HOMOTOPIES FOR THE GENERALIZED BAR COMPLEX ASSOCIATED TO CERTAIN 3-ROWED WEYL MODULES

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Abstract. In [13], Buchsbaum and Rota presented a generalized bar complex associated to certain 3-rowed Weyl modules and proved that this complex is in fact a resolution via an induction on the number of overlaps between the second and third rows and a fundamental exact sequence ([1]). In this paper we study the structure of this resolution by constructing a splitting contracting homotopy for the complexes corresponding to certain shapes. We also use the method of proof to give a basis for the syzygies that is more structured than the basis given in [20]

1. Introduction

This paper is a contribution to the program of understanding resolutions of Weyl modules associated to general shapes, via Letter-Place methods as carried out by Buchsbaum and Rota ([12, 13, 14]).

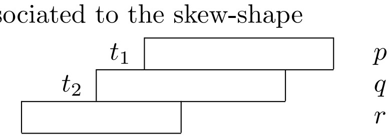
Concretely, we construct an splitting contracting homotopy for the Buschbaum-Rota resolution of 3-rowed Weyl modules for certain skew-shapes. The construction of homotopies is a basic tool in understanding the structure of these resolutions; however they have been constructed only in very special cases: the two rowed case ([12]), some steps for partitions ([14], page 176), hooks ([10]) and skew hooks ([11]).

Let us describe the content of this paper in a more detailed way:

The study of explicit resolutions of Schur and Weyl modules is rooted in the construction of explicit resolutions of determinantal ideals, and the closely related representation theory of $GL(n)$, in a characteristic-free setting (see [3, 4], and the more expository [7, 8, 9]).

In [1], Akin and Buchsbaum constructed a resolution of two-rowed Schur and Weyl modules (the latter called coSchur modules in that paper) using an arithmetic Koszul complex and the exactness of certain sequences of skew-shapes. However, this method becomes too unwieldy when dealing with the many-rowed case. Subsequently, it was realized that the language of Letter-Place could greatly simplify the construction and analysis of resolutions of Weyl modules. In [12], [13], Buchsbaum and Rota used Letter-Place methods and a generalization of the bar resolution to describe resolutions of certain Weyl modules. For two-rowed Weyl modules, they give a complex (the differential bar complex) and construct a splitting contracting homotopy which shows that the given complex is a resolution and allows them to write a basis for the syzygies.

They also consider the following three-rowed case: Weyl modules $K_{\lambda/\mu}$ (where $\lambda/\mu = (p + t_1 + t_2, q + t_2, r)/(t_1 + t_2, t_2, 0)$), associated to the skew-shape



where the number of triple overlaps is at most 1, i.e., $r - t_1 - t_2 \leq 1$. In this case a projective resolution over the Schur algebra

$$\cdots \longrightarrow P_k \xrightarrow{d_k} \cdots \longrightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow K_{\lambda/\mu}$$

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is constructed (see theorem 2 of [13]). The Letter-Place method describes the modules P_i as the free modules generated by a basis of bistandard bitableaux and the boundary maps are explicitly given in terms of polarization operators. All of the terms of these resolutions are direct sums of tensor products of divided powers, and therefore are projective resolutions over the appropiate Schur algebra. A very thorough survey article about this and related topics is [14].

The proof of the fact that the differential bar complex is a resolution in the 3-rowed case depends esentially on a fundamental exact sequence considered in [1] ; this sequence furnishes an induction which works in the case of at most one triple overlap.

In [20], the author constructed and described a basis for the syzygies associated to the resolution the aforementioned 3-rowed Weyl modules satisfying the additional condition $q-p \geq s-t_2-1$ ([21]) where s is the number of overlaps between the second and third rows.

As it is said in that paper, in addition to the intrinsic combinatorial and invariant theoretic interest of such basis, the author thinks of this basis esentially as a stepping stone for the construction of homotopies which lead to resolutions of Weyl modules; but in [20] the author was unable to construct the explicit homotopy. This homotopy is construted in the present paper for the condition $q-p \geq s-1$ (this condition implies at most one triple overlap in the case of skew-shapes)

The techniques used in the construction of basis for the syzygies in [20] is used in a fundamental way for guiding the construction of the homotopy; thus let us describe briefly the techniques of [20]:

This basis is constructed as follows: the canonical basis of bistandard bitableaux of each module P_i in the complex is divided in two complementary subsets, the "essential elements" and the "non-essential elements", so that $P_i = E_{i-1} \oplus N_i$, $P_i \supset E_{i-1} = span(\text{essential elements})$, $P_i \supset N_i = span(\text{non-essential elements})$ in P_i. This partition of the basis satisifes:

- **Completeness condition:** Given a basis element $T_\alpha \in \mathcal{N}_{i+1}$, there exists an explicit $M_\alpha \in E_i$ such that $d_{i+1}(T_\alpha) = d_{i+1}(M_\alpha)$
- **Rank condition:** The submodules E_i satisfy $rank(E_i) = rank(Z_i)$.

Thus, the paritition in to essential and non-essential elements is constructed in such a way that d_i restricted to the span of the essential elements is an isomorphism into its image; thus it also happens that $P_i = E_{i-1} \oplus Z_i$ where $Z_i \subset P_i$ is the i^{th} syzygy, and $d_{i+1} : E_i \to Z_i$ is an isomorphism. The basis for the syzygies is then given by $d_i(\varepsilon)$, ε essential.

Let us remark that the completeness condition in [20] does *not* use the fact that the complex is a resolution; the essential elements M that satisfy $d_i(M) = d_i(N)$ for non-essential elements N are found by hand computation.

The basic construction of this paper is that a stronger form of the completeness condition provides us with an splitting contracting homotopy and dispenses with the rank condition (which does use that the complex is a resolution). The stronger form is as follows:

- **Strong completeness condition:** Given a basis element $T_\alpha \in \mathcal{N}_{i+1}$, there exists an explicit $C \in E_{i+1}$ such that $d_{i+2}(C) = T_\alpha - M'_\alpha$, where $M'_\alpha \in E_i$.

Taking $M_\alpha = M'_\alpha$ in the strong completeness theorem gives us the completeness theorem given in [20]. It is interesting to note that the explicit M' produced by the strong completeness theorem and the M provided in [20] do not look equal; the combinatorial relation between the two completeness concepts is quite interesting and is analyzed in [22].

The construction of the homotopy comes from the following idea: define each homotopy $s_i : P_i \to P_{i+1}$ to be zero on E_{i-1}, and on Z_i is the inverse of the isomorphism $d_{i+1}|_{E_i} : E_i \to Z_i$. This is, of course, cheating, since, first, the construction of the basis in [20] assumes that the complex is a resolution already, (which one wants to avoid since, among other things, the homotopy can be used to prove in a more direct way that the complex is a resolution) and also one wants the homotopy to be written explicitly in terms of the canonical Letter-Place basis that spans each P_i. The result of the present paper comes from taking the neccessary steps so that this cheat becomes legal, that is, we do the neccessary constructions in order to be sure that one does *not* use that the complex is a resolution; and we also carry out the (non-trivial) task of writing the homotopy described above in the canonical basis. Both steps depend on the strong form of the completeness condition.

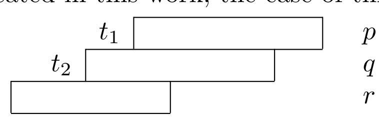
In fact, it seems to the author that the intermediate step of finding basis for the syzygies via the "essential" and "non-essential" elements in the basis at each level, and then go through the weak and strong completeness conditions for the complex, is an implementation of the heuristic "principle of parsimony" described by Buchsbaum and Rota in [14] for finding resolutions of Weyl modules.

Acknowledgments. The author would like to thank D. Buchsbaum for introducing her to the subject and to the specific problem considered in this paper.

2. Notation and Preliminaries

In this section we describe very briefly the generalized bar complex associated to the 3-rowed Weyl modules discussed here. We just describe its Letter-Place basis and the boundary maps; a more in depth description is given in section 3 of [20] for this case and for general Weyl modules, in [14].

Let us now study the case treated in this work, the case of three rows



where the number of triple overlaps is at most 1, i.e., $r - t_1 - t_2 \leq 1$. Here, from [13], we have a resolution

$$\cdots \longrightarrow P_k \xrightarrow{d_k} \cdots \longrightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow K_{\lambda/\mu}$$

modeled on a subquotient of the differential bar complex as follows: consider free bar module $\mathrm{Bar}(Super(L|\{a,b,c\}), A(Z_{ba}, Z_{cb}, Z_{ca}), \{x, y\})$, where $Super(L|\{a,b,c\})$ is the letter-place algebra with the places a, b, c we have been working with, x and y are two separators. The algebra $A(Z_{ba}, Z_{cb}, Z_{ca})$ is the associative noncommutative algebra generated by the variables Z_{ba}, Z_{cb}, Z_{ca}, subject to the commutation relations $Z_{ca}Z_{cb} = Z_{cb}Z_{ca}$ and $Z_{ca}Z_{ba} = Z_{ba}Z_{ca}$. The algebra $A(Z_{ba}, Z_{cb}, Z_{ca})$ acts on the module $Super(L|\{a,b,c\})$ by letting Z_{ba}, Z_{cb} and Z_{ca} act like the polarization operators D_{ba}, D_{cb} and D_{ca}.

Let us impose now the relations

$$Z_{cb}^{(\alpha)} Z_{ba}^{(\beta)} = \sum_{k=0}^{\alpha} Z_{ba}^{(\beta-\alpha+k)} x Z_{cb}^{(k)} Z_{ca}^{(\alpha-k)}$$
$$Z_{ca}x = xZ_{ca}$$
$$Z_{cb}x = xZ_{cb}$$

The module P_k is freely spanned by all elements of the form

$$Z_{cb}^{(\beta_1)} y ... y Z_{cb}^{(\beta_\lambda)} y Z_{ba}^{(\alpha_1)} x ... x Z_{ba}^{(\alpha_\mu)} x \left(\begin{array}{c|ccc} W & a^{(\pi)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where all the integers α_i and β_j are positive, $\beta_1 \geq t_2 + 1$ and $\alpha_1 > t_1 + \sum_j \beta_j$, $\pi = p + \sum_i \alpha_i$, $\sigma_1 + \sigma_2 = q + \sum_j \beta_j - \sum_i \alpha_i$, $\rho_1 + \rho_2 + \rho_3 = r - \sum_j \beta_j$ and $\lambda + \mu = k$.

A remark on notation. Sometimes, such elements -especially linear combinations of them- do not fit in a single line. In such cases what should be a single line splits into several lines in order to fit the printed page.

The boundary operator is $\partial_x + \partial_y$. Let us do an example to describe how this boundary operator works:

$$d_4\left(Z_{cb}^{(\beta_1)} y Z_{cb}^{(\beta_2)} y Z_{ba}^{(\alpha_1)} x Z_{ba}^{(\alpha_2)} x \left(\begin{array}{c|lll} W & a^{(p+\alpha_1+\alpha_2)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right) \right) =$$

$$\binom{\alpha_2+\sigma_1}{\alpha_2} Z_{cb}^{(\beta_1)} y Z_{cb}^{(\beta_2)} y Z_{ba}^{(\alpha_1)} x \left(\begin{array}{c|lll} W & a^{(p+\alpha_1)} & b^{(\alpha_2+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right) -$$

$$\binom{\alpha_1+\alpha_2}{\alpha_1} Z_{cb}^{(\beta_1)} y Z_{cb}^{(\beta_2)} y Z_{ba}^{(\alpha_1+\alpha_2)} x \left(\begin{array}{c|lll} W & a^{(p+\alpha_1+\alpha_2)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right) +$$

$$\sum_{k=0}^{\beta_2} \sum_{j=0}^{k} \binom{\beta_2-j}{\beta_2-k} \binom{\rho_1+\beta_2-j}{\rho_1} \binom{\rho_2+j}{\rho_2} Z_{cb}^{(\beta_1)} y Z_{ba}^{(\alpha_1-\beta_2+k)} x Z_{ba}^{(\alpha_2)} x$$

$$\left(\begin{array}{c|lll} W & a^{(p+\alpha_1+\alpha_2-\beta_2+k)} & b^{(\sigma_1-k+j)} & c^{(\rho_1+\beta_2-j)} \\ W' & b^{(\sigma_2-j)} & c^{(\rho_2+j)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right) -$$

$$\binom{\beta_1+\beta_2}{\beta_1} Z_{cb}^{(\beta_1+\beta_2)} y Z_{ba}^{(\alpha_1)} x Z_{ba}^{(\alpha_2)} x \left(\begin{array}{c|lll} W & a^{(p+\alpha_1+\alpha_2)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right)$$

It is not easy at all to prove that this complex is in fact a resolution; the known proof is done by using the fundamental exact sequence (which only works in the case of one triple overlap, see [13]). The present paper gives a conceptually much simpler proof along the lines of the 2-rowed cased, i.e. by constructing an explicit splitting contracting homotopy.

3. A basis for the syzygies

In this section we describe briefly the contents of [20] for the benefit of the reader; as such, we assume that the Buchsbaum-Rota complex is a resolution.

The i^{th} syzygy $Z_i \subset P_i$ is the kernel of d_i, and by exactness, this is the same as the image of d_{i+1}. Denote by $\mathcal{T}_{i+1} = \{T_1, ..., T_N\}$, $N = rank(P_{i+1})$, the basis of bistandard bitableaux that spans P_{i+1}. Since $Z_i = Im(d_{i+1})$, certainly the set $\{d_{i+1}(T_1), ..., d_{i+1}(T_N)\}$ spans Z_i; of course this set is not linearly independent.

The key element in the construction of the basis will be to discard from $\mathcal{T}_{i+1}$ the elements that are "redundant" for the boundary map. More precisely, the basis of bistandard bitableaux $\{T_1, ..., T_N\}$ spanning P_{i+1} will be divided in two disjoint sets: the *essential elements* $\mathcal{E}_i = \{T_1, ..., T_r\}$ and the *non-essential elements* $\mathcal{N}_{i+1} = \{T_{r+1}, ..., T_N\}$. Set $E_i = span(\mathcal{E}_i)$, $N_{i+1} = span(\mathcal{N}_{i+1})$. This partition $\mathcal{T}_{i+1} = \mathcal{E}_i \cup \mathcal{N}_{i+1}$ will satisfy two conditions:

- **Completeness condition:** Given a basis element $T_\alpha \in \mathcal{N}_{i+1}$, there exists $M_\alpha \in E_i$ such that $d_{i+1}(T_\alpha) = d_{i+1}(M_\alpha)$.
- **Rank condition:** The submodules E_i satisfy $rank(E_i) = rank(Z_i)$.

The completeness condition guarantees that $d_{i+1}|E_i : E_i \to Z_i$ is onto. Then the rank condition shows that $d_{i+1}|E_i : E_i \to Z_i$ is actually an isomorphism.

Therefore the desired basis for the syzygies will be given by:

$$\{d_{i+1}(T_1), ..., d_{i+1}(T_r)\}, T_\alpha \in \mathcal{E}_i \, .$$

Important Remark. The proof of the completeness condition does *not* use the fact that the complex is a resolution; the completeness condition is proven directly: given $T_\alpha \in \mathcal{N}_{i+1}$, an explicit $M_\alpha \in E_i$ satisfying $d_{i+1}(T_\alpha) = d_{i+1}(M_\alpha)$ is found. The stronger form of the completeness condition is similarly done by direct hand computation; we do not use that the complex is a resolution.

Let us now describe the essential elements. This is the first instance of a repeated expository thecnique in this paper, which has been succesful in the past ([19, 20]): describe the first few steps (for P_1, P_2, etc.) and then do the general case; the first modules being much easier to write, the ideas are much clearly exposed there.

3.1. **Construction of E_0.** A basis for P_1 is given by bistandard bitableaux of the form

$$Z_{ba}^{(\alpha)} x \left(\begin{array}{c|ccc} W & a^{(p+\alpha)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right),$$

where $\alpha > t_1$, $\sigma_1 + \sigma_2 = q - \alpha$ and $\rho_1 + \rho_2 + \rho_3 = r$, and

$$Z_{cb}^{(\beta)} y \left(\begin{array}{c|ccc} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right)$$

where $\beta \geq t_2 + 1$, $\sigma_1 + \sigma_2 = q + \beta$ and $\rho_1 + \rho_2 + \rho_3 = r - \beta$.

The essential elements $\mathcal{E}_0$ are the elements of the form

$$Z_{ba}^{(\alpha)} x \left(\begin{array}{c|cc} W & a^{(p+\alpha)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right),$$

where $\alpha \geq t_1 + 1$ and $\rho_1 + \rho_2 + \rho_3 = r$, plus elements of the form

$$Z_{cb}^{(\beta)} y \left(\begin{array}{c|cc} W & a^{(p)} & b^{(\sigma+\beta)} \\ W' & b^{(q-\sigma)} & c^{(\rho_1)} \\ W'' & c^{(\rho_2)} & \end{array} \right),$$

where $\beta \geq t_2 + 1, q - p \leq \sigma \leq t_1$ and $\rho_1 + \rho_2 = r - \beta$.

Thus, the essential elements are those double standard tableaux which

- Have a $Z_{ba}^{(\cdot)} x$ variable and no b in the first row, or
- Have a $Z_{cb}^{(\cdot)} y$ variable and no c in the first row, *plus* the condition $\sigma \leq t_1$.

3.2. **Construction of E_1.** A basis for P_2 is given by bistandard bitableaux of the form

$$Z_{cb}^{(\beta_1)} y \ldots y Z_{cb}^{(\beta_\lambda)} y Z_{ba}^{(\alpha_1)} x \ldots x Z_{ba}^{(\alpha_\mu)} x \left(\begin{array}{c|ccc} W & a^{(\pi)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(\sigma_2)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right)$$

where:

$$\beta_1 \geq t_2 + 1 \quad , \quad \alpha_1 > t_1 + \sum_j \beta_j$$
$$\pi = p + \sum_i \alpha_i \quad , \quad \sigma_1 + \sigma_2 = q + \sum_j \beta_j - \sum_i \alpha_i$$
$$\rho_1 + \rho_2 + \rho_3 = r - \sum_j \beta_j \quad \text{and} \quad \lambda + \mu = 2.$$

Since $\lambda + \mu = 2$, the variables in front of the bitableaux can only be $Z_{ba}^{(\alpha_1)} x Z_{ba}^{(\alpha_2)}$ or $Z_{cb}^{(\beta_1)} y Z_{cb}^{(\beta_2)}$ or $Z_{cb}^{(\beta)} y Z_{ba}^{(\alpha)}$. The essential elements $\mathcal{E}_1$ are the elements of the form

$$Z_{ba}^{(\alpha_1)} x Z_{ba}^{(\alpha_2)} x \left(\begin{array}{c|cc} W & a^{(p+\alpha_1+\alpha_2)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha_1-\alpha_2)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right)$$

where $\alpha_1 > t_1$, $\alpha_2 > 0$ and $\rho_1 + \rho_2 + \rho_3 = r$, plus elements of the form

$$Z_{cb}^{(\beta_1)} y Z_{cb}^{(\beta_2)} y \left(\begin{array}{c|cc} W & a^{(p)} & b^{(\sigma+\beta_1+\beta_2)} \\ W' & b^{(q-\sigma)} & c^{(\rho_1)} \\ W'' & c^{(\rho_2)} & \end{array} \right)$$

where $\beta_1 \geq t_2 + 1$, $\beta_2 > 0$, $\rho_1 + \rho_2 = r - \beta_1 - \beta_2$, $q - p \leq \sigma \leq t_1$, plus elements of the form

$$Z_{cb}^{(\beta)} y Z_{ba}^{(\alpha)} x \left(\begin{array}{c|cc} W & a^{(p+\alpha)} & c^{(\rho_1)} \\ W' & b^{(q+\beta-\alpha)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right)$$

where $\beta \geq t_2 + 1$, $\alpha > t_1 + \beta$ and $\rho_1 + \rho_2 + \rho_3 = r - \beta$. Thus, the essential elements in E_1 are those

- Having $Z_{ba}^{(\cdot)} x Z_{ba}^{(\cdot)} x$ or $Z_{cb}^{(\cdot)} y Z_{ba}^{(\cdot)} x$ variables and no b in the first row, or
- Having$Z_{cb}^{(\cdot)} y Z_{cb}^{(\cdot)} y$ variables, no c in the first row plus the condition $\sigma \leq t_1$.

And the nonessential elements $\mathcal{N}_2$ are given by the complement of $\mathcal{E}_1$; that is, a basis element T is non-essential if:

- T has $Z_{ba}^{(\cdot)} x Z_{ba}^{(\cdot)} x$ or $Z_{cb}^{(\cdot)} y Z_{ba}^{(\cdot)} x$ variables and b appears in the first row, or
- T has $Z_{cb}^{(\cdot)} y Z_{cb}^{(\cdot)} y$ variables, and $\sigma > t_1$; there might or might not be a c in the first row, or
- T has $Z_{cb}^{(\cdot)} y Z_{cb}^{(\cdot)} y$ variables, $\sigma \leq t_1$; and there is a c in the first row.

3.3. **The general case.** In general, $\mathcal{E}_i$ is the set formed by the basis elements of the form

$$Z_{ba}^{(\alpha_1)} x Z_{ba}^{(\alpha_2)} x \ldots x Z_{ba}^{(\alpha_{i+1})} x \left(\begin{array}{c|cc} W & a^{(p+|\alpha|)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right)$$

where $\alpha_1 > t_1$, $\alpha_j > 0$ for $j = 2, 3, \ldots i+1$, $\rho_1 + \rho_2 + \rho_3 = r$, $|\alpha| = \sum_{k=1}^{i+1} \alpha_k$; plus elements of the form

$$Z_{cb}^{(\beta_1)} y Z_{cb}^{(\beta_2)} y \ldots y Z_{cb}^{(\beta_{i+1})} y \left(\begin{array}{c|cc} W & a^{(p)} & b^{(\sigma+|\beta|)} \\ W' & b^{(q-\sigma)} & c^{(\rho_1)} \\ W'' & c^{(\rho_2)} & \end{array} \right)$$

where $\beta_1 \geq t_2 + 1$, $\beta_j > 0 for j = 2, 3, \ldots, i+1$, $|\beta| = \sum_{k=1}^{i+1} \beta_k$, $\rho_1 + \rho_2 = r - |\beta|$, $q - p \leq \sigma \leq t_1$; plus elements of the form

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\eta)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\tau)}x\left(\begin{array}{c|cc} W & a^{(p+|\alpha|)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

where $\eta+\tau=i+1$, $|\beta|=\sum_{j=1}^{\eta}\beta_j$, $|\alpha|=\sum_{j=1}^{\tau}\alpha_j$, $\beta_1\geq t_2+1$, $\beta_j>0 for j=2,...,\eta$, $\alpha_1>t_1+|\beta|$, $\alpha_j>0 for j=2,...,\tau$ and $\rho_1+\rho_2+\rho_3=r-|\beta|$.

In a similar fashion as in the previous cases the way of defining of essential elements depends on the variable in front: the essential elements are those which

- Have at least one $Z_{ba}^{(\cdot)}$ variable in front, and no b in the first row.
- Have only $Z_{cb}^{(\cdot)}$ variables in front, no c in the first row, and $\sigma\leq t_1$.

And $\mathcal{N}_i$ is the complement of the $\mathcal{E}_{i-1}$ in the canonical basis of P_i, that is, non-essential elements are those which

- Have at least one $Z_{ba}^{(\cdot)}$ variable, and b appears in the first row.
- Have only $Z_{cb}^{(\cdot)}$ variables, and either c appears in the first row, or $\sigma\geq t_1$.

4. Construction of the homotopy

The general strategy is described as follows:

Let us assume for a moment that the Buchsbaum-Rota differential bar complex is a resolution. Then by the completeness and rank theorems of [20], we know that $P_i=E_{i-1}\oplus N_i=E_{i-1}\oplus Z_i$, and $d_{i+1}|_{E_i}:E_i\to Z_i$ is an isomorphism. Then let us define $s_i:P_i\to P_{i+1}$ as follows: any element $p\in P_i$ is written uniquely in each decompostion as $p=\varepsilon+n=\varepsilon+z$, where $\varepsilon\in E_{i-1}, n\in N_i$ and $z\in Z_i$. Then we define $s_i(p)$ as the unique element $\tilde{\varepsilon}\in E_i$ such that $d_i(\tilde{\varepsilon})=z$. Note that in this map the span of the essential elements is mapped to zero; with such definition, it is trivial to show that $d_{i+1}s_i+s_{i-1}d_i=Id_{P_i}$.

Now, as said in the introduction, we have cheated in two ways: first this is not written in terms of the standard basis, and second, we assumed that we already have a resolution in order to use $P_i=E_{i-1}\oplus N_i=E_{i-1}\oplus Z_i$, and $d_{i+1}|_{E_i}:E_i\to Z_i$ is an isomorphism.

How do we straighten this cheat? First, writing it in terms of a basis: if T is a basis element of P_i, we will define $s_i(T)$ to be zero if T is essential; if T is not essential we will construct the strong form of the completeness condition: explicitly find an element ε, which is a linear combination of essential basis elements, such that that $d_{i+1}(\varepsilon)=T-$ linear combinations of essential elements. We then define $s_i(T)=\varepsilon$.

Then we need to show by hand that the maps s_i form a splitting contracting homotopy. This amounts to proving that for each basis element $T\in P_i$, $(d_{i+1}s_i+s_{i-1}d_i)(T)=T$.

Note that if T is essential, then we have to show that $s_{i-1}d_i(T)=T$ since $s_i(T)=0$. If T is non-essential, then the strong completeness theorem implies that there is nothing to prove. Indeed, if T is non-essential and $s_i(T)=\varepsilon$ is such that $d_{i+1}(\varepsilon)=T-M$, where M is a linear combination of essential elements, we have

$$d_{i+1}s_i(T)+s_{i-1}d_i(T)=d_{i+1}(\varepsilon)+s_{i-1}d_i(M)\,,$$

since $d_i(M)=d_i(T)$ because $d^2=0$. But if we prove the homotopy relation for essential elements, $s_{i-1}d_i(M)=M$, and by definition we have $d_{i+1}(\varepsilon)=T-M$. Thus

$$d_{i+1}s_i(T)+s_{i-1}d_i(T)=T-M+M=T\,,$$

which shows that we only need to prove the homotopy equation $s_{i-1}d_i(T)=T$ for essential basis elements T. By construction the homotopy is splitting.

We carry out this program in 4.1, 4.2 and 4.3; again, we give the maps $s_i:P_i\to P_{i+1}$ in a pedagogical sequence, first the s_0 and s_1 and then the general case.

Remark. In the previous work [20], given a non-essential element T_α, a linear combination $\tilde{M}_\alpha$ of essential elementes such that $d(T_\alpha) = d(\tilde{M}_\alpha)$ is found, and thus there is (in principle, assuming that the Buchsbaum-Rota complex is a resolution) a linear combination ϵ of essential elements such that $d(\epsilon) = T_\alpha - \tilde{M}_\alpha$. Again assuming that this complex is a resolution, this ϵ has to be the same as the explicit ε constructed above, and thus $\tilde{M}_\alpha = M_\alpha$. But $\tilde{M}_\alpha$ and M_α do not look the same at all; it takes a lot of combinatorial identities to prove that they are equal. But proving these identities makes $d(T_\alpha - M_\alpha) = 0$ independent of $d^2 = 0$, therefore giving a more elementary and precise undersatanding of the Buchsbaum-Rota resolution. We prove these combinatorial equalities in [22], as part of comparing the two completeness theorems involved.

4.1. **Construction of s_0.** Before we construct of homotopy $s_0 : P_0 \to P_1$ we will construct the essential and non-essential elements of P_0, which were not given in [20] (since they were not needed).

A basis of P_0 is given by bistandard bitableaux of the form

$$\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $t_1 = \sigma_1 + 1 + \sigma$ and $\rho_1 + \rho_2 + \rho_3 = r$.

The non-essential elements $\mathcal{N}_0$ are the elements of the form

$$\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $\sigma_1 > t_1$ and $\rho_1 + \rho_2 + \rho_3 = r$, plus elements of the form

$$\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $q - p \leq \sigma_1 \leq t_1$, $\rho_1 \geq t_2 + 1$ and $\rho_1 + \rho_2 + \rho_3 = r$.

And the essential elements $\mathcal{E}_{-1}$ are given by the complement of $\mathcal{N}_0$. We call $E\_1 = span(\mathcal{E}_{-1})$ and $N_0 = span(\mathcal{N}_0)$.

Now we define $s_0 : P_0 \to P_1$ in the following way: Given a canonical basis elements T in P_0 we define $s_0(T)$ to be zero if T is essential and if T is non-essential we will have two cases corresponding to the different way in which an element can be non-essential.

Case 1.Non-essential elements with $\sigma_1 > t_1$ and c may or may not appear in the first row. If $\rho_1 + \rho_2 + \rho_3 = r$, we define

$$s_0\left(\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right) = Z_{ba}^{(\sigma_1)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

Case 2.Non-essential elements with $q - p \leq \sigma_1 \leq t_1$ and $\rho_1 \geq t_2 + 1$. If $t_1 = \sigma_1 + 1 + \sigma$ and $\rho_1 + \rho_2 + \rho_3 = r$, we define

$$s_0\left(\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right) = \sum_{k=0}^{\gamma} (-1)^k \binom{\rho_2 + k}{\rho_2} Z_{cb}^{(\rho_1 - k)} y \left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right) -$$

$$\sum_{l=1}^{\rho_1-\sigma-1}(-1)^{\gamma}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\gamma}Z_{ba}^{(t_1+l)}x\left(\begin{array}{c|ll} W & a^{(p+t_1+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

where γ is equal to $\sigma+1$ if $\rho_1-\sigma-1\geq t_2+1$ and γ is equal to ρ_1-t_2-1 if $\rho_1-\sigma-1<t_2+1$.

4.2. **Construction of s_1.** Given a canonical basis element T in P_1 we define $s_1(T)$ to be zero if T is essential and if T is non-essential we will have three cases corresponding to the different ways in which an element can be non-essential.

Case 1.Non-essential elements having $Z_{ba}^{(\cdot)}$ variable and b appearing in the first row. If $\alpha>t_1$, $\sigma_1>0$ and $\rho_1+\rho_2+\rho_3=r$, we have define

$$s_1\left(Z_{ba}^{(\alpha)}x\left(\begin{array}{c|lll} W & a^{(p+\alpha)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=Z_{ba}^{(\alpha)}xZ_{ba}^{(\sigma_1)}x\left(\begin{array}{c|ll} W & a^{(p+\alpha+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

Case 2.Non-essential elements having $Z_{cb}^{(\cdot)}$ variable, there may or may not be a c in the top row but $\sigma_1>t_1$. If $\beta\geq t_2+1$, $\rho_1\geq 0$ and $\rho_1+\rho_2+\rho_3=r-\beta$, we define

$$s_1\left(Z_{cb}^{(\beta)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=Z_{cb}^{(\beta)}yZ_{ba}^{(\sigma_1+\beta)}x\left(\begin{array}{c|ll} W & a^{(p+\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

Case 3.Non-essential elements having $Z_{cb}^{(\cdot)}$ variable and c appearing in the first row and $q-p\leq\sigma_1\leq t_1$. If $\beta\geq t_2+1$, $\rho_1>0$, $t_1=\sigma_1+1+\sigma$ and $\rho_1+\rho_2+\rho_3=r-\beta$, we define

$$s_1\left(Z_{cb}^{(\beta)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=\sum_{k=0}^{\sigma+1}(-1)^kZ_{cb}^{(\beta)}yZ_{cb}^{(\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

$$-\sum_{l=1}^{\rho_1-\sigma-1}(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}Z_{cb}^{(\beta)}yZ_{ba}^{(t_1+\beta+l)}x\left(\begin{array}{c|ll} W & a^{(p+t_1+\beta+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

Let us now show that the defined map is a homotopy. For the essential elements E of the canonical basis, we have to show that

$$s_0d_1(E)=E\,,$$

since we have defined s_1 to be zero on such elements. For that we consider the following cases:

Case 1.Essential elements having $Z_{ba}^{(\cdot)}$ variable and b doesn't appear in the first row. If $\alpha>t_1$ and $\rho_1+\rho_2+\rho_3=r$ then

$$s_0d_1(E)=s_0d_1\left(Z_{ba}^{(\alpha)}x\left(\begin{array}{c|ll} W & a^{(p+\alpha)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\right)=s_0\left(\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\alpha)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$

$$Z_{ba}^{(\alpha)}x\left(\begin{array}{c|ll} W & a^{(p+\alpha)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

Case 2.Essential elements having $Z_{cb}^{(\cdot)}$ variable and c doesn't appear in the first row. If $\beta \geq t_2+1$, $q-p \leq \sigma_1 \leq t_1$, $t_1 = \sigma_1 + 1 + \sigma$ and $\rho_1 + \rho_2 = r - \beta$ then

$$s_0 d_1(E) = s_0 d_1 \left(Z_{cb}^{(\beta)} y \left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_1)} \\ W'' & c^{(\rho_2)} & \end{array} \right) \right) =$$

$$s_0 \left(\sum_{i=0}^{\beta} \binom{\rho_1 + i}{\rho_1} \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+i)} & c^{(\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_1+i)} & \\ W'' & c^{(\rho_2)} & & \end{array} \right) \right) =$$

$$s_0 \left(\sum_{i=0}^{\alpha} \binom{\rho_1 + i}{\rho_1} \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+i)} & c^{(\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_1+i)} & \\ W'' & c^{(\rho_2)} & & \end{array} \right) + \sum_{i=\alpha+1}^{\sigma+1} \binom{\rho_1 + i}{\rho_1} \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+i)} & c^{(\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_1+i)} & \\ W'' & c^{(\rho_2)} & & \end{array} \right) \right.$$

$$\left. + \sum_{i=\sigma+2}^{\beta} \binom{\rho_1 + i}{\rho_1} \left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+i)} & c^{(\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_1+i)} & \\ W'' & c^{(\rho_2)} & & \end{array} \right) \right)$$

where α is such that $\beta - \alpha \geq t_2 + 1$ if $\beta - \sigma - 1 \geq t_2 + 1$ and $\beta - \alpha = t_2 + 1$ if $\beta - \sigma - 1 < t_2 + 1$. Hence, we have to consider two cases.

Case 2.1.If $\beta - \sigma - 1 \geq t_2 + 1$ then

$$s_0 d_1(E) = s_0 d_1 \left(Z_{cb}^{(\beta)} y \left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_1)} \\ W'' & c^{(\rho_2)} & \end{array} \right) \right) =$$

$$\sum_{i=0}^{\sigma+1} \binom{\rho_1 + i}{\rho_1} s_0 \left(\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+i)} & c^{(\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_1+i)} & \\ W'' & c^{(\rho_2)} & & \end{array} \right) \right) +$$

$$\sum_{i=\sigma+2}^{\beta} \binom{\rho_1 + i}{\rho_1} s_0 \left(\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+i)} & c^{(\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_1+i)} & \\ W'' & c^{(\rho_2)} & & \end{array} \right) \right) =$$

$$\sum_{i=0}^{\sigma+1} \sum_{k=0}^{\sigma+1-i} (-1)^k \binom{\rho_1 + i}{\rho_1} \binom{\rho_1 + i + k}{\rho_1 + i} Z_{cb}^{(\beta-i-k)} y \left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta)} \\ W' & b^{(q-\sigma_1-i-k)} & c^{(\rho_1+i+k)} \\ W'' & c^{(\rho_2)} & \end{array} \right) -$$

$$\sum_{i=0}^{\sigma+1} \sum_{l=1}^{\beta-1-\sigma} (-1)^{\sigma+1-i} \binom{\rho_1 + 1 + \sigma + l}{\rho_1 + i} \binom{\rho_1 + i}{\rho_1} \binom{\sigma - i + l}{\sigma - i + 1} Z_{ba}^{(t_1+l)} x \left(\begin{array}{c|ll} W & a^{(p+t_1+l)} & c^{(\beta-1-\sigma-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_1+1+\sigma+l)} \\ W'' & c^{(\rho_2)} & \end{array} \right) +$$

$$\sum_{i=\sigma+2}^{\beta} \binom{\rho_1 + i}{\rho_1} Z_{ba}^{(\sigma_1+i)} x \left(\begin{array}{c|ll} W & a^{(p+\sigma+i)} & c^{(\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_1+i)} \\ W'' & c^{(\rho_2)} & \end{array} \right) = A - B + C$$

Claim: $A = E$ and $C = B$.

Let's do first $A = E$,

.-If $k + i = t$ and $0 \leq t \leq \sigma + 1$ then

$$\sum_{k=0}^{t}(-1)^k\binom{\rho_1+t}{\rho_1+i}\binom{\rho+i}{\rho_1} = \binom{\rho_1+t}{\rho_1}\sum_{k=0}^{t}(-1)^k\binom{t}{i} = \binom{\rho_1+t}{\rho_1}\sum_{k=0}^{t}(-1)^k\binom{t}{k}$$

which is equal to 1 if $t=0$ and 0 if t is different to 0.

Now, we will prove that $C=B$. For each l, $1\le l\le \beta-1-\sigma$, we have that

$$\sum_{i=0}^{\sigma+1}(-1)^{\sigma+1-i}\binom{\rho_1+1+\sigma+l}{\rho_1+i}\binom{\rho_1+i}{\rho_1}\binom{\sigma-i+l}{\sigma-i+1} = \binom{\rho_1+1+\sigma+l}{\rho_1}\sum_{i=0}^{\sigma+1}(-1)^{\sigma+1-i}\binom{1+\sigma+l}{i}\binom{\sigma-i+l}{\sigma-i+1}$$

Set $\sigma+1-i=j$

$$\binom{\rho_1+1+\sigma+l}{\rho_1}\sum_{j=0}^{\sigma+1}(-1)^j\binom{1+\sigma+l}{\sigma+1-j}\binom{l+j-1}{j} = \binom{\rho_1+1+\sigma+l}{\rho_1}\sum_{j=0}^{\sigma+1}\binom{1+\sigma+l}{\sigma+1-j}\binom{-l}{j} = \binom{\rho_1+1+\sigma+l}{\rho_1}\binom{1+\sigma}{\sigma+1} = \binom{\rho_1+1+\sigma+l}{\rho_1}$$

Case 2.2. If $\beta-\sigma-1<t_2+1$ then

$$s_0d_1(E) = s_0d_1\left(Z_{cb}^{(\beta)}y\left(\begin{array}{c|cc} W & a^{(p)} & b^{(\sigma_1+\beta)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_1)} \\ W'' & c^{(\rho_2)} & \end{array}\right)\right) =$$

$$\sum_{i=0}^{\alpha}\sum_{k=0}^{\beta-i-t_2-1}(-1)^k\binom{\rho_1+i}{\rho_1}\binom{\rho_1+i+k}{\rho_1+i}Z_{cb}^{(\beta-i-k)}y\left(\begin{array}{c|cc} W & a^{(p)} & b^{(\sigma_1+\beta)} \\ W' & b^{(q-\sigma_1-i-k)} & c^{(\rho_1+i+k)} \\ W'' & c^{(\rho_2)} & \end{array}\right) -$$

$$\sum_{i=0}^{\alpha}\sum_{l=1}^{\beta-\sigma-1}(-1)^{\beta-i-t_2-1}\binom{\rho_1+i}{\rho_1}\binom{\rho_1+\sigma+1+l}{\rho_1+i}\binom{\sigma-i+l}{\beta-i-t_2-1}Z_{ba}^{(t_1+l)}x\left(\begin{array}{c|cc} W & a^{(p+t_1+l)} & c^{(\beta-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_1+1+\sigma+l)} \\ W'' & c^{(\rho_2)} & \end{array}\right) +$$

$$\sum_{i=\sigma+2}^{\beta}\binom{\rho_1+i}{\rho_1}Z_{ba}^{(\sigma_1+i)}x\left(\begin{array}{c|cc} W & a^{(p+\sigma_1+i)} & c^{(\beta-i)} \\ W' & b^{(q-\sigma_1-i)} & c^{(\rho_1+i)} \\ W'' & c^{(\rho_2)} & \end{array}\right) = A-B+C$$

Claim: $A=E$ and $C=B$.

We will do first $A=E$,

.-If $k+i=t$ and $0\le t\le \beta-t_2-1$ then

$$\sum_{k=0}^{t}(-1)^k\binom{\rho_1+t}{\rho_1+i}\binom{\rho_1+i}{\rho_1} = \binom{\rho_1+t}{\rho_1}\sum_{k=0}^{t}(-1)^k\binom{t}{k}$$

which is equal to 1 if $t=0$ and 0 if t is different to 0.

Now, we will prove that $C = B$. For each l, $1 \leq l \leq \beta - \sigma - 1$, we have that

$$\sum_{i=0}^{\alpha}(-1)^{\beta-t_2-1+i}\binom{\rho_1+\sigma+1+l}{\rho_1+i}\binom{\rho_1+i}{\rho_1}\binom{\sigma-i+l}{\beta-t_2-1-i} =$$
$$\binom{\rho_1+\sigma+1+l}{\rho_1}\sum_{i=0}^{\alpha}(-1)^{\beta-t_2-1-i}\binom{\sigma+1+l}{i}\binom{\sigma-i+l}{\beta-t_2-1-i}$$

Set $\beta - t_2 - 1 - i = j$

$$\binom{\rho_1+\sigma+1+l}{\rho_1}\sum_{j=\beta-t_2-1-\alpha}^{\beta-t_2-1}(-1)^j\binom{\sigma+1+l}{\beta-t_2-1-j}\binom{l+\sigma-\beta+t_2+1+j}{j} =$$
$$\binom{\rho_1+\sigma+1+l}{\rho_1}\sum_{j=\beta-t_2-1-\alpha}^{\beta-t_2-1}\binom{\sigma+1+l}{\beta-t_2-1-j}\binom{\beta-t_2-1-l-\sigma-1}{j}$$

Set $n = j - \beta + t_2 + 1 + \alpha$

$$\binom{\rho_1+\sigma+1+l}{\rho_1}\sum_{n=0}^{\alpha}\binom{\sigma+1+l}{\alpha-n}\binom{\beta-t_2-1-l-\sigma-1}{\beta-t_2-1-\alpha+n} = \binom{\rho_1+\sigma+1+l}{\rho_1}\binom{\beta-t_2-1}{\beta-t_2-1} =$$
$$\binom{\rho_1+\sigma+1+l}{\rho_1}$$

4.3. **General construction of s_i.** Let us define the homotopy in general now. Given a canonical basis element T in P_i we define $s_i(T)$ to be zero if T is essential; if T is non-essential we will have four cases corresponding to the different ways in which an element can be non-essential.

Case 1.Non-essential elements having only $Z_{ba}^{(\cdot)}$ variables and b appearing in the first row. If $\sigma_1 > 0$, we define

$$s_i\left(Z_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_i)}x\left(\begin{array}{c|lll} W & a^{(p+|\alpha|)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$
$$Z_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_i)}xZ_{ba}^{(\sigma_1)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

Case 2.The mixed case with both $Z_{ba}^{(\cdot)}$ and $Z_{cb}^{(\cdot)}$ variables and b in the first row. If $\rho > 0$, $\lambda + \mu = i$, $\beta_1 \geq t_2 + 1$ and $\beta_j > 0$ for $j = 2, ..., \lambda$, $\alpha_1 > t_1 + |\beta|$ and $\alpha_j > 0$ for $j = 2, ..., \mu$, $|\beta| = \sum \beta_j$, $|\alpha| = \sum \alpha_j$, and $\rho_1 + \rho_2 + \rho_3 = r - |\beta|$, we define

$$s_i\left(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}x\left(\begin{array}{c|lll} W & a^{(p+|\alpha|)} & b^{(\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}xZ_{ba}^{(\rho)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|+\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

Case 3.Non-essential elements having $Z_{cb}^{(\cdot)}$ variables in front, and $\sigma_1 > t_1$; there might or might not be a c in the first row. If $\sigma_1 > t_1$, $\beta_1 \geq t_2+1$, $\beta_j > 0$ for $j = 2, ..., i$, and $\rho_1 \geq 0$. We define

$$s_i\left(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}yZ_{ba}^{(\sigma_1+|\beta|)}x\left(\begin{array}{c|ll} W & a^{(p+\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

Case 4.This is the case of non-essential elements N having no $Z_{ba}^{(\cdot)}$ variable in front and $q-p \leq \sigma_1 \leq t_1$ and c appears in the first row. If $\rho_1 > 0$, $\beta_1 \geq t_2+1$, $\beta_j > 0$ for $j = 2, ..., i$, $\rho_1+\rho_2+\rho_3 = r-|\beta|$ and $t_1 = \sigma_1+1+\sigma$, we define

$$s_i\left(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$

$$\sum_{k=0}^{\sigma+1}(-1)^k Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-k)}\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-$$

$$\sum_{l=1}^{\rho_1-\sigma-1}(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}yZ_{ba}^{(t_1+|\beta|+l)}x\left(\begin{array}{c|ll} W & a^{(p+t_1+|\beta|+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

Let us now show that the map defined is a homotopy. For the essential elements E of the canonical basis, we have to show that

$$s_{i-1}d_i(E) = E\,,$$

since we have defined s_i to be zero on such elements. We do the proof for each type of essential elements.

Case 1.Essential elements having only $Z_{ba}^{(\cdot)}$ variables and b doesn't appear in the first row. If $\alpha_1 > t_1$, $\alpha_j > 0$ for $j = 2, ..., i$, $|\alpha| = \sum_{k=1}^{i}$ and $\rho_1+\rho_2+\rho_3 = r$ then

$$s_{i-1}d_i(E) = s_{i-1}d_i\left(Z_{ba}^{(\alpha_1)}xZ_{ba}^{(\alpha_2)}x...xZ_{ba}^{(\alpha_i)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\right)=$$

$$s_{i-1}\left(Z_{ba}^{(\alpha_1)}xZ_{ba}^{(\alpha_2)}x...xZ_{ba}^{(\alpha_{i-1})}x\left(\begin{array}{c|lll} W & a^{(p+|\alpha|-\alpha_i)} & b^{(\alpha_i)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)+\right.$$

$$\left.\sum_{m=i-1}^{1}(-1)^{i-m}\binom{\alpha_m+\alpha_{m+1}}{\alpha_m}Z_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_{m-1})}xZ_{ba}^{(\alpha_m+\alpha_{m+1})}xZ_{ba}^{(\alpha_{m+2})}x...xZ_{ba}^{(\alpha_i)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\right)=$$

$$s_{i-1}\Bigg(Z_{ba}^{(\alpha_1)}xZ_{ba}^{(\alpha_2)}x...xZ_{ba}^{(\alpha_{i-1})}x\left(\begin{array}{c|lll} W & a^{(p+|\alpha|-\alpha_i)} & b^{(\alpha_i)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\Bigg)=$$

$$Z_{ba}^{(\alpha_1)}xZ_{ba}^{(\alpha_2)}x...xZ_{ba}^{(\alpha_i)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)=E$$

Case 2.The mixed case with both $Z_{ba}^{(\cdot)}$ and $Z_{cb}^{(\cdot)}$ variables and b doesn't appear in the first row. If $\lambda+\mu=i$, $\beta_1\geq t_2+1$, $\beta_j>0$ for $j=2,...,\lambda$, $\alpha_1>t_1+|\beta|$, $|\beta|=\sum_{j=1}^{\lambda}\beta_j$, $\alpha_j>0$ for $j=2,...,\mu$, $|\alpha|=\sum_{j=1}^{\mu}\alpha_j$ and $\rho_1+\rho_2+\rho_3=r-|\beta|$ then

$$s_{i-1}d_i(E)=s_{i-1}d_i\Bigg(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\Bigg)=$$

$$s_{i-1}\Bigg(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_{\mu-1})}x\left(\begin{array}{c|lll} W & a^{(p+|\alpha|-\alpha_\mu)} & b^{(\alpha_\mu)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)+$$

$$\sum_{k=\mu-1}^{1}(-1)^{\mu-k}\binom{\alpha_k+\alpha_{k+1}}{\alpha_k}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_k+\alpha_{k+1})}x...xZ_{ba}^{(\alpha_\mu)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)+$$

$$\sum_{l=\lambda-1}^{1}(-1)^{\mu-\lambda-l}\binom{\beta_l+\beta_{l+1}}{\beta_l}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_l+\beta_{l+1})}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)+$$

$$(-1)^{\mu}\sum_{k_1=0}^{\beta_\lambda}\sum_{i=2}^{\mu}\sum_{k_i=0}^{k_{i-1}}\sum_{m=0}^{k_\mu}M_{\lambda,\mu,k}\binom{\beta_\lambda+\rho_1-k_\mu}{\rho_1}\binom{\beta_\lambda+\rho_1-m}{k_\mu-m}\binom{\rho_2+m}{\rho_2}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{\lambda-1})}yZ_{ba}^{(\alpha_1-\beta_\lambda+k_1)}xZ_{ba}^{(\alpha_2-k_1+k_2)}x...xZ_{ba}^{(\alpha_{\mu-1})}-k_{\mu-2}+k_{\mu-1}xZ_{ba}^{(\alpha_\mu-k_{\mu-1}+m)}x$$

$$\left(\begin{array}{c|ll} W & a^{(p+|\alpha|-\beta_\lambda+m)} & c^{(\beta_\lambda+\rho_1-m)} \\ W' & b^{(q+|\beta|-|\alpha|-m)} & c^{(\rho_2+m)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\Bigg)$$

where

$$M_{\lambda,\mu,k}=\binom{\beta_\lambda-k_\mu}{\beta_\lambda-k_1,k_1-k_2,...,k_{\mu-1}-k_\mu}=\frac{(\beta_\lambda-k_\mu)!}{(\beta_\lambda-k_1)!(k_1-k_2)!...(k_{\mu-1}-k_\mu)!}$$

is the monomial coefficient.

Thus

$$s_{i-1}d_i\Bigg(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\Bigg)=$$

$$s_{i-1}\Bigg(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_{\mu-1})}x\left(\begin{array}{c|lll} W & a^{(p+|\alpha|-\alpha_\mu)} & b^{(\alpha_\mu)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\Bigg)=$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right) = E$$

Case 3.Essential elements having only $Z_{cb}^{(.)}$ variables and c doesn't appear in the first row. If $\beta_1 \geq t_2+1$, $\beta_j > 0$ for $j = 2,3,...,i$; $q-p \leq \sigma_1 \leq t_1$; $t_1 = \sigma_1+1+\sigma$; $|\beta| = \sum_{j=1}^{i}\beta_j$ and $\rho_1+\rho_2+\rho_3 = r-|\beta|$ then

$$s_{i-1}d_i(E) = s_{i-1}d_i\left(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_1)} \\ W'' & c^{(\rho_2)} & \end{array}\right)\right) = s_{i-1}\left(\sum_{k=0}^{\sigma+1}\binom{\rho_1+k}{\rho_1}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}y\right.$$

$$\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+k+|\beta|-\beta_i)} & c^{(\beta_i-k)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_1+k)} & \\ W'' & c^{(\rho_2)} & & \end{array}\right) + \sum_{k=\sigma+2}^{\beta_i}\binom{\rho_1+k}{\rho_1}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}y$$

$$\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+k+|\beta|-\beta_i)} & c^{(\beta_i-k)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_1+k)} & \\ W'' & c^{(\rho_2)} & & \end{array}\right) + \sum_{m=i-1}^{1}(-1)^{i-m}\binom{\beta_m+\beta_{m+1}}{\beta_m}$$

$$\left.Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_1)} \\ W'' & c^{(\rho_2)} & \end{array}\right)\right) = s_{i-1}\left(\sum_{k=0}^{\sigma+1}\binom{\rho_1+k}{\rho_1}\right.$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+k+|\beta|-\beta_i)} & c^{(\beta_i-k)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_1+k)} & \\ W'' & c^{(\rho_2)} & & \end{array}\right) +$$

$$\left.\sum_{k=\sigma+2}^{\beta_i}\binom{\rho_1+k}{\rho_1}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+k+|\beta|-\beta_i)} & c^{(\beta_i-k)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_1+k)} & \\ W'' & c^{(\rho_2)} & & \end{array}\right)\right) =$$

$$\sum_{k=0}^{\sigma+1}\sum_{j=0}^{\sigma+1-k}(-1)^j\binom{\rho_1+k}{\rho_1}\binom{\rho_1+k+j}{\rho_1+k}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{cb}^{(\beta_i-k-j)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} \\ W' & b^{(q-\sigma_1-k-j)} & c^{(\rho_1+k+j)} \\ W'' & c^{(\rho_2)} & \end{array}\right) -$$

$$\sum_{k=0}^{\sigma+1}\sum_{l=1}^{\beta_i-k-\sigma+k-1}(-1)^{\sigma+1-k}\binom{\rho_1+k}{\rho_1}\binom{\rho_1+k+\sigma+1-k+l}{\rho_1+k}\binom{\sigma-k+l}{\sigma-k+1}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{ba}^{(t_1+|\beta|-\beta_i+l)}x$$

$$\left(\begin{array}{c|ll} W & a^{(p+t_1+|\beta|-\beta_i+l)} & c^{(\beta_i-k-\sigma-1+k-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_1+k+\sigma-k+1+l)} \\ W'' & c^{(\rho_2)} & \end{array}\right) + \sum_{k=\sigma+2}^{\beta_i}\binom{\rho_1+k}{\rho_1}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{ba}^{(\sigma_1+k+|\beta|-\beta_i)}x$$

$$\left(\begin{array}{c|ll} W & a^{(p+\sigma_1+k+|\beta|-\beta_i)} & c^{(\beta_i-k)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_1+k)} \\ W'' & c^{(\rho_2)} & \end{array}\right) = A-B+C$$

Claim: $A = E$ and $C = B$.

Let's do first $A = E$,

.-If $k+j=t$ and $0 \leq t \leq \sigma+1$ then

$$\sum_{k=0}^{t}(-1)^{t-k}\binom{\rho_1+t}{\rho_1+k}\binom{\rho_1+k}{\rho_1} = (-1)^t\sum_{k=0}^{t}(-1)^k\binom{\rho_1+t}{\rho_1}\binom{t}{k} =$$

$$(-1)^t\binom{\rho_1+t}{\rho_1}\sum_{k=0}^{t}(-1)^k\binom{t}{k}$$

which is equal to 1 if $t=0$ and 0 if t is different to 0.

Now, we will prove that $C=B$. For each l, $1 \leq l \leq \beta_i-\sigma-1$, we have that

$$\sum_{k=0}^{\sigma+1}(-1)^{\sigma+1-k}\binom{\rho_1+\sigma+1+l}{\rho_1+k}\binom{\rho_1+k}{\rho_1}\binom{\sigma+l-k}{\sigma+1-k} =$$

$$\binom{\rho_1+\sigma+1+l}{\rho_1}\sum_{k=0}^{\sigma+1}(-1)^{\sigma+1-k}\binom{\sigma+1+l}{k}\binom{\sigma+l-k}{\sigma+1-k}$$

Set $\sigma+1-k=j$

$$\binom{\rho_1+\sigma+1+l}{\rho_1}\sum_{j=0}^{\sigma+1}(-1)^j\binom{\sigma+1+l}{\sigma+1-j}\binom{l+j-1}{j} = \binom{\rho_1+\sigma+1+l}{\rho_1}\sum_{j=0}^{\sigma+1}\binom{\sigma+1+l}{\sigma+1-j}\binom{-l}{j} =$$

$$\binom{\rho_1+\sigma+1+l}{\rho_1}\binom{\sigma+1}{\sigma+1} = \binom{\rho_1+\sigma+1+l}{\rho_1},$$

which finishes the proof.

5. Another basis for the syzygies

In [20], the basis for the syzygies was given as $d_i(\epsilon)$, ϵ essential. Here we give a basis that is, in the author's opinion, more easily indexable, and is given along the lines of the 2-rowed case explained in [10]: a basis of the syzygies is given by considering elements of a subset $\{x_{i_\alpha}\}$ of the canonical basis $\{x_i\}$ such that $\{s_i(x_{i_\alpha}\})$ is linearly independent. Then the set $d_{i+1}s_i(x_{i_\alpha})$ will be the basis for the syzygies.

In our case, the non-essential elements satisfy this condition; therefore a basis for the syzygies will be given by the $d_{i+1}s_i(N_\alpha)$, N_α non-essential.

This basis has the advantage of being indexed by the non-essential elements N_α themselves, indeed recall that $d_{i+1}s_i(N_\alpha)=N_\alpha-M_\alpha$, where M_α is a linear combination of essential elements. Thus the basis is given by applying a transformation of the form $Id+P$, where P sends essential elements to (say) zero and non-essential elements to essential elements.

Concretely, the basis for the syzygies is given by

5.1. **Basis of the syzygies in P_0:** For non-essential elements of the form

$$\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $\sigma_1 > t_1$ and $\rho_1+\rho_2+\rho_3 = r$, the corresponding basis of the syzygies is

$$d_1 s_0\Bigg(\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\Bigg) = d_1\Bigg(Z_{ba}^{(\sigma_1)}x\left(\begin{array}{c|ll} W & a^{(p+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\Bigg) =$$

$$\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

For non-essential elements of the form

$$\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $q-p \le \sigma_1 \le t_1$, $\rho_1 \ge t_2+1$ and $\rho_1+\rho_2+\rho_3 = r$, the corresponding basis of the syzygies is

$$d_1 s_0\Bigg(\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\Bigg) = d_1\Bigg(\sum_{k=0}^{\gamma}(-1)^k\binom{\rho_2+k}{\rho_2}Z_{cb}^{(\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right) -$$

$$\sum_{l=1}^{\rho_1-\sigma-1}(-1)^{\gamma}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\gamma}Z_{ba}^{(t_1+l)}x\left(\begin{array}{c|ll} W & a^{(p+t_1+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\Bigg) =$$

$$\sum_{k=0}^{\gamma}\sum_{j=0}^{\rho_1-k}(-1)^k\binom{\rho_2+k+j}{j}\binom{\rho_2+k}{\rho_2}\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+k+j)} & c^{(\rho_1-k-j)} \\ W' & b^{(q-\sigma_1-k-j)} & c^{(\rho_2+k+j)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right) -$$

$$\sum_{l=1}^{\rho_1-\sigma-1}(-1)^{\gamma}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\gamma}\left(\begin{array}{c|lll} W & a^{(p+t_1+l)} & b^{(t_1+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where γ is equal to $\sigma+1$ if $\rho_1-\sigma-1 \ge t_2+1$ and γ is equal to ρ_1-t_2-1 if $\rho_1-\sigma-1 < t_2+1$.

5.2. **Basis for the syzygies in P_1:** For non-esssential elements of the form

$$Z_{ba}^{(\alpha)}x\left(\begin{array}{c|lll} W & a^{(p+\alpha)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $\alpha > t_1$, $\sigma_1 > 0$ and $\rho_1+\rho_2+\rho_3 = r$, the corresponding basis of the syzygies is

$$d_2 s_1\Bigg(Z_{ba}^{(\alpha)}x\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\Bigg) =$$

$$d_2\left(Z_{ba}^{(\alpha)} x Z_{ba}^{(\sigma_1)} x \left(\begin{array}{c|cc} W & a^{(p+\alpha+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\right) = Z_{ba}^{(\alpha)} x \left(\begin{array}{c|ccc} W & a^{(p)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right) -$$
$$\binom{\alpha+\sigma_1}{\alpha} Z_{ba}^{(\alpha+\sigma_1)} x \left(\begin{array}{c|cc} W & a^{(p+\alpha+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-\alpha-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

For non-essential elements of the form

$$Z_{cb}^{(\beta)} y \left(\begin{array}{c|ccc} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $\sigma_1 \geq t_1$, $\beta \geq t_2+1$, $\rho_1 \geq 0$ and $\rho_1+\rho_2+\rho_3 = r-\beta$, the corresponding basis of the syzygies is

$$d_2 s_1\left(Z_{cb}^{(\beta)} y \left(\begin{array}{c|ccc} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right) =$$
$$d_2\left(Z_{cb}^{(\beta)} y Z_{ba}^{(\sigma_1+\beta)} x \left(\begin{array}{c|cc} W & a^{(p+\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\right) = Z_{cb}^{(\beta)} y \left(\begin{array}{c|ccc} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right) -$$
$$\sum_{j=0}^{\beta} \binom{\rho_1+\beta-j}{\beta-j}\binom{\rho_2+j}{\rho_2} Z_{ba}^{(\sigma_1+j)} x \left(\begin{array}{c|cc} W & a^{(p+\sigma_1+j)} & c^{(\rho_1+\beta-j)} \\ W' & b^{(q-\sigma_1-j)} & c^{(\rho_2+j)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

For non-essential elements of the form

$$Z_{cb}^{(\beta)} y \left(\begin{array}{c|ccc} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $q-p \leq \sigma_1 \leq t_1$, $\rho_1 > 0$ and $\rho_1 > \sigma+1$, $\beta \geq t_2+1$, $t_1 = \sigma_1+1+\sigma$, $\rho_1+\rho_2+\rho_3 = r-\beta$, the corresponding basis of the syzygies is

$$d_2 s_1\left(Z_{cb}^{(\beta)} y \left(\begin{array}{c|ccc} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right) =$$
$$d_2\left(\sum_{k=0}^{\sigma+1}(-1)^k \binom{\rho_2+k}{\rho_2} Z_{cb}^{(\beta)} y Z_{cb}^{(\rho_1-k)} y \left(\begin{array}{c|cc} W & a^{(p)} & b^{(\sigma_1+\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right) -\right.$$
$$\left.\sum_{l=1}^{\rho_1-\sigma-1} (-1)^{\sigma+1} \binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1} Z_{cb}^{(\beta)} y Z_{ba}^{(t_1+\beta+l)} x \left(\begin{array}{c|cc} W & a^{(p+t_1+\beta+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\right) =$$
$$\sum_{k=0}^{\sigma+1}\sum_{j=0}^{\rho_1-k} (-1)^k \binom{\rho_2+k}{\rho_2}\binom{\rho_2+k+j}{\rho_2+k} Z_{cb}^{(\beta)} y \left(\begin{array}{c|ccc} W & a^{(p)} & b^{(\sigma_1+\beta+k+j)} & c^{(\rho_1-k-j)} \\ W' & b^{(q-\sigma_1-k-j)} & c^{(\rho_2+k+j)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right) -$$
$$\sum_{k=0}^{\sigma+1}(-1)^k \binom{\rho_2+k}{\rho_2}\binom{\beta+\rho_1-k}{\beta} Z_{cb}^{(\beta+\rho_1-k)} y \left(\begin{array}{c|cc} W & a^{(p)} & b^{(\sigma_1+\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right) -$$

$$\sum_{l=1}^{\rho_1-\sigma-1}(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}Z_{cb}^{(\beta)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(t_1+\beta+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)+$$

$$\sum_{l=1}^{\rho_1-\sigma-1}\sum_{n=0}^{\beta}(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}\binom{\beta+\rho_1-\sigma-1-n-l}{\beta-n}\binom{\rho_2+\sigma+1+l+n}{\rho_2+\sigma+1+l}$$

$$Z_{ba}^{(t_1+l+n)}x\left(\begin{array}{c|ll} W & a^{(p+t_1+l+n)} & c^{(\beta+\rho_1-\sigma-1-n-l)} \\ W' & b^{(q-t_1-n)} & c^{(\rho_2+\sigma+1+n+l)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

For non-essential elements of the form

$$Z_{cb}^{(\beta)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $q-p\leq\sigma_1\leq t_1$, $\rho_1>0$ and $\rho_1\leq\sigma+1$, $\beta\geq t_2+1$, $t_1=\sigma_1+1+\sigma$, $\rho_1+\rho_2+\rho_3=r-\beta$, the corresponding basis of the syzygies is

$$d_2s_1\left(Z_{cb}^{(\beta)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$

$$d_2\left(\sum_{k=0}^{\rho_1-1}(-1)^k\binom{\rho_2+k}{\rho_2}Z_{cb}^{(\beta)}yZ_{cb}^{(\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\right)=$$

$$\sum_{k=0}^{\rho_1-1}\sum_{j=0}^{\rho_1-k}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\rho_2+k+j}{\rho_2+k}Z_{cb}^{(\beta)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+\beta+k+j)} & c^{(\rho_1-k-j)} \\ W' & b^{(q-\sigma_1-k-j)} & c^{(\rho_2+k+j)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)-$$

$$\sum_{k=0}^{\rho_1-1}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\beta+\rho_1-k}{\beta}Z_{cb}^{(\beta+\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+\beta+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

5.3. **Basis for the syzygies in general P_i:** For non-essential elements of the form

$$Z_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_i)}x\left(\begin{array}{c|lll} W & a^{(p+|\alpha|)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $\sigma_1>0$ and $\rho_1+\rho_2+\rho_3=r$, the corresponding basis of the syzygies is

$$d_{i+1}s_i\left(Z_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_i)}x\left(\begin{array}{c|lll} W & a^{(p+|\alpha|)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$

$$d_{i+1}\left(Z_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_i)}xZ_{ba}^{(\sigma_1)}x\left(\begin{array}{c|ll} W & a^{(p+|\alpha|+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\right)=$$

$$Z_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_i)}x\left(\begin{array}{c|lll} W & a^{(p+|\alpha|)} & b^{(\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)-$$

$$\binom{\alpha_i+\sigma_1}{\alpha_i} Z_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_i+\sigma_1)}x \left(\begin{array}{c|cc} W & a^{(p+|\alpha|+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right) -$$

$$\sum_{m=i-1}^{1} (-1)^{i-m} \binom{\alpha_m+\alpha_{m+1}}{\alpha_m} Z_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_{m-1})}xZ_{ba}^{(\alpha_m+\alpha_{m+1})}xZ_{ba}^{(\alpha_{m+2})}x...xZ_{ba}^{(\alpha_i)}xZ_{ba}^{(\sigma_1)}x$$

$$\left(\begin{array}{c|cc} W & a^{(p+|\alpha|+\sigma_1)} & c^{(\rho_1)} \\ W' & b^{(q-|\alpha|-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right)$$

For non-essential elements of the form

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}x \left(\begin{array}{c|ccc} W & a^{(p+|\alpha|)} & b^{(\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right)$$

where $\rho > 0$, $\lambda + \mu = i$, $\beta_1 \geq t_2 + 1$ and $\beta_j > 0$ for $j = 2, ..., \lambda$, $\alpha_1 > t_1 + |\beta|$ and $\alpha_j > 0$ for $j = 2, ..., \mu$, $|\beta| = \sum \beta_j$, $|\alpha| = \sum \alpha_j$, and $\rho_1 + \rho_2 + \rho_3 = r - |\beta|$, the corresponding basis of the syzygies is

$$d_{i+1}s_i\left(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}x \left(\begin{array}{c|ccc} W & a^{(p+|\alpha|)} & b^{(\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right) \right)$$

$$d_{i+1}\left(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}xZ_{ba}^{(\rho)}x \left(\begin{array}{c|cc} W & a^{(p+|\alpha|+\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right) \right) =$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}x \left(\begin{array}{c|ccc} W & a^{(p+|\alpha|)} & b^{(\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array} \right) -$$

$$\binom{\alpha_\mu+\rho}{\alpha_\mu} Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_{\mu-1})}xZ_{ba}^{(\alpha_\mu+\rho)}x \left(\begin{array}{c|cc} W & a^{(p+|\alpha|+\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right) -$$

$$\sum_{k=\mu-1}^{1} (-1)^{\mu-k} \binom{\alpha_k+\alpha_{k+1}}{\alpha_k} Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_k+\alpha_{k+1})}x...xZ_{ba}^{(\alpha_\mu)}xZ_{ba}^{(\rho)}x$$

$$\left(\begin{array}{c|cc} W & a^{(p+|\alpha|+\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right) -$$

$$\sum_{l=\lambda-1}^{1} (-1)^{\mu+\lambda-l} \binom{\beta_l+\beta_{l+1}}{\beta_l} Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_l+\beta_{l+1})}y...yZ_{cb}^{(\beta_\lambda)}yZ_{ba}^{(\alpha_1)}x...xZ_{ba}^{(\alpha_\mu)}xZ_{ba}^{(\rho)}x$$

$$\left(\begin{array}{c|cc} W & a^{(p+|\alpha|+\rho)} & c^{(\rho_1)} \\ W' & b^{(q+|\beta|-|\alpha|-\rho)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array} \right) -$$

$$(-1)^{\mu} \sum_{k_1=0}^{\beta_\lambda} \sum_{i=2}^{\mu} \sum_{k_i=0}^{k_{i-1}} \sum_{m=0}^{k_\mu} M_{\lambda,\mu,k} \binom{\beta_\lambda+\rho_1-k_\mu}{\rho_1} \binom{\beta_\lambda+\rho_1-m}{k_\mu-m} \binom{\rho_2+m}{\rho_2}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{\lambda-1})}yZ_{ba}^{(\alpha_1-\beta_\lambda+k_1)}xZ_{ba}^{(\alpha_2-k_1+k_2)}x...xZ_{ba}^{(\alpha_\mu-k_{\mu-1}+k_\mu)}xZ_{ba}^{(\sigma_1-k_\mu+m)}x$$

$$\left(\begin{array}{c|ll} W & a^{(p+|\alpha|-\beta_\lambda+\sigma_1+m)} & c^{(\beta_\lambda+\rho_1-m)} \\ W' & b^{(q+|\beta|-|\alpha|-\sigma_1-m)} & c^{(\rho_2+m)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

where

$$M_{\lambda,\mu,k} = \binom{\beta_\lambda - k_\mu}{\beta_\lambda - k_1, k_1 - k_2, ..., k_{\mu-1} - k_\mu} = \frac{(\beta_\lambda - k_\mu)!}{(\beta_\lambda - k_1)!(k_1 - k_2)!...(k_{\mu-1} - k_\mu)!}$$

is the monomial coefficient.

For non-essential elements of the form

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $\sigma_1 > t_1$, $\beta_1 \geq t_2 + 1$, $\beta_j > 0$ for $j = 2, ..., i$, $t_1 = \sigma_1 + 1 + \sigma$ and $\rho_1 \geq 0$, the corresponding basis of the syzygies is

$$d_{i+1}s_i\left(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$

$$d_{i+1}\left(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}yZ_{ba}^{(\sigma_1+|\beta|)}x\left(\begin{array}{c|ll} W & a^{(p+\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\right)=$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)-$$

$$\sum_{j=0}^{\beta_i}\binom{\beta_i+\rho_1-j}{\rho_1}\binom{\rho_2+j}{\rho_2}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{ba}^{(\sigma_1+|\beta|-\beta_i+j)}\left(\begin{array}{c|ll} W & a^{(p+\sigma_1+|\beta|-\beta_i+j)} & c^{(\beta_i+\rho_1-j)} \\ W' & b^{(q-\sigma_1-j)} & c^{(\rho_2+j)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-$$

$$\sum_{l=i-1}^{1}(-1)^{i-l}\binom{\beta_l+\beta_{l+1}}{\beta_l}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_l+\beta_{l+1})}y...yZ_{cb}^{(\beta_i)}yZ_{ba}^{(\sigma_1+|\beta|)}x\left(\begin{array}{c|ll} W & a^{(p+\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

For non-essential elements of the form

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $q - p \leq \sigma_1 \leq t_1$, $\rho_1 > 0$ and $\rho_1 > \sigma + 1$, $\beta_1 \geq t_2 + 1$, $\beta_j > 0$ for $j = 2, ..., i$, $t_1 = \sigma_1 + 1 + \sigma$ and $\rho_1 + \rho_2 + \rho_3 = r - |\beta|$, the corresponding basis for the syzygies is

$$d_{i+1}s_i\left(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$

$$d_{i+1}\left(\sum_{k=0}^{\sigma+1}(-1)^k\binom{\rho_2+k}{\rho}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-\right.$$

$$\sum_{l=1}^{\rho_1-\sigma-1}(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}yZ_{ba}^{(t_1+|\beta|+l)}x\left(\begin{array}{c|lll} W & a^{(p+t_1+|\beta|+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\Bigg)=$$

$$\sum_{k=0}^{\sigma+1}\sum_{j=0}^{\rho_1-k}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\rho_2+k+j}{\rho_2+k}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|+k+j)} & c^{(\rho_1-k-j)} \\ W' & b^{(q-\sigma_1-k-j)} & c^{(\rho_2+k+j)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)-$$

$$\sum_{k=0}^{\sigma+1}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\beta_i+\rho_1-k}{\beta_i}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{cb}^{(\beta_i+\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-$$

$$\sum_{k=0}^{\sigma+1}\sum_{m=i-1}^{1}(-1)^{k+i-m}\binom{\rho_2+k}{\rho_2}\binom{\beta_m+\beta_{m+1}}{\beta_m}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-k)}y$$

$$\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-\sum_{l=1}^{\rho_1-\sigma-1}(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(t_1+|\beta|+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)+$$

$$\sum_{l=1}^{\rho_1-\sigma-1}\sum_{n=0}^{\beta_i}(-1)^{\sigma+1}\binom{\rho_2+\sigma+1+l+n}{\rho_2+\sigma+1+l}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}\binom{\beta_i+\rho_1-\sigma-1-n-l}{\beta_i-n}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{ba}^{(t_1+|\beta|-\beta_i+l+n)}x\left(\begin{array}{c|ll} W & a^{(p+t_1+|\beta|-\beta_i+l+n)} & c^{(\beta_i+\rho_1-\sigma-1-n-l)} \\ W' & b^{(q-t_1-l-n)} & c^{(\rho_2+\sigma+1+l+n)} \\ W'' & c^{(\rho_3)} & \end{array}\right)+$$

$$\sum_{l=1}^{\rho_1-\sigma-1}\sum_{m=i-1}^{1}(-1)^{\sigma+1+i-m}\binom{\rho_2+\sigma+1+l}{\rho_2}\binom{\sigma+l}{\sigma+1}\binom{\beta_m+\beta_{m+1}}{\beta_m}$$

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_m+\beta_{m+1})}y...yZ_{cb}^{(\beta_i)}yZ_{ba}^{(t_1+|\beta|+l)}x\left(\begin{array}{c|ll} W & a^{(p+t_1+|\beta|+l)} & c^{(\rho_1-\sigma-1-l)} \\ W' & b^{(q-t_1-l)} & c^{(\rho_2+\sigma+1+l)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

For non-essential elements of the form

$$Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)$$

where $q-p\leq\sigma_1\leq t_1$, $\rho_1>0$ and $\rho_1\leq\sigma+1$, $\beta_1\geq t_2+1$, $\beta_j>0$ for $j=2,...,i$, $t_1=\sigma_1+1+\sigma$ and $\rho_1+\rho_2+\rho_3=r-|\beta|$, the corresponding basis for the syzygies is

$$d_{i+1}s_i\left(Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|)} & c^{(\rho_1)} \\ W' & b^{(q-\sigma_1)} & c^{(\rho_2)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)\right)=$$

$$d_{i+1}\left(\sum_{k=0}^{\rho_1-1}(-1)^k\binom{\rho_2+k}{\rho_2}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)\right)=$$

$$\sum_{k=0}^{\rho_1-1}\sum_{j=0}^{\rho_1-k}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\rho_2+k+j}{\rho_2+k}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_i)}y\left(\begin{array}{c|lll} W & a^{(p)} & b^{(\sigma_1+|\beta|+k+j)} & c^{(\rho_1-k-j)} \\ W' & b^{(q-\sigma_1-k-j)} & c^{(\rho_2+k+j)} & \\ W'' & c^{(\rho_3)} & & \end{array}\right)-$$

$$\sum_{k=0}^{\rho_1-1}(-1)^k\binom{\rho_2+k}{\rho_2}\binom{\beta_i+\rho_1-k}{\beta_i}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{i-1})}yZ_{cb}^{(\beta_i+\rho_1-k)}y\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)-$$

$$\sum_{k=0}^{\rho_1-1}\sum_{m=i-1}^{1}(-1)^{k+i-m}\binom{\rho_2+k}{\rho_2}\binom{\beta_m+\beta_{m+1}}{\beta_m}Z_{cb}^{(\beta_1)}y...yZ_{cb}^{(\beta_{m-1})}yZ_{cb}^{(\beta_m+\beta_{m+1})}yZ_{cb}^{(\beta_{m+2})}y...yZ_{cb}^{(\beta_i)}yZ_{cb}^{(\rho_1-k)}y$$
$$\left(\begin{array}{c|ll} W & a^{(p)} & b^{(\sigma_1+|\beta|+\rho_1)} \\ W' & b^{(q-\sigma_1-k)} & c^{(\rho_2+k)} \\ W'' & c^{(\rho_3)} & \end{array}\right)$$

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