le 2\left(\|\mathbf{f}(\mathbf{d})\|_{L^2}^2 + \|\partial_t \widetilde{\mathbf{d}}\|_{L^2}^2\right), \quad (8)$$

where the right hand side is bounded in $L^1(0,T)$ using that $\partial_t \widetilde{\mathbf{d}} \in L^2(L^2)$.

3 The Main Result

Theorem 3.1 *Let* $T > 0$ *and* $\Omega \subset \mathbb{R}^N$ *(*$N = 2$ *or* 3*) an open bounded domain with lipschitz boundary. Assume* $\boldsymbol{h} \in H^1(0,T; H^{1/2}(\partial\Omega))$ *such that* $|\boldsymbol{h}(t,x)| \le 1$ *a.e.* $(t,x) \in (0,T) \times \partial\Omega$ *and* $\boldsymbol{h}(0) = \boldsymbol{h}(T)$ *in* Ω*. Then there exists a weak reproductive solution* $(\boldsymbol{u}, p, \boldsymbol{d})$ *of problem* (1)–(3) □

Remark. Hypothesis of regularity for boundary data $\mathbf{h}$ can be relaxed by

$$\mathbf{h} \in L^\infty(0,T;H^{1/2}(\partial\Omega)) \quad \text{with} \quad \mathbf{h}_t \in L^2(0,T;L^2(\partial\Omega)).$$

Indeed, this regularity implies the following regularity for the lifting function

$$\widetilde{\mathbf{d}} \in L^\infty(0,T;H^1(\Omega)) \quad \text{with} \quad \widetilde{\mathbf{d}}_t \in L^2(0,T;L^2(\Omega)),$$

which will be sufficient in the sequel. Notice that regularity for time derivative can be proved using the "transposition solution" of the Laplace-Dirichlet problem with boundary data equal to $\mathbf{h}_t$ (due to $\mathbf{h}_t$ has not a trace sense). By the contrary, further regularity for the domain Ω is necessary (which implies H^2-regularity for the adjoint problem), in order to well define the transposition solution.

□

In the proof of this theorem, the Galerkin method will be used. Firstly, we consider the initial-boundary problem associated to arbitrary initial data. Afterwards, the key is to find certain initial data that are "reproduced" at final time.

3.1 The Approximate Initial-Boundary Problem

Let $\{\phi_i\}_n \geq 1$ and $\{\varphi_i\}_n \geq 1$ "special" basis of V and $H^1_0(\Omega)$, respectively, formed by eigenfunctions of the Stokes problem

$$(\nabla\phi_i, \nabla\mathbf{v}) = \lambda_i(\phi_i, \mathbf{v}) \qquad \forall\, \mathbf{v} \in V,\ \phi_i \in V, \quad \text{con } \|\phi_i\|_{L^2} = 1, \quad \lambda_i \nearrow +\infty$$

and of the Poisson problem

$$(\nabla\varphi_i, \nabla w) = \mu_i(\varphi_i, w) \qquad \forall\, w \in H^1_0,\ \varphi_i \in H^1, \quad \text{con } \|\varphi_i\|_{L^2} = 1, \quad \mu_i \nearrow +\infty$$

Let V^m and W^m be the finite-dimensional subspaces spanned by $\{\phi_1, \phi_2, \ldots, \phi_n\}$ and $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ respectively.

Given $\mathbf{u}_0 \in H$ and $\mathbf{d}_0 \in H^1$ (verifying the compatibility condition $\mathbf{h}_0|_{\partial\Omega} = \mathbf{h}(0)$), for each $m \geq 1$, we seek an approximate solution $(\mathbf{u}_m, \mathbf{d}_m)$, with $\mathbf{d}_m = \widehat{\mathbf{d}}_m + \widetilde{\mathbf{d}}$, such that:

$$\mathbf{u}_m : [0,T] \mapsto V^m, \qquad \mathbf{u}_m(t) = \sum_{j=1}^m \xi_{i,m}(t)\phi_i,$$

$$\widehat{\mathbf{d}}_m : [0,T] \mapsto W^m, \qquad \widehat{\mathbf{d}}_m(t) = \sum_{j=1}^m \zeta_{i,m}(t)\varphi_i,$$

verifying the following variational formulation a.e. in t:

$$\begin{cases} (\partial_t \mathbf{u}_m(t), \mathbf{v}_m) + ((\mathbf{u}_m(t)\cdot\nabla)\mathbf{u}_m(t), \mathbf{v}_m) + (\nabla \mathbf{u}_m(t), \nabla \mathbf{v}_m) \\ \qquad +(\nabla \mathbf{d}_m^t(t)\Delta\widehat{\mathbf{d}}_m(t), \mathbf{v}_m) = 0 \quad \forall\, \mathbf{v}_m \in V^m \\ (\partial_t \widehat{\mathbf{d}}_m(t), \mathbf{e}_m) + ((\mathbf{u}_m(t)\cdot\nabla)\mathbf{d}_m(t), \mathbf{e}_m) + (\mathbf{f}(\mathbf{d}_m(t)), \mathbf{e}_m) \\ \qquad +(\nabla\widehat{\mathbf{d}}_m(t), \nabla \mathbf{e}_m) = -(\partial_t \widetilde{\mathbf{d}}(t), \mathbf{e}_m) \quad \forall\, \mathbf{e}_m \in W^m \\ \mathbf{u}_m(0) = \mathbf{u}_{0m} = P_m(\mathbf{u}_0), \quad \mathbf{d}_m(0) = \mathbf{d}_{0m} = Q_m(\mathbf{d}_0), \end{cases} \tag{9}$$

Here, P_m denotes the orthogonal projection operator of H onto V^m, $P_m : H \mapsto V^m$ and Q_m the orthogonal projection operator of L^2 onto W^m, $Q_m : L^2 \mapsto W^m$. In particular, $\mathbf{u}_{0m} \to \mathbf{u}_0$ and $\mathbf{d}_{0m} \to \mathbf{d}_0$ as $m \to 0$.

Re-writing (9) as a first order ordinary differential system (in normal form) associated to the unknowns $(\xi_{i,m}(t), \zeta_{i,m}(t))$, one has the existence of a maximal solution (defined in some interval $[0, \tau_m) \subset [0, T]$) of the related Cauchy problem. Moreover, from a priori estimates (independent on m) which will be obtained below, in particular one has that $\tau_m = T$. Finally, using regularity of the chosen spectral basis, uniqueness of approximate solution will be proved in section 3.3.

Remark: Since a discretization in space has been done in definition of approximate solution, the maximum principle is not always verified, therefore the constraint $|\mathbf{d}_m| \le 1$ is not true in general.

3.2 "A priori" estimates

Taking $\mathbf{u}_m(t) \in V^m$ as test function in the $\mathbf{u}$-system of (9) and $-\Delta\widehat{\mathbf{d}}_m(t) \in W^m$ in the $\mathbf{d}$-system (latter is possible due to consider the special eigenfunction basis), and following the argument that yields to energy inequality (Section 2.4), one has

$$\frac{d}{dt}\Big(\|\mathbf{u}_m\|_{L^2}^2 + \|\nabla\widehat{\mathbf{d}}_m\|_{L^2}^2\Big) + 2\|\nabla\mathbf{u}_m\|_{L^2}^2 + \|\Delta\widehat{\mathbf{d}}_m\|_{L^2}^2 \le 2\Big(\|\mathbf{f}(\mathbf{d}_m)\|_{L^2}^2 + \|\partial_t\widetilde{\mathbf{d}}\|_{L^2}^2\Big),$$

hence, using the initial estimates $\|\mathbf{u}_m(0)\|_{L^2}^2 \le C$ and $\|\nabla\widehat{\mathbf{d}}_m(0)\|_{L^2}^2 \le C$, Gronwall's lemma implies

$$(\mathbf{u}_m) \text{ is uniformly bounded in } L^\infty(H) \cap L^2(V)$$

and

$$(\widehat{\mathbf{d}}_m) \text{ is uniformly bounded in } L^\infty(H_0^1) \cap L^2(H^2)$$

Therefore, since $\mathbf{d}_m = \widehat{\mathbf{d}}_m + \widetilde{\mathbf{d}}$ with $\widetilde{\mathbf{d}} \in L^\infty(H^1)$ and $\Delta\mathbf{d}_m = \Delta\widehat{\mathbf{d}}_m$, one has

$$(\mathbf{d}_m) \text{ is uniformly bounded in } L^\infty(H^1)$$

and

$$(\Delta \mathbf{d}_m) \text{ is uniformly bounded in } L^2(L^2).$$

Using previous estimations in (9), one has the following estimations:

$$(\partial_t \mathbf{u}_m) \text{ is uniformly bounded in } L^2((V \cap L^\infty)')$$

and

$$(\partial_t \widehat{\mathbf{d}}_m) \text{ is uniformly bounded in } L^2(L^{3/2})$$

Using compactness results for time spaces with values in Banach spaces (see [9]) with the triplet $V \hookrightarrow H \hookrightarrow (V \cap L^\infty)'$ and $H^2 \cap H_0^1 \hookrightarrow H_0^1 \hookrightarrow L^{3/2}$, one has

$$(\mathbf{u}_m) \text{ is relatively compact in } L^2(H)$$

and

$$(\widehat{\mathbf{d}}_m) \text{ is relatively compact in } L^2(H_0^1).$$

Consequently, $(\mathbf{d}_m)$ is relatively compact in $L^2(H^1)$.

In fact, this compactness is sufficient in the pass to the limit in (9) in order to control the nonlinear terms.

Remark. Notice that if h and Ω are regular enough, then $\widetilde{\mathbf{d}} \in L^2(H^2)$ and $(\mathbf{d}_m)$ is bounded in $L^2(H^2)$.

3.3 Uniqueness of Approximate Solution

Without less of generality, in this section only the $3D$ case ($N = 3$) will be considered. Let $(\mathbf{u}_m^1, \mathbf{d}_m^1)$ and $(\mathbf{u}_m^2, \mathbf{d}_m^2)$, two solutions of (9), and we denote $\mathbf{u}_m = \mathbf{u}_m^1 - \mathbf{u}_m^2$ and $\mathbf{d}_m = \mathbf{d}_m^1 - \mathbf{d}_m^2$ (notice that $\mathbf{d}_m = \widehat{\mathbf{d}}_m$). Making the difference between (9) for $(\mathbf{u}_m^1, \mathbf{d}_m^1)$ and $(\mathbf{u}_m^2, \mathbf{d}_m^2)$, considering $\mathbf{u}_m$ and $-\Delta \mathbf{d}_m$ as test functions, and taking into account that

$$\begin{aligned} \Big((\mathbf{u}_m^1 \cdot \nabla)\mathbf{u}_m^1 - (\mathbf{u}_m^2 \cdot \nabla)\mathbf{u}_m^2, \mathbf{u}_m\Big) &= \Big((\mathbf{u}_m^1 \cdot \nabla)\mathbf{u}_m + (\mathbf{u}_m \cdot \nabla)\mathbf{u}_m^2, \mathbf{u}_m\Big) \\ &= \Big((\mathbf{u}_m \cdot \nabla)\mathbf{u}_m^2, \mathbf{u}_m\Big) \end{aligned}$$

$$\begin{aligned} & \Big(\Delta \mathbf{d}_m^1 \cdot \nabla \mathbf{d}_m^1 - \Delta \mathbf{d}_m^2 \cdot \nabla \mathbf{d}_m^2, \mathbf{u}_m\Big) + \Big((\mathbf{u}_m^1 \cdot \nabla)\mathbf{d}_m^1 - (\mathbf{u}_m^2 \cdot \nabla)\mathbf{d}_m^2, -\Delta \mathbf{d}_m\Big) \\ = & \Big(\Delta \mathbf{d}_m^1 \cdot \nabla \mathbf{d}_m + \Delta \mathbf{d}_m \cdot \nabla \mathbf{d}_m^2, \mathbf{u}_m\Big) + \Big((\mathbf{u}_m^1 \cdot \nabla)\mathbf{d}_m + (\mathbf{u}_m \cdot \nabla)\mathbf{d}_m^2, -\Delta \mathbf{d}_m\Big) \\ = & \Big(\Delta \mathbf{d}_m^1 \cdot \nabla \mathbf{d}_m, \mathbf{u}_m\Big) + \Big((\mathbf{u}_m^1 \cdot \nabla)\mathbf{d}_m, -\Delta \mathbf{d}_m\Big) \end{aligned}$$

the following equality holds

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}_m\|_{L^2}^2+\|\nabla\mathbf{d}_m\|_{L^2}^2\right)+\|\nabla\mathbf{u}_m\|_{L^2}^2+\|\Delta\mathbf{d}_m\|_{L^2}^2\\
&=-((\mathbf{u}_m\cdot\nabla)\mathbf{u}_m^2,\mathbf{u}_m)-((\Delta\mathbf{d}_m^1\cdot\nabla)\mathbf{d}_m,\mathbf{u}_m)\\
&\ +((\mathbf{u}_m^1\cdot\nabla)\mathbf{d}_m,\Delta\mathbf{d}_m)-(\mathbf{f}(\mathbf{d}_m^1)-\mathbf{f}(\mathbf{d}_m^2),\Delta\mathbf{d}_m).
\end{aligned}
\tag{10}
$$

Bounding each term on the right hand side of (10):

$$
\begin{aligned}
|((\mathbf{u}_m\cdot\nabla)\mathbf{u}_m,\mathbf{u}_m^2)| &\le \|\mathbf{u}_m\|_{L^4}\cdot\|\nabla\mathbf{u}_m\|_{L^2}\cdot\|\mathbf{u}_m^2\|_{L^4}\\
&\le \|\mathbf{u}_m\|_{L^2}^{1/4}\cdot\|\mathbf{u}_m\|_{H^1}^{7/4}\cdot\|\mathbf{u}_m^2\|_{L^4}\\
&\le \bar{\varepsilon}\|\mathbf{u}_m\|_{H^1}^2+C_{\bar{\varepsilon}}\|\mathbf{u}_m^2\|_{L^4}^8\cdot\|\mathbf{u}_m\|_{L^2}^2.
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\left|((\Delta\mathbf{d}_m^1\cdot\nabla)\mathbf{d}_m,u_m)\right| &\le \|\Delta\mathbf{d}_m^1\|_{L^3}\cdot\|\nabla\mathbf{d}_m\|_{L^6}\cdot\|\mathbf{u}_m\|_{L^2}\\
&\le C\|\Delta\mathbf{d}_m^1\|_{L^3}\cdot\|\nabla\mathbf{d}_m\|_{H^1}\cdot\|\mathbf{u}_m\|_{L^2}\\
&\le \bar{\varepsilon}\|\Delta\mathbf{d}_m\|_{L^2}^2+C_{\bar{\varepsilon}}\|\Delta\mathbf{d}_m^1\|_{L^3}^2\cdot\|\mathbf{u}_m\|_{L^2}^2.
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
|((\mathbf{u}_m^1\cdot\nabla)\mathbf{d}_m,\Delta\mathbf{d}_m)| &\le \|\mathbf{u}_m^1\|_{L^4}\cdot\|\nabla\mathbf{d}_m\|_{L^4}\cdot\|\Delta\mathbf{d}_m\|_{L^2}\\
&\le C\|\mathbf{u}_m^1\|_{L^4}\cdot\|\Delta\mathbf{d}_m\|_{L^2}^{7/4}\cdot\|\nabla\mathbf{d}_m\|_{L^4}^{1/4}\\
&\le \bar{\varepsilon}\|\Delta\mathbf{d}_m\|_{L^2}^2+C_{\bar{\varepsilon}}\|\mathbf{u}_m^1\|_{L^4}^8\cdot\|\nabla\mathbf{d}_m\|_{L^2}^2.
\end{aligned}
\tag{13}
$$

In order to bound $(\mathbf{f}(\mathbf{d}_m^1)-\mathbf{f}(\mathbf{d}_m^2),\Delta\mathbf{d}_m)$ we will use the following expression of $\mathbf{f}$:

$$
\mathbf{f}(\mathbf{d}_m^1)-\mathbf{f}(\mathbf{d}_m^2)=\frac{1}{\varepsilon^2}(|\mathbf{d}_m^1|^2\mathbf{d}_m^1-|\mathbf{d}_m^2|^2\mathbf{d}_m^2-\mathbf{d}_m).
$$

Adding and differentiating $|\mathbf{d}_m^1|^2\mathbf{d}_m^2$, we can write

$$
\mathbf{f}(\mathbf{d}_m^1)-\mathbf{f}(\mathbf{d}_m^2)=\frac{1}{\varepsilon^2}g(\mathbf{d}_m^1,\mathbf{d}_m^2)\mathbf{d}_m
$$

being $g(\mathbf{d}_m^1,\mathbf{d}_m^2)=|\mathbf{d}_m^1|^2+(\mathbf{d}_m^1+\mathbf{d}_m^2)\mathbf{d}_m^2-1$, which verifies $|g(\mathbf{d}_m^1,\mathbf{d}_m^2)|\le 4$. Therefore

$$
\begin{aligned}
|(\mathbf{f}(\mathbf{d}_m^1)-\mathbf{f}(\mathbf{d}_m^2),\Delta\mathbf{d}_m)| &\le \left|\left(\frac{1}{\varepsilon^2}g(\mathbf{d}_m^1,\mathbf{d}_m^2)\mathbf{d}_m,\Delta\mathbf{d}_m\right)\right|\\
&\le \frac{4}{\varepsilon^2}\|\mathbf{d}_m\|_{L^2}\|\Delta\mathbf{d}_m\|_{L^2}\\
&\le \frac{C(\bar{\varepsilon})}{\varepsilon^4}\|\nabla\mathbf{d}_m\|_{L^2}^2+\bar{\varepsilon}\|\Delta\mathbf{d}_m\|_{L^2}^2
\end{aligned}
\tag{14}
$$

Accordingly (12)–(14), choosing $\bar{\varepsilon}$ small enough, one has

$$\begin{cases} \dfrac{d}{dt}\left(\|\mathbf{u}_m\|^2_{L^2} + \|\nabla \mathbf{d}_m\|^2_{L^2}\right) \leq a(t)(\|\mathbf{u}_m\|^2_{L^2} + \|\nabla \mathbf{d}_m\|^2_{L^2}) \\ \|\mathbf{u}_m(0)\|^2_{L^2} + \|\nabla \mathbf{d}_m(0)\|^2_{L^2} = 0. \end{cases}$$

with $a(t) \in L^1(0,T)$ (using that $\|\Delta \mathbf{d}^1_m\|^2_{L^3}$, $\|\mathbf{u}^2_m\|^8_{L^4}$ and $\|\mathbf{u}^1_m\|^8_{L^4} \in L^1(0,T)$). Applying Gronwall's lemma, one has $\mathbf{u}_m = 0$ and $\nabla \mathbf{d}_m = 0$. Finally, since $\mathbf{d}_m = 0$ on $\partial\Omega$, then $\mathbf{d}_m = 0$. Therefore, uniqueness of approximate solution for the initial-boundary problem is finished.

3.4 Existence of approximate reproductive solution.

From energy inequality (8) associated to approximate problem (9):

$$\frac{d}{dt}\left(\|\mathbf{u}_m\|^2_{L^2} + \|\nabla \widehat{\mathbf{d}}_m\|^2_{L^2}\right) + C\left(\|\mathbf{u}_m\|^2_{L^2} + \|\nabla \widehat{\mathbf{d}}_m\|^2_{L^2}\right) \leq C^\star\left(\|\mathbf{f}(\mathbf{d}_m)\|^2_{L^2} + \|\partial_t \widetilde{\mathbf{d}}_m\|^2_{L^2}\right).$$

Multiplying by e^{Ct} and integrating in $[0,T]$:

$$\begin{aligned} e^{CT}\left(\|\mathbf{u}_m(T)\|^2_{L^2} + \|\nabla \widehat{\mathbf{d}}_m(T)\|^2_{L^2}\right) \;&\leq\; \|\mathbf{u}_m(0)\|^2_{L^2} + \|\nabla \widehat{\mathbf{d}}_m(0)\|^2_{L^2} \\ &+\; C^\star \int_0^T e^{Ct}\left(\|\mathbf{f}(\mathbf{d}_m)\|^2_{L^2} + \|\partial_t \widetilde{\mathbf{d}}_m\|^2_{L^2}\right). \end{aligned} \tag{15}$$

Given $(\mathbf{u}^m_0, \mathbf{d}^m_0) \in V^m \times W^m$, we define the map

$$\begin{aligned} L^m : [0,T] \;&\mapsto\; \mathbb{R}^m \times \mathbb{R}^m \\ t \;&\mapsto\; (\xi_{1m}(t), ..., \xi_{mm}(t), \zeta_{1m}(t), ..., \zeta_{mm}(t)) \end{aligned}$$

where $(\xi_{1m}(t), ..., \xi_{mm}(t))$ and $(\zeta_{1m}(t), ..., \zeta_{mm}(t))$ are coefficients of $\mathbf{u}_m(t)$ and $\widehat{\mathbf{d}}_m(t)$ respect to V^m and W^m respectively, being $(\mathbf{u}_m(t), \widehat{\mathbf{d}}_m(t))$ the (unique) approximate solution corresponding to the initial data $(\mathbf{u}^m_0, \mathbf{d}^m_0)$.

Now, varying the initial data, we are going to define a new map $\Phi^m : \mathbb{R}^m \times \mathbb{R}^m \mapsto \mathbb{R}^m \times \mathbb{R}^m$ as follows: given $L^m_0 \in \mathbb{R}^m \times \mathbb{R}^m$, then $\Phi^m(L^m_0) = L^m(T)$, where $L^m(t)$ is related to the solution of problem (9) with initial data $L^m_0 (= L^m(0))$.

By uniqueness of approximate solution of the initial-boundary problem, this map is well-defined. Moreover, using regularity of the corresponding ordinary differential system, this map is continuous.

In order to prove existence of fixed point of Φ^m, we will use Leray-Schauder theorem. Consequently, we have to prove that for all $\lambda \in [0,1]$, solutions $L^m_0(\lambda)$ of

$$L^m_0(\lambda) = \lambda \Phi^m(L^m_0(\lambda))$$

are uniformly bounded (independent of λ). Since $L_0^m(0) = \{0\}$, it is suffices to analyse $\lambda \in (0,1]$ and the equation

$$\frac{1}{\lambda} L_0^m(\lambda) = \Phi^m(L_0^m(\lambda)).$$

Considering the norm $\|L^m(t)\|_{\mathbf{R}^m \times \mathbf{R}^m} = \left(\|\mathbf{u}_m\|_{L^2}^2 + \|\nabla \widehat{\mathbf{d}}_m\|_{L^2}^2\right)^{1/2}$ in $\mathbb{R}^m \times \mathbb{R}^m$, inequality (15) yields

$$e^{CT} \|\frac{1}{\lambda} L_0^m(\lambda)\|_{\mathbf{R}^m \times \mathbf{R}^m}^2 \leq \|L_0^m(\lambda)\|_{\mathbf{R}^m \times \mathbf{R}^m}^2 + C(T),$$

hence, since $\lambda \in (0,1]$, one has

$$\|L_0^m(\lambda)\|_{\mathbf{R}^m \times \mathbf{R}^m}^2 \leq \frac{C(T)}{e^{CT} - 1}$$

which is a bound independent of λ (and m).

In particular, for each approximate reproductive solution $(\mathbf{u}_m, \mathbf{d}_m)$, their corresponding initial-end data is bounded in the $L^2 \times H^1$-norm, i.e $\|(\mathbf{u}_m, \widehat{\mathbf{d}}_m)(0)\|_{L^2 \times H^1} \leq C$ (independent of m). Therefore, the estimations obtained in Section 3.2 hold for the approximate reproductive solutions. □

3.5 Pass to the limit in reproductive approximate solutions

The pass to the limit in variational formulation (9) can be done as in [6], using estimations and compactness obtained in Section 3.2 (independents of m) in order to control nonlinear terms. Consequently, here we will only write the pass to the limit in reproductive conditions.

From estimations of $(\mathbf{d}_m)$ in $L^\infty(H^1)$ and $(\partial_t \mathbf{d}_m)$ in $L^2(L^{3/2})$ and using the triplet of spaces $H^2 \hookrightarrow L^2 \hookrightarrow L^{3/2}$, one has ([9]) that $(\mathbf{d}_m)$ is relatively compact in $C([0,T]; L^2)$, hence $\mathbf{d}_m(T) \to \mathbf{d}(T)$ and $\mathbf{d}_m(0) \to \mathbf{d}(0)$ in $L^2(\Omega)$. Since $\mathbf{d}_m(T) = \mathbf{d}_m(0)$, then $\mathbf{d}(T) = \mathbf{d}(0)$ in $L^2(\Omega)$. Moreover, it is easy to see that $\mathbf{d} \in C_w([0,T]; H^1)$ (i.e. $\mathbf{d}$ is continuous from $[0,T]$ onto H^1, respect to the weak topology in H^1), therefore $\mathbf{d}(T) = \mathbf{d}(0)$ in $H^1(\Omega)$. The argument for $\mathbf{u}$ is similar.

Consequently, we have found a weak reproductive solution of problem (1).

4 Relation between reproductive and periodic solutions

4.1 Any unique reproductive solution is periodic.

In the $2D$ case, uniqueness of weak solution for the initial-boundary problem (associated to $\mathbf{u}(0)$ and $\mathbf{d}(0)$) can be showed.

Consequently, given a reproductive solution $(\mathbf{u},\mathbf{d})$ associated to $\mathbf{u}(0)=\mathbf{u}(T):=\mathbf{u}_0$ and $\mathbf{d}(0)=\mathbf{d}(T):=\mathbf{d}_0$, then $(\mathbf{u},\mathbf{d})$ is the (unique) solution of the initial-boundary problem associated to the initial data $(\mathbf{u}_0,\mathbf{d}_0)$, which is defined in all time $t\in(0,\infty)$. Therefore, this solution is T-periodic, because for instance in $(T,2T)$ must be equal to the reproductive solution such that $\mathbf{u}(T)=\mathbf{u}(2T)=\mathbf{u}_0$ and $\mathbf{d}(T)=\mathbf{d}(2T)=\mathbf{d}_0$, etc.

4.2 Regularity of reproductive solutions

In the $2D$ case and under more regularity conditions on boundary data $\mathbf{h}$ and on the domain Ω (and therefore on the lift function $\widetilde{\mathbf{d}}$), it will be possible to find regular solution of problem (1)–(3). Consequently, taking into account previous subsection, we will have existence (and uniqueness) of regular time periodic solution.

More concretely, assuming Ω regular enough, $\widetilde{\mathbf{d}}\in L^\infty(H^3)\cap L^2(H^4)$ such that $\partial_t\widetilde{\mathbf{d}}\in L^\infty(H^1)\cap L^2(H^2)$, $\partial_{tt}\widetilde{\mathbf{d}}\in L^2(L^2)$ and initial data $\mathbf{u}_0\in H^2$ and $\mathbf{d}_0\in H^3$, then the reproductive solution $(\mathbf{u},p,\mathbf{d})$ verifies:

$$(\mathbf{u},p,\mathbf{d})\in L^\infty(H^2\times H^1\times H^3)\cap L^2(H^3\times H^2\times H^4),$$
$$(\partial_t\mathbf{u},\partial_t\mathbf{d})\in L^\infty(H\times H^1)\cap L^2(V\times H^2).$$

Indeed, if we derive respect to the time equations in the approximate problem (9) and consider $\partial_t\mathbf{u}_m$ and $-\partial_t\Delta\widehat{\mathbf{d}}_m$ as test functions respectively, we have

$$\begin{aligned}\frac{1}{2}\ \frac{d}{dt}\ &\Big(\|\partial_t\mathbf{u}_m\|_{L^2}^2+\|\nabla\partial_t\widehat{\mathbf{d}}_m\|_{L^2}^2\Big)+\|\nabla\partial_t\mathbf{u}_m\|_{L^2}^2+\|\Delta\partial_t\widehat{\mathbf{d}}_m\|_{L^2}^2\\ =\ &-((\partial_t\mathbf{u}_m\cdot\nabla)\mathbf{u}_m,\partial_t\mathbf{u}_m)-(\nabla\partial_t\mathbf{d}_m\Delta\widehat{\mathbf{d}}_m,\partial_t\mathbf{u}_m)\\ &+((\mathbf{u}_m\cdot\nabla)\partial_t\mathbf{d}_m,\Delta\partial_t\widehat{\mathbf{d}}_m)+(\partial_t\mathbf{f}(\mathbf{d}_m),\Delta\partial_t\mathbf{d}_m)+(\partial_{tt}\widetilde{\mathbf{d}},\Delta\partial_t\widehat{\mathbf{d}}_m).\end{aligned}\tag{16}$$

Here, the equality $\Delta\partial_t\mathbf{d}_m=\Delta\partial_t\widehat{\mathbf{d}}_m$ has been crucial, in order to vanish the terms:

$$-(\nabla\mathbf{d}_m\Delta\partial_t\widehat{\mathbf{d}}_m,\partial_t\mathbf{u}_m)+((\partial_t\mathbf{u}_m\cdot\nabla)\mathbf{d}_m,\Delta\partial_t\widehat{\mathbf{d}}_m)=0.$$

Moreover, since $\mathbf{d}_m=\widehat{\mathbf{d}}_m+\widetilde{\mathbf{d}}$ the following decomposition has been done:

$$-(\partial_{tt}\mathbf{d}_m,\Delta\partial_t\widehat{\mathbf{d}}_m)=\frac{1}{2}\frac{d}{dt}\|\nabla\partial_t\widehat{\mathbf{d}}_m\|_{L^2}^2-(\partial_{tt}\widetilde{\mathbf{d}},\Delta\partial_t\widehat{\mathbf{d}}_m).$$

In order to bound the right hand side of (16) we can follows [6] (although now some new terms related to $\widetilde{\mathbf{d}}$ appear). For instance, the first and second terms of the right hand-side of (16) are bounded (using Gagliardo-Niremberg's inequality) by

$$\begin{aligned}\|\nabla\mathbf{u}_m\|_{L^2}\|\partial_t\mathbf{u}_m\|_{L^4}^2 &\leq C\|\nabla\mathbf{u}_m\|_{L^2}\|\partial_t\mathbf{u}_m\|_{L^2}\|\nabla\partial_t\mathbf{u}_m\|_{L^2}\\ &\leq \bar\varepsilon\|\nabla\partial_t\mathbf{u}_m\|_{L^2}^2+\frac{C}{\bar\varepsilon}\|\nabla\mathbf{u}_m\|_{L^2}^2\|\partial_t\mathbf{u}_m\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}&\|\Delta\widehat{\mathbf{d}}_m\|_{L^2}\|\nabla\partial_t\mathbf{d}_m\|_{L^4}\|\partial_t\mathbf{u}_m\|_{L^4}\\ &\leq C\|\Delta\widehat{\mathbf{d}}_m\|_{L^2}\|\nabla\partial_t\mathbf{d}_m\|_{L^2}^{1/2}\|\nabla\partial_t\mathbf{d}_m\|_{H^1}^{1/2}\|\nabla\partial_t\mathbf{u}_m\|_{L^2}^{1/2}\|\partial_t\mathbf{u}_m\|_{L^2}^{1/2}\\ &\leq \bar\varepsilon\Big(\|\nabla\partial_t\mathbf{u}_m\|_{L^2}^2+\|\nabla\partial_t\mathbf{d}_m\|_{H^1}^2\Big)+\frac{C}{\bar\varepsilon}\|\Delta\widehat{\mathbf{d}}_m\|_{L^2}^2\Big(\|\nabla\partial_t\widehat{\mathbf{d}}_m\|_{L^2}^2+\|\nabla\partial_t\widetilde{\mathbf{d}}\|_{L^2}^2+\|\partial_t\mathbf{u}_m\|_{L^2}^2\Big),\end{aligned}$$

hence $\|\nabla\partial_t\widetilde{\mathbf{d}}\|_{L^2}^2\in L^\infty(0,T)$ is necessary.

In order to bound third term of the right hand-side of (16), we have

$$\begin{aligned}&\|\mathbf{u}_m\|_{L^4}\|\nabla\partial_t\mathbf{d}_m\|_{L^4}\|\Delta\partial_t\widehat{\mathbf{d}}_m\|_{L^2}\\ &\leq \|\mathbf{u}_m\|_{L^2}^{1/2}\|\nabla\mathbf{u}_m\|_{L^2}^{1/2}\|\nabla\partial_t\mathbf{d}_m\|_{L^2}^{1/2}\|\nabla\partial_t\mathbf{d}_m\|_{H^1}^{1/2}\|\Delta\partial_t\widehat{\mathbf{d}}_m\|_{L^2}\\ &\leq \bar\varepsilon\|\Delta\partial_t\widehat{\mathbf{d}}_m\|_{L^2}^2+\frac{C}{\bar\varepsilon}\|\nabla\mathbf{u}_m\|_{L^2}\|\nabla\partial_t\mathbf{d}_m\|_{L^2}\|\nabla\partial_t\mathbf{d}_m\|_{H^1},\end{aligned}$$

hence $\|\nabla\partial_t\widetilde{\mathbf{d}}_m\|_{L^2}\in L^\infty(0,T)$ and $\|\nabla\partial_t\widetilde{\mathbf{d}}_m\|_{H^1}\in L^2(0,T)$ are necessary.

The fourth term can be bounded by

$$C\|\partial_t\mathbf{d}_m\|_{L^2}\|\Delta\partial_t\mathbf{d}_m\|_{L^2}\leq\bar\varepsilon\|\Delta\partial_t\mathbf{d}_m\|_{L^2}^2+\frac{C}{\bar\varepsilon}(\|\partial_t\widehat{\mathbf{d}}_m\|_{L^2}^2+\|\partial_t\widetilde{\mathbf{d}}_m\|_{L^2}^2)$$

hence hypothesis $\|\partial_t\widetilde{\mathbf{d}}_m\|_{L^2}^2\in L^1(0,T)$ is necessary. Finally,

$$(\partial_{tt}\widetilde{\mathbf{d}}_m,\Delta\partial_t\mathbf{d}_m)\leq\bar\varepsilon\|\Delta\partial_t\mathbf{d}_m\|_{L^2}^2+\frac{C}{\bar\varepsilon}\|\partial_{tt}\widetilde{\mathbf{d}}_m\|_{L^2}^2$$

hence $\|\partial_{tt}\widetilde{\mathbf{d}}_m\|_{L^2}^2\in L^1(0,T)$ is necessary.

Considering all previous estimations, the Gronwall lemma and hypotheses of regularity on initial data $\mathbf{u}_0\in H^2$ and $\mathbf{d}_0\in H^3$ (in particular $(\partial_t\mathbf{u}_m(0),\partial_t\widehat{\mathbf{d}}_m(0))$ is bounded in $L^2(0,T)\times H^1(0,T)$), we arrive at

$$\partial_t\mathbf{u}_m \text{ is uniformly bounded in } L^\infty(H)\cap L^2(V)$$

and

$$\partial_t\widehat{\mathbf{d}}_m \text{ is uniformly bounded in } L^\infty(H_0^1)\cap L^2(H^2)$$

In particular

$$\partial_t \mathbf{d}_m \text{ is uniformly bounded in } L^\infty(H^1) \cap L^2(H^2).$$

All these new estimates, implying the corresponding regularity on the limit functions $(\partial_t \mathbf{u}, \partial_t \mathbf{d})$, which yields, using regularity for Stokes problem and for Poisson problem verified for $(\mathbf{u}, p)$ and $\mathbf{d}$ respectively (passing the others terms to the second hand side) jointly with a "bootstrap" technique, to the following regularity in space:

$$(\mathbf{u}, p, \widehat{\mathbf{d}}) \in L^\infty(H^2 \times H^1 \times H^3) \cap L^2(H^3 \times H^2 \times H^4).$$

Finally, since $\widetilde{\mathbf{d}} \in L^\infty(H^3) \cap L^2(H^4)$, we obtain the required regularity.

Remark. In the $3D$ case, it is not clear how regularity of reproductive solution can be proved, because global in time regularity cannot be expected, excepting small initial data or big viscosity ([6]). But, these latter results are not showed using previous argument done for the $2D$ case (i.e. deriving system respect to t and taking $\partial_t \mathbf{u}$ and $-\partial_t \Delta \mathbf{d}$ as tests functions), which is possible for the Galerkin approximation. By the contrary, in the argument used in [6] is important to consider the limit system verified for d (doing for instance a semi-Galerkin approximation). Nevertheless, it is not easy to find reproductive solution with a semi-Galerkin approximation. Therefore, the problem of the regularity for reproductive solutions in $3D$ domains remains open.

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