Minimal flows on Compact Manifolds

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Abstract

In this paper we compute the minimal number of non-degenerate singularities that can be realized on some manifold with non-empty boundary in terms only of abstract homological boundary information. We specify the index and the types (connecting or disconnecting) of the singularities realizing the minimum. The Euler characteristics of manifolds realizing the minimum are obtained and the associated Lyapunov graphs of Morse type are described and shown to have the lowest topological complexity.

Introduction

In this paper we introduce a notion of minimal flows on compact manifolds. A flow on a compact manifold M with total number of singularities h is minimal if there exists no other flow realizable on M with fewer singularities than h. In [Rei] techniques are developed to continue a gradient flow to one with the minimal number of critical points. However, the approach is quite distinct from ours and our results are of a different nature.

Let M be any compact manifold of dimension n such that $\partial M = \partial M^+ \cup \partial M^-$, with ∂M^+ and ∂M^- non-empty[1] where $\partial M^+ (\partial M^-)$ is the disjoint union of $e^+ (e^-)$ components of ∂M, and denote it by $\partial M^\pm = \bigcup_{i=1}^{e^\pm} M_i^\pm$. Also, consider the sum of the Betti numbers, $\beta_j(M_i^\pm)$, of these components, i.e. $B_j^\pm = \sum_{i=1}^{e^\pm} \beta_j(M_i^\pm)$ where $j = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$.

We now consider that we have abstract information without reference to a specific manifold, i.e., positive integers e^+, e^- and integers corresponding to the differences (of Betti numbers) $B_j^+ - B_j^-$. A *minimum number of singularities* $h_{\min}$ can be determined depending only on e^+, e^- and the differences $B_j^+ - B_j^-$ where $j = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$. This minimum number of singularities $h_{\min}$ has a topological-dynamical meaning. Given any compact manifold M with $e^+ + e^-$ boundary components, $e^\pm$ components $M_i^\pm, i = 1, \ldots, e^\pm$ labelled with $\{\beta_j(M_i^\pm); \ j = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$, there exists no Morse flow realizable on M entering through the (M_i^+)'s and exiting through the (M_i^-)'s with fewer singularities than $h_{\min}$. Hence, in this sense, a flow realizing $h_{\min}$ is a minimal flow on some compact manifold M respecting the given homological restrictions on the entering and exiting boundaries for the flow. Of course, there may be many such compact manifolds realizing this minimal flow. On the other hand, there are many compact manifolds with the same boundary specification given above which possess minimal flows with total number of singularities greater than $h_{\min}$.

[1]we will make no further mention of this fact and will assume it throughout the article

The simple example in Figure 1, on compact 2-manifolds illustrates this point, i.e., given one entering boundary component and two exiting boundary components, (in this case these components must be circles) $h_{\min} = 1$ hence $h_1 = 1$. However, there are other minimal flows on other compact 2-manifold with the same homological boundary specification which possess a greater number of singularities than $h_{\min}$. Of course, in dimension two, the number of boundary components, $e^+ + e^-$ and the genus g completely determines $h_{\min}$, i.e., the number of singularities of index one, h_1, by the formula $2 - 2g - (e^+ + e^-) = h_1$. If we define topological complexity in terms of the genus, note that $h_{\min}$ is realized on the manifold of lowest complexity. However, in higher dimensions we can also measure topological complexity in terms of the presence of dual pairs and hence a similar phenomenon is observed.
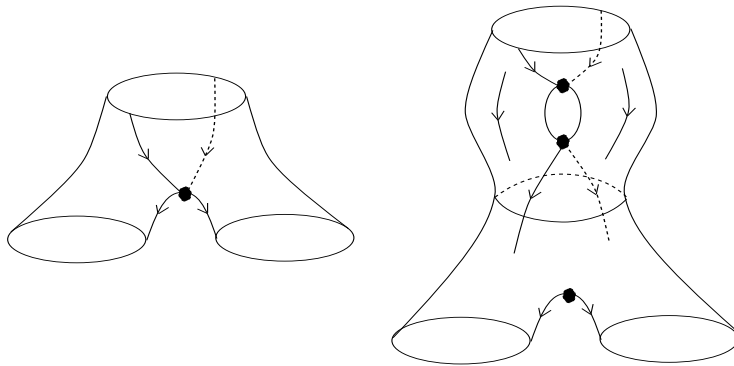

Figure 1: Minimal flows and topological complexity

Theorem 1 asserts that a minimum number of singularities $h_{\min}$ can be determined depending only on e^+, e^- and the differences (of Betti numbers) $B_j^+ - B_j^-$. Moreover, the set

$$\mathcal{H} = \{(h_1, \ldots, h_{n-1}) : \sum_{i=1}^{n-1} h_i = h_{\min} : \text{ Poincaré-Hopf inequalities are satisfied}\}$$

is completely determined.

One can also determine the set

$$\mathcal{H}^{cd}(h_1, \ldots, h_{n-1}) = \{(h_1^c, h_1^d, \ldots, h_{n-1}^c, h_{n-1}^d) : \text{ the } h^{cd} - \text{system is satisfied}\}.$$

We also prove that given $(h_1, \ldots, h_{n-1}) \in \mathcal{H}$, the set $\mathcal{H}^{cd}(h_1, \ldots, h_{n-1})$ is a singleton.

Moreover, for each element in $\mathcal{H}^{cd}(h_1, \ldots, h_n)$, a family $\mathcal{F}_{L(\underline{h})}$ of Lyapunov semi-graphs of Morse type $L(\underline{h})$ is determined.

With the previous notation we can state the theorem:

Theorem 1. *Given positive integers e^+ and e^- and integers $B_j^+ - B_j^-$,*

1. *there exists a number $h_{\min}$ which is the lower bound on the number of singularities of any Morse flow realizable on any compact manifold with e^+ entering boundaries and e^- exiting boundaries with Betti numbers satisfying the given differences $B_j^+ - B_j^-$.*

2. *the set $\mathcal{H}$ is completely determined and each element in $\mathcal{H}$ determines a set $\mathcal{H}^{cd}(h_1, \ldots, h_n)$ which is a singleton.*

3. *the range of the Euler characteristics, $\chi_{\min}(M, \partial M)$ of the compact manifolds M realizing the minimal flows is obtained.*

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4. given $\mathcal{H}^{cd}(h_1, \ldots, h_n)$ the corresponding family of Lyapunov semi-graphs of Morse type is completely determined; all graphs are explicitly described and possess lowest topological complexity.

Section 1 contains background information. In Section 2 we prove item 1, which follows from Propositions 2.1.1, 2.2.1 and 2.3.1, and item 2 which follows from Propositions 2.1.2, 2.2.2, 2.3.2, 2.1.5, 2.2.5 and 2.3.5. In Section 3 we prove item 3, which follows from Propositions 3.1.2, and item 4, which follows from Subsection 3.2.

1 Background

In this section we will introduce basic definitions and results obtained in [BMRez].

In order to bookkeep dynamical and topological information of a given flow on a given manifold, Franks introduced in [F] Lyapunov graphs. Here we make use of *abstract Lyapunov graphs*.

An *abstract Lyapunov graph (semi-graph)*[2] is an oriented graph with no oriented cycles such that each vertex v is labelled with a list of non-negative integers $\{h_0(v) = k_0, \ldots, h_n(v) = k_n\}$. Also, the labels on each edge $\{\beta_0 = 1, \beta_1, \ldots, \beta_{n-2}, \beta_{n-1} = 1\}$ must be a collection of non-negative integers satisfying Poincaré duality and if $n-1$ is even then $\beta_{\frac{n-1}{2}}$ is even.

An *abstract Lyapunov graph (semi-graph) of Morse type* will be defined subsequently, but roughly speaking, it is an abstract Lyapunov graph with all vertices labelled with non-degenerate singularities of index j, i. e., $\{h_j(v) = 1\}$.

Let N^- be an $(n-1)$-dimensional closed manifold, H an n-handle and $N^+ = \partial((N^- \times [0,1]) \cup H)$. The following definition distinguishes the effect on the Betti numbers of N^+ and N^- once the handle H has been attached to $N^- \times [0,1]$.

A handle containing a singularity of index ℓ or respectively, the corresponding vertex on L is called ℓ-*disconnecting*, in short ℓ-d, if this handle has the algebraic effect of increasing the ℓ-th Betti number of N^+ or respectively, the corresponding β_ℓ label on the incoming edge. A handle containing a singularity of index ℓ or the corresponding vertex on L is called $(\ell-1)$-*connecting*, in short $(\ell-1)$-c, if this handle has the algebraic effect of decreasing the $(\ell-1)$-th Betti number of N^+ or respectively, the corresponding $\beta_{\ell-1}$ label on the incoming edge. A handle containing a singularity of index ℓ or the corresponding vertex on L is called β-*invariant*, in short β-i, if all Betti numbers are kept constant (see Figure 2). Details can be found in [CrRez].

An *abstract Lyapunov graph of Morse type L* is an abstract Lyapunov graph that satisfies the following:

1. every vertex is labelled with $h_j = 1$ for some $j = 0, \ldots, n$.

2. the number of incoming edges, e^+, and the number of outgoing edges, e^-, of a vertex must satisfy:

   (a) if $h_j = 1$ for $j \neq 0, 1, n-1, n$ then $e^+ = 1$ and $e^- = 1$;

   (b) if $h_1 = 1$ then $e^+ = 1$ and $0 < e^- \leq 2$; if $h_{n-1} = 1$ then $e^- = 1$ and $0 < e^+ \leq 2$;

   (c) if $h_0 = 1$ then $e^- = 0$ and $e^+ = 1$; if $h_n = 1$ then $e^+ = 0$ and $e^- = 1$.

3. every vertex labelled with $h_\ell = 1$ must be of type ℓ-d or $(\ell-1)$-c. Furthermore if $n = 2i = 0$ mod 4 and $h_i = 1$ then v may be labelled with β-i.

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[2]Given a finite set V we define a *directed semi-graph* $G' = (V', E')$ as a pair of sets $V' = V \cup \{\infty\}$, $E' \subset V' \times V'$. As usual, we call the elements of V' vertices and since we regard the elements of E' as ordered pairs, these are called *directed edges*. Furthermore the edges of the form (∞, v) and (v, ∞) are called *semi-edges* (or dangling edges as in [Rez]). Note that whenever G' does not contain semi-edges, G' is a graph in the usual sense. The graphical representation of the graph will have the semi-edges cut short.
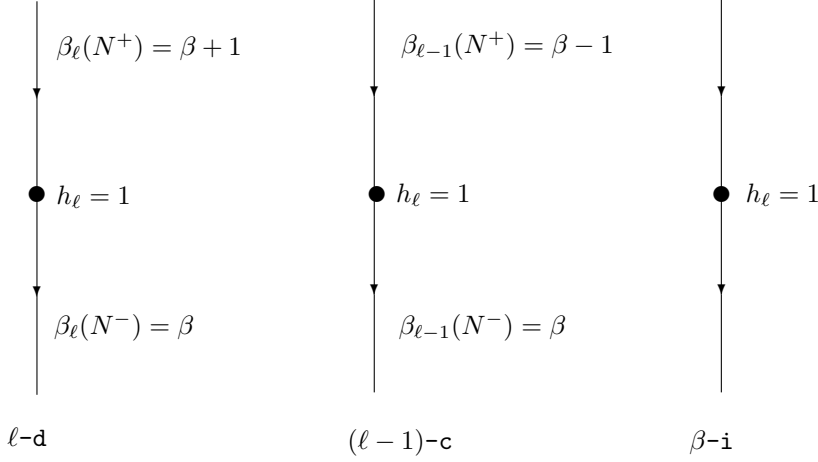
Figure 2: The three possible algebraic effects.

In [BMRez] we prove a continuation result for abstract Lyapunov semi-graphs to abstract Lyapunov semi-graphs of Morse type. This was done by presenting an algorithm which not only constructs the continuation but also provides the number of possible continuations. The main theorem in that paper asserts that every abstract Lyapunov semi-graph that satisfies the Poincaré-Hopf inequalities at each vertex can be continued to an abstract Lyapunov semi-graph of Morse type. The Poincaré-Hopf inequalities are deduced from an analysis of long exact sequences of index pairs. See [Co] for more details on Conley index theory.

Consider N^+ the entering set and N^- the exiting set for the flow defined on N, where (N, N^-) is an index pair for an isolated invariant set Λ. The Poincaré-Hopf inequalities were obtained in [BMRez] by analyzes of the long exact sequences for the pairs (N, N^-) and (N, N^+). The Poincaré-Hopf inequalities[3] are the collection of the inequalities below, where rank $H_j(N, N^-) = h_j$, rank $H_j(N, N^+) = h_{n-j}$, rank $H_0(N^-) = e^-$, rank $H_0(N^+) = e^+$, rank $H_0(N) = 1$ and rank $(H_j(N^\pm)) = B_j^\pm$.

$$
\left\{
\begin{array}{l}
\left\{
\begin{array}{l}
h_j \geq -(B_{j-1}^+ - B_{j-1}^-) + (B_{j-2}^+ - B_{j-2}^-) + - \ldots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\
\quad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \ldots \\
\quad \pm(h_{n-1} - h_1) \pm [(h_n - h_0) + (e^+ - e^-)] \\
\\
h_{n-j} \geq -\big[-(B_{j-1}^+ - B_{j-1}^-) + (B_{j-2}^+ - B_{j-2}^-) + - \ldots \pm (B_2^+ - B_2^-) \pm (B_1^+ - B_1^-) \\
\quad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \ldots \\
\quad \pm(h_{n-1} - h_1) \pm [(h_n - h_0) + (e^+ - e^-)]\big] \\
\end{array}
\right. \\
\quad \vdots \\
\left\{
\begin{array}{l}
h_2 \geq -(B_1^+ - B_1^-) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-) \\
h_{n-2} \geq -\big[-(B_1^+ - B_1^-) - (h_{n-1} - h_1) + (h_n - h_0) + (e^+ - e^-)\big] \\
\end{array}
\right. \\
\left\{
\begin{array}{l}
h_1 \geq h_0 - 1 + e^- \\
h_{n-1} \geq h_n - 1 + e^+ \\
\end{array}
\right.
\end{array}
\right.
\tag{1}
$$

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[3]In the setting of Morse flows, the h_j's are the numbers of non-degenerate singularities of Morse index j.

Moreover, the equality

$$\mathcal{B}^+ - \mathcal{B}^- = e^- - e^+ + \sum_{j=0}^{2i+1} (-1)^j h_j \tag{2}$$

where

$$\mathcal{B}^+ = \frac{(-1)^i}{2} B_i^+ \pm B_{i-1}^+ \pm \ldots - B_1^+$$

$$\mathcal{B}^- = \frac{(-1)^i}{2} B_i^- \pm B_{i-1}^- \pm \ldots - B_1^-$$

must hold in the odd case, and

$$h_i - \sum_{j=1}^{i-1} (-1)^{j+1} (B_j^+ - B_j^-) - \sum_{j=0}^{i-1} (-1)^j (h_{2i-j} - h_j) + (e^- - e^+) \quad \text{be even, for } 2i = 2 \mod 4 \tag{3}$$

in the even case $2i = 2 \mod 4$.

Also in [BMRez] was shown that the Poincaré-Hopf inequalities are equivalent to the linear system below, which we refer to as the $h^{cd} -$ system.

$$\begin{cases} e^- - 1 - h_1^c = 0 \\ \{h_j = h_j^c + h_j^d + \beta^i, \;\; j = 1, \ldots, n-1, \beta^i = 0 \text{ if } n \neq 0 \mod 4 \\ e^+ - 1 - h_{n-1}^d = 0 \\ \{-(B_k^+ - B_k^-) + h_k^d - h_{k+1}^c - h_{n-k}^c + h_{n-k+1}^d = 0 \;\; k = 1, \ldots, \lfloor \frac{n-2}{2} \rfloor \end{cases} \tag{4}$$

If $n = 2i+1$ we have an additional equation of the linear system above is $\frac{-(B_i^+ - B_i^-)}{2} + h_i^d - h_{i+1}^c = 0$.

2 Minimal flow and its singularities

In this section we deal with minimality: we compute the number of singularities of minimal flows, their indices and their types. Although the technical guiding line is the same, we need to distinguish three situations according to the parity of the dimension of the underlying manifold(s) to develop details.

2.1 Odd dimension $n = 2i + 1$

2.1.1 Computation of $h_{\min}$

Notation For $j = 1 \ldots n$ we denote by PH_j the right hand side expressions in the Poincaré-Hopf inequalities (1), and by OPH_j their optimal value, i.e. $OPH_j = \max\{0, PH_j\}$. Hence, the general solution $\underline{h} \in \mathbf{N}_0^{n+1}$ of the Poincaré-Hopf inequalities (1) has coordinates of the form

$$\begin{cases} h_0 &= \alpha_0 \\ h_n &= \alpha_n \\ h_1 &= \alpha_0 - 1 + e^- + \alpha_1 \\ h_{n-1} &= \alpha_n - 1 + e^+ + \alpha_{n-1} \\ h_j &= OPH_j + \alpha_j \end{cases} \tag{5}$$

where $\alpha_j \in \mathbf{N}_0$ for all $j = 0 \ldots n$.

Proposition 2.1.1. *In dimension $n = 2i + 1$, given positive integers e^+ and e^- and integers $B_j^+ - B_j^-$, $j = 1, \ldots, i$, the minimal number of singularities needed in order to have continuation is*

$$h_{\min} = \min_{\substack{\{\underline{h} \in \mathbf{N}_0^{n+1} \\ satisfying \ (1) \ and \ (2)\}}} \sum_{j=0}^n h_j = e^+ + e^- - 2 + \sum_{j=2}^i \mid B_{j-1}^+ - B_{j-1}^- \mid + \mid \frac{B_i^+ - B_i^-}{2} \mid$$

Proof Continuation is possible if and only if the Poincaré-Hopf inequalities (1) and (2) hold. For $j = 2 \ldots i$ we have

$$PH_j = -(B_{j-1}^+ - B_{j-1}^-) - PH_{j-1} - (OPH_{n-(j-1)} - OPH_{j-1}) - (\alpha_{n-(j-1)} - \alpha_{j-1})$$

$$OPH_{n-(j-1)} - OPH_{j-1} = -PH_{j-1}$$

(if $OPH_{j-1} = PH_{j-1}$ then $OPH_{n-(j-1)} = 0$, else if $OPH_{j-1} = 0$ then $OPH_{n-(j-1)} = -PH_{j-1}$). Hence,

$$PH_j = -(B_{j-1}^+ - B_{j-1}^-) - (\alpha_{n-(j-1)} - \alpha_{j-1}) \qquad \forall j = 2 \ldots i \tag{6}$$

Let us first consider equation (2).

$$(\mathcal{B}^+ - \mathcal{B}^-) = -\alpha_1 + \alpha_{n-1} + \sum_{j=2}^{n-2} (-1)^j (OPH_j + \alpha_j)$$

$$\sum_{j=2}^i (-1)^{j-1} (B_{j-1}^+ - B_{j-1}^-) + \frac{(-1)^i (B_i^+ - B_i^-)}{2} = \sum_{j=1}^{n-1} (-1)^j \alpha_j + \sum_{j=2}^{n-2} (-1)^j OPH_j$$

For $j = 2 \ldots i$ either $OPH_j = PH_j$ and $OPH_{n-j} = 0$, or $OPH_j = 0$ and $OPH_{n-j} = PH_{n-j} = -PH_j$. In any case we have

$$(-1)^j OPH_j + (-1)^{n-j} OPH_{n-j} = (-1)^{j+1} \left[(B_{j-1}^+ - B_{j-1}^-) + (\alpha_{n-(j-1)} - \alpha_{j-1}) \right]$$

and equation (2) becomes

$$\frac{(-1)^i (B_i^+ - B_i^-)}{2} = (-1)^i \alpha_i + (-1)^{i+1} \alpha_{i+1}$$

We are now ready to minimize the sum of the coordinates of the general solution $\underline{h} \in \mathbf{N}_0^{n+1}$ of the Poincaré-Hopf inequalities (1).

$$
\begin{aligned}
\sum_{j=0}^n h_j &= \sum_{j=0}^n (OPH_j + \alpha_j) = \\
&= \alpha_0 + \alpha_n + e^+ + e^- - 2 + \sum_{j=2}^i \mid PH_j \mid + \sum_{j=0}^n \alpha_j = \\
&= \alpha_0 + \alpha_n + e^+ + e^- - 2 + \sum_{j=2}^i \mid -(B_{j-1}^+ - B_{j-1}^-) - (\alpha_{n-(j-1)} - \alpha_{j-1}) \mid + \sum_{j=0}^n \alpha_j \geq \\
&\geq e^+ + e^- - 2 + \sum_{j=2}^i (\mid B_{j-1}^+ - B_{j-1}^- \mid - \mid \alpha_{n-(j-1)} - \alpha_{j-1} \mid) + \sum_{j=0}^n \alpha_j \geq \\
&\geq e^+ + e^- - 2 + \sum_{j=2}^i (\mid B_{j-1}^+ - B_{j-1}^- \mid - \mid \alpha_{n-(j-1)} \mid - \mid \alpha_{j-1} \mid) + \sum_{j=1}^{n-1} \alpha_j = \\
&= e^+ + e^- - 2 + \sum_{j=2}^i \mid B_{j-1}^+ - B_{j-1}^- \mid + \alpha_i + \alpha_{i+1} \geq \\
&\geq e^+ + e^- - 2 + \sum_{j=2}^i \mid B_{j-1}^+ - B_{j-1}^- \mid + \mid \frac{B_i^+ - B_i^-}{2} \mid
\end{aligned}
$$

To show that the lower bound is taken, let $\alpha_j = \alpha_{n-j} = 0$ for $j = 0 \ldots i - 1$. If $B_i^+ \geq B_i^-$ then choose $\alpha_i = \frac{B_i^+ - B_i^-}{2}$ and $\alpha_{i+1} = 0$, else choose $\alpha_i = 0$ and $\alpha_{i+1} = \frac{B_i^+ - B_i^-}{2}$. $\triangle$

The meaning of the formula computing $h_{\min}$ is that there is a straightforward way of making all the differences $B_j^+ - B_j^-$ vanish. The presence of each singularity is justified by the fact that either it will make one of the $B_j^+ - B_j^-$ smaller or it will decrease the edges contributions: we can say there is no waste of singularities.

2.1.2 Distribution of the $h_{\min}$ singularities according to the index

Convention From now on $h_0 = h_n = 0$.

Once $h_{\min}$ is known, we want to describe the set

$$\mathcal{H} = \{(h_1, \ldots, h_{n-1}) : \sum_{i=1}^{n-1} h_i = h_{\min} : \text{ Poincaré-Hopf inequalities are satisfied}\}.$$

This is done in Proposition 2.1.2.

Notation We shall denote by $\underline{v}_e \in \mathbf{N}_0^{n-1}$ the following vector, associated with the edges contribution to the number of singularities:

$$\underline{v}_e = (\quad e^- - 1, \quad 0, \ldots, 0, \quad e^+ - 1 \quad)$$

Let us first consider the singularities altering the first Betti number and denote by $\mathcal{V}_1$ the set of vectors

$$\mathcal{V}_1 = \{\underline{v}_1(k_1)\}_{k_1=0}^{|B_1^+ - B_1^-|}$$

where $\underline{v}_1(k_1)$ is defined as below:

$$\begin{cases} \underline{v}_1(k_1) = (\quad k_1, \quad 0, \quad 0, \quad \ldots, \quad 0, \quad |B_1^+ - B_1^-| - k_1, \quad\quad 0 \quad\quad) & \text{if } B_1^+ \geq B_1^- \\ \underline{v}_1(k_1) = (\quad 0, \quad k_1, \quad 0, \quad \ldots, \quad 0, \quad\quad 0, \quad\quad |B_1^+ - B_1^-| - k_1 \quad) & \text{otherwise} \end{cases}$$

In general, as for the singularities altering the j-th Betti number, let us denote by $\mathcal{V}_j$ the set of vectors

$$\mathcal{V}_j = \{\underline{v}_j(k_j)\}_{k_j=0}^{|B_j^+ - B_j^-|} \qquad \text{for } j = 2 \ldots i - 1$$

where $\underline{v}_j(k_j)$ is defined as below:

$$\begin{cases} \underline{v}_j(k_j) = (\quad 0,\ldots,0, \quad \overbrace{k_j}^{j}, \quad 0, \quad 0,\ldots,0, \quad \overbrace{|B_j^+ - B_j^-| - k_j,}^{n-j-1} \quad\quad 0, \quad\quad 0,\ldots,0 \quad) & \text{if } B_j^+ \geq B_j^- \\ \underline{v}_j(k_j) = (\quad 0,\ldots,0, \quad 0, \quad \underbrace{k_j}_{j+1}, \quad 0,\ldots,0, \quad\quad 0, \quad\quad \underbrace{|B_j^+ - B_j^-| - k_j,}_{n-j} \quad 0,\ldots,0 \quad) & \text{otherwise} \end{cases}$$

Last, as for the singularities altering the middle dimension Betti number, let us define the vector $\underline{v}_i$ as

$$\begin{cases} \underline{v}_i = (\quad 0,\ldots,0, \quad \overbrace{\dfrac{|B_i^+ - B_i^-|}{2}}^{i}, \quad 0, \quad 0,\ldots,0 \quad) & \text{if } B_i^+ \geq B_i^- \\ \underline{v}_i = (\quad 0,\ldots,0, \quad 0, \quad \underbrace{\dfrac{|B_i^+ - B_i^-|}{2}}_{i+1}, \quad 0,\ldots,0 \quad) & \text{otherwise} \end{cases}$$

Proposition 2.1.2. *In dimension $n = 2i + 1$, given positive integers e^+ and e^- and integers $B_j^+ - B_j^-$, $j = 1, \ldots, i$, a vector $\underline{h} \in \mathbf{N}_0^{n-1}$ satisfies the Poincaré-Hopf inequalities (1) and (2) and realizes $h_{\min}$ if and only if it can be written as*

$$\underline{h} = \underline{v}_e + \sum_{j=1}^{i-1} \underline{v}_j(k_j) + \underline{v}_i \qquad \text{where } \underline{v}_j(k_j) \in \mathcal{V}_j \quad \forall j = 1, \ldots, i - 1$$

($\mathcal{V}_j$ as in the previous notation)

Proof The if part is a straightforward computation. As for the converse, recall that when a vector $\underline{h} = (h_1, \ldots, h_{n-1})$ realizes $h_{\min}$, each singularity h_j must either reduce the contribution of the edges or reduce the difference between a couple of Betti numbers (just look at the formula for $h_{\min}$). First, we must use $e^+ + e^- - 2$ singularities to get rid of the edges contribution: $e^- - 1$ of them must be of type h_1^c and the remaining $e^+ - 1$ of type h_{n-1}^d. If we disconsider the type of such singularities, it is easy to recognize the role of $\underline{v}_e$. Next, for each j from 1 to $i-1$, we must use exactly $\mid B_j^+ - B_j^- \mid$ singularities to make the difference between the j-th Betti numbers vanish. If $B_j^+ \geq B_j^-$, this can be done by singularities of type h_j^d and/or h_{n-j-1}^d; if $B_j^+ \leq B_j^-$, this can be done by singularities of type h_{j+1}^c and/or h_{n-j}^c. Again, if we disconsider the type, it is easy to recognize the role of each $\underline{v}_j$. Last, we are left with $\frac{|B_i^+ - B_i^-|}{2}$ singularities associated with the middle dimension Betti numbers. We have no choice: if $B_i^+ \geq B_i^-$, we must use singularities of type h_i^d, else of type h_{i+1}^c. The presence of $\underline{v}_i$ is hence explained. $\triangle$

Note that there is no restriction to the choice of the k_j's in $\{0 \ldots \mid B_j^+ - B_j^- \mid\}$.

Corollary 2.1.3. *The total number of vectors $\underline{h}$ realizing $h_{\min}$ is $\prod_{j=1}^{i-1}(|B_j^+ - B_j^-| + 1)$.*

By reading the proof above, one realizes that knowing a decomposition of $\underline{h}$ is knowing a vector of type of singularities realizing $\underline{h}$, that is a vector of

$$\mathcal{H}^{cd}(h_1, \ldots, h_n) = \{(h_1^c, h_1^d, \ldots, h_{n-1}^c, h_{n-1}^d) \text{ such that the } h^{cd}\text{-system is satisfied}\}$$

In order to be more explicit, let us define a map

$$g: \quad \mathbf{N}_0^{n-1} \quad \longrightarrow \quad \mathbf{N}_0^{2n-2}$$

$$\underline{v}_e \quad \longrightarrow \quad (h_1^c = e^- - 1, h_1^d = 0, \ldots, h_{n-1}^c = 0, h_{n-1}^d = e^+ - 1,)$$

$$\underline{v}_j(k_j) \quad \longrightarrow \quad (h_1^c = 0, \ldots, h_j^c = 0, \overbrace{h_j^d = k_j}^{2j}, h_{j+1}^c = 0, \ldots, \overbrace{h_{n-j-1}^d = \mid B_j^+ \geq B_j^- - k_j \mid}^{2n-2j-2}, \ldots, h_{n-1}^d = 0)$$
$$\text{if } B_j^+ \geq B_j^- \quad \forall j = 1, \ldots, i-1$$

$$(h_1^c = 0, \ldots, h_j^c = 0, \underbrace{h_{j+1}^c = k_j}_{2j+1}, h_{j+1}^d = 0, \ldots, \underbrace{h_{n-j}^c = \mid B_j^+ \geq B_j^- - k_j \mid}_{2n-2j-1}, \ldots, h_{n-1}^d = 0)$$
$$\text{if } B_j^+ \leq B_j^- \quad \forall j = 1, \ldots, i-1$$

$$\underline{v}_i \quad \longrightarrow \quad (h_1^c = 0, \ldots, h_j^c = 0, \ldots, h_i^c = 0, \overbrace{h_i^d = \mid B_i^+ \geq B_i^- \mid}^{2i}, h_{i+1}^c = 0, \ldots, h_{n-1}^d = 0)$$
$$\text{if } B_i^+ \geq B_i^-$$

$$(h_1^c = 0, \ldots, h_j^c = 0, \ldots, h_i^d = 0, \overbrace{h_{i+1}^c = \mid B_i^+ \geq B_i^- \mid}^{2i+1}, h_{i+1}^d = 0, \ldots, h_{n-1}^d = 0)$$
$$\text{if } B_i^+ \leq B_i^-$$

Corollary 2.1.4. *Following the notation right above, if $\underline{h} = \underline{v}_e + \sum_{j=1}^{i-1} \underline{v}_j(k_j) + \underline{v}_i$ then $g(\underline{v}_e) + \sum_{j=1}^{i-1} g(\underline{v}_j(k_j)) + g(\underline{v}_i)$ belongs to $\mathcal{H}^{cd}(\underline{h})$.*

2.1.3 Finding all the possible types of singularities

Now that we have the distributions of the $h_{\min}$ singularities according to the index, we can apply to each one of them the algorithm of [BMRez], and find all the possible matching types. We find that for each $\underline{h}$ in $\mathcal{H}$, the set $\mathcal{H}^{cd}(\underline{h})$ is a singleton.

Proposition 2.1.5. *In dimension $n = 2i+1$, each vector $\underline{h} \in \mathbf{N}_0^{n-1}$ realizing $h_{\min}$ determines the types of singularities uniquely.*

Proof Let us recall that, according to [BMRez], for a given $\underline{h}$ the number of solutions is given by the product $\prod_{j=1}^{i-1} n_j$, where n_j is given by

$$
\left\{
\begin{array}{ll}
n_j = \min\{h_{j+1} - \hat{\beta}_j, h_{n-(j+1)}\} - \max\{0, -\hat{\beta}_j\} + 1 & \text{if } j \text{ is odd} \\
n_j = \min\{h_{n-(j+1)} - \hat{\beta}_j, h_{j+1}\} - \max\{0, -\hat{\beta}_j\} + 1 & \text{if } j \text{ is even}
\end{array}
\right.
$$

where

$$
\hat{\beta}_j = \sum_{\ell=1}^{j}(-1)^\ell(B_\ell^+ - B_\ell^-) - \sum_{\ell=1}^{j+1}(-1)^\ell(h_\ell - h_{n-\ell}) - (e^- - e^+).
$$

Computation gives $n_j = 1$ for all $j = 1, \ldots, i-1$ and we are done:

for j odd we have
if $B_j^+ \geq B_j^-$ then $\quad \hat{\beta}_j = -[(B_j^+ - B_j^-) - k_j]$
$\qquad\qquad\qquad\qquad \max\{0, -\hat{\beta}_j\} = (B_j^+ - B_j^-) - k_j$
$\qquad\qquad\qquad\qquad$ if $B_{j+1}^+ \geq B_{j+1}^-$ $\quad$ then $\qquad\qquad \min\{h_{j+1} - \hat{\beta}_j, h_{n-(j+1)}\} = h_{n-(j+1)} = (B_j^+ - B_j^-) - k_j$
$\qquad\qquad\qquad\qquad$ else $\qquad\qquad\qquad\qquad\qquad \min\{h_{j+1} - \hat{\beta}_j, h_{n-(j+1)}\} = h_{j+1} - \hat{\beta}_j = (B_j^+ - B_1^-) - k_j$
else $\qquad\qquad\qquad\qquad \hat{\beta}_j = k_j$
$\qquad\qquad\qquad\qquad \max\{0, -\hat{\beta}_j\} = 0$
$\qquad\qquad\qquad\qquad$ if $B_{j+1}^+ \geq B_{j+1}^-$ $\quad$ then $\qquad\qquad \min\{h_{j+1} - \hat{\beta}_j, h_{n-(j+1)}\} = h_{n-(j+1)} = 0$
$\qquad\qquad\qquad\qquad$ else $\qquad\qquad\qquad\qquad\qquad \min\{h_{j+1} - \hat{\beta}_j, h_{n-(j+1)}\} = h_{j+1} - \hat{\beta}_j = 0$

$\qquad$ for j even we have
$\qquad$ if $B_j^+ \geq B_j^-$ then $\quad \hat{\beta}_j = (B_j^+ - B_j^-) - k_j$
$\qquad\qquad\qquad\qquad\quad \max\{0, -\hat{\beta}_j\} = 0$
$\qquad\qquad\qquad\qquad\quad$ if $B_{j+1}^+ \geq B_{j+1}^-$ $\quad$ then $\quad \min\{h_{n-(j+1)} - \hat{\beta}_j, h_{j+1}\} = h_{n-(j+1)} - \hat{\beta}_j = 0$
$\qquad\qquad\qquad\qquad\quad$ else $\qquad\qquad\qquad\qquad \min\{h_{n-(j+1)} - \hat{\beta}_j, h_{j+1}\} = h_{j+1} = 0$
$\qquad$ else $\qquad\qquad\qquad\qquad \hat{\beta}_j = -k_j$
$\qquad\qquad\qquad\qquad\qquad \max\{0, -\hat{\beta}_j\} = k_j$
$\qquad\qquad\qquad\qquad\qquad$ if $B_{j+1}^+ \geq B_{j+1}^-$ $\quad$ then $\quad \min\{h_{n-(j+1)} - \hat{\beta}_j, h_{j+1}\} = h_{n-(j+1)} - \hat{\beta}_j = k_j$
$\qquad\qquad\qquad\qquad\qquad$ else $\qquad\qquad\qquad\qquad \min\{h_{n-(j+1)} - \hat{\beta}_j, h_{j+1}\} = h_{j+1} = k_j \qquad \triangle$

Now that we know that we have uniqueness, we do not need the algorithm of [BMRez] anymore (in this special case):

Corollary 2.1.6. *Let $n = 2i + 1$, $\underline{h} = \underline{v}_e + \sum_{j=1}^{i-1} \underline{v}_j(k_j) + \underline{v}_i$ and g as in Corollary 2.1.4. Then the unique element of $\mathcal{H}^{cd}(\underline{h})$ can be written as $g(\underline{v}_e) + \sum_{j=1}^{i-1} g(\underline{v}_j(k_j)) + g(\underline{v}_i)$.*

2.1.4 Example in dimension 5

Consider the following homological boundary information in dimension 5:

$$
\{e^+ = 2, e^- = 3, B_1^+ - B_1^- = -2, B_2^+ - B_2^- = -2\}
$$

We have in this case $h_{\min} = 6$. As for the distribution of the six singularities we have (Proposition 2.1.2)

$$
\begin{array}{llll}
\underline{v}_e = (& 2, \ 0 & , \ 0, \ 1 &) \\
\underline{v}_1 = (& 0, \ k_1 & , \ 0, \ 2 - k_1 &) \qquad k_1 \in \{0, 1, 2\} \\
\underline{v}_2 = (& 0, \ 0 & , \ 1, \ 2 &)
\end{array}
$$

hence, the set of vectors $\underline{h} = (h_1, h_2, h_3, h_4)$ satisfying the Poincaré-Hopf inequalities and realizing $h_{\min}$ are

$$
\{(2, 0, 1, 3), (2, 1, 1, 2), (2, 2, 1, 1)\}
$$

Concerning their types of singularities, which are uniquely determined by $\underline{h}$, we have

$$
\begin{aligned}
\mathcal{H}^{cd}((2,0,1,3)) &= \{(h_1^c = 2, h_1^d = 0, h_2^c = 0, h_2^d = 0, h_3^c = 1, h_3^d = 0, h_4^c = 2, h_4^d = 1)\} \\
\mathcal{H}^{cd}((2,1,1,2)) &= \{(h_1^c = 2, h_1^d = 0, h_2^c = 1, h_2^d = 0, h_3^c = 1, h_3^d = 0, h_4^c = 1, h_4^d = 1)\} \\
\mathcal{H}^{cd}((2,2,1,1)) &= \{(h_1^c = 2, h_1^d = 0, h_2^c = 2, h_2^d = 0, h_3^c = 1, h_3^d = 0, h_4^c = 0, h_4^d = 1)\}
\end{aligned}
$$

as we can obtain either from applying the algorithm of [BMRez] or, in a more direct way, from Corollary 2.1.6.

2.2 Even dimension $n = 0 \mod 4$

2.2.1 Computation of $h_{\min}$

Proposition 2.2.1. *In even dimension n, $n = 0 \mod 4$, given positive integers e^+ and e^- and integers $B_j^+ - B_j^-$, $j = 1, \ldots, i-1$, the minimal number of singularities needed in order to have continuation is*

$$
h_{\min} = \min_{\substack{\{\underline{h} \in \mathbf{N}_0^{n+1} \\ satisfying\ (1)\ \}}} \sum_{j=0}^{n} h_j = e^+ + e^- - 2 + \sum_{j=1}^{i-1} | B_j^+ - B_j^- |
$$

Proof Computation is slightly different from the one in the odd case. Here follow the details. We use the same notation (5) established in Section 2 for the odd case. Also, let us recall equation (6) in the proof of Proposition 2.1.1, which still holds in the even case, that is

$$
PH_j = -(B_{j-1}^+ - B_{j-1}^-) - (\alpha_{n-(j-1)} - \alpha_{j-1}) \qquad \forall j = 2 \ldots i
$$

Let us minimize the sum of the coordinates of the general solution $\underline{h} \in \mathbf{N}_0^{n+1}$ of the Poincaré-Hopf inequalities (1).

$$
\begin{aligned}
\sum_{j=0}^{n} h_j \ &= \sum_{j=0}^{n} (OPH_j + \alpha_j) = \\
&= \alpha_0 + \alpha_n + e^+ + e^- - 2 + \sum_{j=2}^{i} | PH_j | + \sum_{j=0}^{n} \alpha_j = \\
&= \alpha_0 + \alpha_n + e^+ + e^- - 2 + \sum_{j=2}^{i} | -(B_{j-1}^+ - B_{j-1}^-) - (\alpha_{n-(j-1)} - \alpha_{j-1}) | + \sum_{j=0}^{n} \alpha_j \geq \\
&\geq e^+ + e^- - 2 + \sum_{j=2}^{i} (| B_{j-1}^+ - B_{j-1}^- | - | \alpha_{n-(j-1)} - \alpha_{j-1} |) + \sum_{j=0}^{n} \alpha_j \geq \\
&\geq e^+ + e^- - 2 + \sum_{j=2}^{i} (| B_{j-1}^+ - B_{j-1}^- | - | \alpha_{n-(j-1)} | - | \alpha_{j-1} |) + \sum_{j=1}^{n-1} \alpha_j \geq \\
&\geq e^+ + e^- - 2 + \sum_{j=2}^{i} | B_{j-1}^+ - B_{j-1}^- |
\end{aligned}
$$

To show that the lower bound is taken, let $\alpha_j = \alpha_{n-j} = 0$ for $j = 0 \ldots i$. $\triangle$

Again there is no waste of singularities in the sense that each singularity is necessary to make the edges contributions and the Betti numbers vanish.

2.2.2 Distribution of the $h_{\min}$ singularities according to the index

Convention From now on $h_0 = h_n = 0$.

Notation We shall denote by $\underline{v}_e \in \mathbf{N}_0^{n-1}$ the following vector, associated with the edges contribution to the number of singularities:

$$
\underline{v}_e = (\quad e^- - 1, \quad 0, \ldots, 0, \quad e^+ - 1 \quad)
$$

Let us first consider the singularities altering the first Betti number and denote by $\mathcal{V}_1$ the set of vectors

$$
\mathcal{V}_1 = \{\underline{v}_1(k_1)\}_{k_1=0}^{|B_1^+ - B_1^-|}
$$

where $\underline{v}_1(k_1)$ is defined as below:

$$
\begin{cases}
\underline{v}_1(k_1) = (\quad k_1, \quad 0, \quad 0, \quad \ldots, \quad 0, \quad |B_1^+ - B_1^-| - k_1, \qquad 0 \qquad) & \text{if } B_1^+ \geq B_1^- \\
\underline{v}_1(k_1) = (\quad 0, \quad k_1, \quad 0, \quad \ldots, \quad 0, \qquad 0, \qquad |B_1^+ - B_1^-| - k_1 \ ) & \text{otherwise}
\end{cases}
$$

In general, as for the singularities altering the j-th Betti number, let us denote by $\mathcal{V}_j$ the set of vectors

$$
\mathcal{V}_j = \left\{ \underline{v}_j(k_j) \right\}_{k_j=0}^{|B_j^+ - B_j^-|} \qquad \text{for } j = 2 \ldots i - 1
$$

where $\underline{v}_j(k_j)$ is defined as below:

$$
\begin{cases}
\underline{v}_j(k_j) = (\quad 0,\ldots,0, \quad \overbrace{k_j}^{j}, \quad 0, \quad 0,\ldots,0, \quad \overbrace{|B_j^+ - B_j^-| - k_j}^{n-j-1}, \qquad 0, \qquad 0,\ldots,0 \ ) & \text{if } B_j^+ \geq B_j^- \\
\underline{v}_j(k_j) = (\quad 0,\ldots,0, \quad 0, \quad \underbrace{k_j}_{j+1}, \quad 0,\ldots,0, \qquad 0, \qquad \underbrace{|B_j^+ - B_j^-| - k_j}_{n-j}, \quad 0,\ldots,0 \ ) & \text{otherwise}
\end{cases}
$$

Observe that, for $j = i - 1$, the vector $\underline{v}_{i-1}(k_{i-1})$ has coordinates

$$
\begin{cases}
\underline{v}_{i-1}(k_{i-1}) = (\quad 0,\ldots,0, \quad \overbrace{k_{i-1}}^{i-1}, \quad \overbrace{|B_{i-1}^+ - B_{i-1}^-| - k_{i-1}}^{i}, \qquad 0, \qquad 0,\ldots,0 \ ) & \text{if } B_{i-1}^+ \geq B_{i-1}^- \\
\underline{v}_{i-1}(k_{i-1}) = (\quad 0,\ldots,0, \quad 0, \quad \underbrace{k_{i-1}}_{i}, \quad \underbrace{|B_{i-1}^+ - B_{i-1}^-| - k_{i-1}}_{i+1}, \quad 0,\ldots,0 \ ) & \text{otherwise}
\end{cases}
$$

Proposition 2.2.2. *In even dimension n, $n = 0 \mod 4$, given positive integers e^+ and e^- and integers $B_j^+ - B_j^-$, $j = 1,\ldots,i-1$, a vector $\underline{h} \in \mathbf{N}_0^{n-1}$ satisfies the Poincaré-Hopf inequalities (1) and realizes the minimum if and only if it can be written as*

$$
\underline{h} = \underline{v}_e + \sum_{j=1}^{i-1} \underline{v}_j(k_j) \qquad \text{where } \underline{v}_j(k_j) \in \mathcal{V}_j \quad \forall j = 1,\ldots,i-1
$$

($\mathcal{V}_j$ as in the previous notation)

Proof The same as in Proposition 2.1.2, without the difficulty of the middle dimension. $\triangle$

Note that there is no restriction to the choice of the k_j's in $\{0 \ldots |B_j^+ - B_j^-|\}$, hence

Corollary 2.2.3. *the total number of vectors $\underline{h}$ realizing $h_{\min}$ is $\prod_{j=1}^{i-1}(|B_j^+ - B_j^-| + 1)$.*

Also in the even case, we can deduce that knowing a decomposition of $\underline{h}$, we know a sequence of type of singularities realizing $\underline{h}$. In fact, as we did in the odd case, we can define a map g as follows

$$g : \quad \mathbf{N}_0^{n-1} \quad \longrightarrow \quad \mathbf{N}_0^{2n-2}$$

$$\underline{v}_e \quad \longrightarrow \quad (h_1^c = e^- - 1, h_1^d = 0, \ldots, h_{n-1}^c = 0, h_{n-1}^d = e^+ - 1,)$$

$$\underline{v}_j(k_j) \quad \longrightarrow \quad (h_1^c = 0, \ldots, h_j^c = 0, \overbrace{h_j^d = k_j}^{2j}, h_{j+1}^c = 0, \ldots, \overbrace{h_{n-j-1}^d = |B_j^+ \geq B_j^- - k_j|}^{2n-2j-2}, \ldots, h_{n-1}^d = 0)$$
$$\text{if } B_j^+ \geq B_j^- \quad \forall j = 1,\ldots,i-1$$

$$(h_1^c = 0, \ldots, h_j^c = 0, \underbrace{h_{j+1}^c = k_j}_{2j+1}, h_{j+1}^d = 0, \ldots, \underbrace{h_{n-j}^c = |B_j^+ \geq B_j^- - k_j|}_{2n-2j-1}, \ldots, h_{n-1}^d = 0)$$
$$\text{if } B_j^+ \leq B_j^- \quad \forall j = 1,\ldots,i-1$$

Corollary 2.2.4. *Following the notation right above, if $\underline{h} = \underline{v}_e + \sum_{j=1}^{i-1} \underline{v}_j(k_j)$ then $g(\underline{v}_e) + \sum_{j=1}^{i-1} g(\underline{v}_j(k_j))$ belongs to $\mathcal{H}^{cd}(\underline{h})$.*

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2.2.3 Finding all the possible types of singularities

As we did in the odd case, now that we have the distributions of the $h_{\min}$ singularities according to the index, we can apply to each one of them the algorithm of [BMRez], and find all the possible matching types.

Proposition 2.2.5. *In even dimension n, $n = 0 \mod 4$, each vector $\underline{h} \in \mathbf{N}_0^{n-1}$ realizing the minimum determines the types of singularities uniquely.*

Proof Let us recall that, according to [BMRez], for a given $\underline{h}$ the number of solutions is given by the product $\prod_{j=1}^{i-1} n_j$, where n_j is given by

$$
\begin{cases}
n_j & = \min\{h_{j+1} - \hat{\beta}_j, h_{n-(j+1)}\} - \max\{0, -\hat{\beta}_j\} + 1 & \text{if } j \text{ is odd, } j \neq i-1 \\
n_{i-1} & = \frac{(h_i - \hat{\beta}_{i-1})}{2} - \max\{0, -\hat{\beta}_{i-1}\} + 1 & \text{if } j \text{ is odd, } j = i-1 \\
n_j & = \min\{h_{n-(j+1)} - \hat{\beta}_j, h_{j+1}\} - \max\{0, -\hat{\beta}_j\} + 1 & \text{if } j \text{ is even}
\end{cases}
$$

where

$$
\begin{cases}
\hat{\beta}_j & = \sum_{\ell=1}^{j}(-1)^{\ell}(B_\ell^+ - B_\ell^-) - \sum_{\ell=1}^{j+1}(-1)^{\ell}(h_\ell - h_{n-\ell}) - (e^- - e^+) \\
\hat{\beta}_{i-1} & = -(B_{i-1}^+ - B_{i-1}^-) + \hat{\beta}_{i-2} - h_{i+1} + h_{i-1}
\end{cases}
$$

For all $j = 1, \ldots, i-2$, computation gives $n_j = 1$ exactly in the same way as in Proposition 2.1.2. As for n_{i-1}, first remark that $i - 2$ is even, hence

$$
\begin{array}{ll}
\text{if } B_{i-2}^+ \geq B_{i-2}^- \text{ then} & \hat{\beta}_{i-2} = (B_{i-2}^+ - B_{i-2}^-) - k_{i-2} \\
\text{else} & \hat{\beta}_{i-2} = -k_{i-2}
\end{array}
$$

Further computation yields

$$
\begin{array}{ll}
\text{if } B_{i-1}^+ \geq B_{i-1}^- \quad \text{then} & \hat{\beta}_{i-1} = -[(B_{i-1}^+ - B_{i-1}^-) - k_{i-1}] \\
& h_i = -\hat{\beta}_{i-1} \\
& \frac{(h_i - \hat{\beta}_{i-1})}{2} = \max\{0, -\hat{\beta}_{i-1}\} = (B_{i-1}^+ - B_{i-1}^-) - k_{i-1} \\
\text{else} & \hat{\beta}_{i-1} = k_{i-1} \\
& h_i = k_{i-1} = \hat{\beta}_{i-1} \\
& \frac{(h_i - \hat{\beta}_{i-1})}{2} = \max\{0, -\hat{\beta}_{i-1}\} = 0
\end{array}
$$

that is, $n_{i-1} = 1$, and we are done. $\triangle$

Corollary 2.2.6. *Let n be even, $n = 0 \mod 4$, $\underline{h} = \underline{v}_e + \sum_{j=1}^{i-1} \underline{v}_j(k_j)$ and g as in Corollary 2.2.4. Then the unique element of $\mathcal{H}^{cd}(\underline{h})$ can be written as $g(\underline{v}_e) + \sum_{j=1}^{i-1} g(\underline{v}_j(k_j))$.*

2.3 Even dimension $n = 2 \mod 4$

2.3.1 Computation of $h_{\min}$

Proposition 2.3.1. *In even dimension n, $n = 2 \mod 4$, given positive integers e^+ and e^- and integers $B_j^+ - B_j^-$, $j = 1, \ldots, i-1$, the minimal number of singularities needed in order to have continuation is*

$$
h_{\min} = \min_{\substack{\{\underline{h} \in \mathbf{N}_0^{n+1} \\ satisfying \ (1) \ and \ (3)\}}} \sum_{j=0}^{n} h_j = e^+ + e^- - 2 + \sum_{j=1}^{i-1} |\, B_j^+ - B_j^- \,|
$$

Proof We use the same notation (5) established in Section 2 for the odd case. In exactly the same way followed in the proof of Proposition 2.1.1 we get the same estimation

$$\sum_{j=0}^{n} h_j \geq e^+ + e^- - 2 + \sum_{j=2}^{i} \mid B_{j-1}^+ - B_{j-1}^- \mid$$

Up to now, we haven't used condition (3) yet. To show that the lower bound is taken, let $\alpha_j = \alpha_{n-j} = 0$ for $j = 0 \ldots i$. To show that condition (3) holds for such a solution, just observe that the parity of that expression is equivalent to the parity of

$$h_i + (B_{i-1}^+ - B_{i-1}^-) - (\alpha_{i+1} - \alpha_{i-1})$$

(use notation (5) and the fact that i is odd). $\triangle$

Again there is no waste of singularities, each singularity being necessary in order to reduce the differences of the Betti numbers and the edges contributions.

2.3.2   Distribution of the $h_{\min}$ singularities according to the index

Convention From now on $h_0 = h_n = 0$.

Notation We keep the same notation as in Subsection 2.2.2: the definitions of the vectors $\underline{v}_j(k_j) \in \mathcal{V}_j$ corresponding to the action on the Betti numbers are the same.

Proposition 2.3.2. *In even dimension n, $n = 2 \mod 4$, given positive integers e^+ and e^- and integers $B_j^+ - B_j^-$, $j = 1, \ldots, i-1$, a vector $\underline{h} \in \mathbf{N}_0^{n-1}$ satisfies the Poincaré-Hopf inequalities (1) and (3) and realizes the minimum if and only if it can be written as*

$$\underline{h} = \underline{v}_e + \sum_{j=1}^{i-1} \underline{v}_j(k_j) \qquad \text{where } \underline{v}_j(k_j) \in \mathcal{V}_j \quad \forall j = 1, \ldots, i-1$$

($\mathcal{V}_j$ as in the previous notation)

Proof The same as in Proposition 2.1.2, except that also condition (3) must hold. This is a straightforward check, the parity of the expression being that of

$$\begin{array}{ll} 2(B_{i-1}^+ - B_{i-1}^-) & \text{if } B_{i-1}^+ \geq B_{i-1}^- \\ -2(\mid B_{i-1}^+ - B_{i-1}^- \mid -k_{i-1}) & \text{otherwise} \end{array} \qquad \triangle$$

There is no restriction to the choice of the k_j's in $\{0 \ldots \mid B_j^+ - B_j^- \mid\}$.

Corollary 2.3.3. *The total number of vectors $\underline{h}$ realizing $h_{\min}$ is $\prod_{j=1}^{i-1}(|B_j^+ - B_j^-| + 1)$.*

As in the previous situations, knowing a decomposition of $\underline{h}$ is knowing the type of singularities realizing $\underline{h}$.

Corollary 2.3.4. *For g as in Corollary 2.2.4, if $\underline{h} = \underline{v}_e + \sum_{j=1}^{i-1} \underline{v}_j(k_j)$ then $g(\underline{v}_e) + \sum_{j=1}^{i-1} g(\underline{v}_j(k_j))$ belongs to $\mathcal{H}^{cd}(\underline{h})$.*

2.3.3   Finding all the possible types of singularities

As we did in the odd case, now that we have the distributions of the $h_{\min}$ singularities according to the index, we can apply to each one of them the algorithm of [BMRez], and find all the possible matching types.

Proposition 2.3.5. *In even dimension n, $n = 2 \mod 4$, each vector $\underline{h} \in \mathbf{N}_0^{n-1}$ realizing the minimum determines the types of singularities uniquely.*

Proof Let us recall that, according to [BMRez], for a given $\underline{h}$ the number of solutions is given by the product $\prod_{j=1}^{i-1} n_j$, where n_j is given by

$$
\begin{cases}
n_j & = \min\{h_{j+1} - \hat{\beta}_j, h_{n-(j+1)}\} - \max\{0, -\hat{\beta}_j\} + 1 & \text{if } j \text{ is odd,} \\
n_j & = \min\{h_{n-(j+1)} - \hat{\beta}_j, h_{j+1}\} - \max\{0, -\hat{\beta}_j\} + 1 & \text{if } j \text{ is even}
\end{cases}
$$

where

$$
\hat{\beta}_j = \sum_{\ell=1}^{j}(-1)^\ell(B_\ell^+ - B_\ell^-) - \sum_{\ell=1}^{j+1}(-1)^\ell(h_\ell - h_{n-\ell}) - (e^- - e^+)
$$

Hence the proof is exactly the same as that of Proposition 2.1.2. $\triangle$

Corollary 2.3.6. *Let n be even, $n = 2 \mod 4$, $\underline{h} = \underline{v}_e + \sum_{j=1}^{i-1} \underline{v}_j(k_j)$ and g as in Corollary 2.2.4. Then the unique element of $\mathcal{H}^{cd}(\underline{h})$ can be written as $g(\underline{v}_e) + \sum_{j=1}^{i-1} g(\underline{v}_j(k_j))$.*

3 Topological aspects

In this section we will consider the realization of minimal flows on compact manifolds making use of abstract Lyapunov semi-graphs. We also obtain the formulas of the Euler characteristic for these minimal flows. In the odd case, $\chi(M, \partial M)$ depends only on homological boundary information i.e., e^+, e^- and the differences $B_j^+ - B_j^-$ and not on the singularities realizing $h_{\min}$. In the even case, although $\chi(M, \partial M)$ depends on $h_{\min}$ and hence on homological boundary information, it also depends on the singularities realizing $h_{\min}$.

3.1 Euler characteristic

It is well known that $\chi(M, \partial M) = \sum_{j=0}^{n}(-1)^j h_j$.

In general we prove the following formulas for the Euler characteristic of compact manifolds.

Proposition 3.1.1. *Given $(M, \partial M)$ abstractly in terms of positive integers e^+, e^- and integers corresponding to the differences (of Betti numbers) $B_j^+ - B_j^-$ where $j = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ (n being the dimension), we have:*

1. if $n = 2i + 1$, then

$$
\chi(M, \partial M) = \sum_{j=1}^{i-1}(B_j^+ - B_j^-) + \frac{B_i^+ - B_i^-}{2} + (e^+ - e^-)
$$

2. if $n = 2 \mod 4$, then

$$
\chi(M, \partial M) = \sum_{j=1}^{i}(B_j^+ - B_j^-) + (e^+ - e^-) \mod 2
$$

Proof Consider the definition of $\chi(M, \partial M)$ given above. Item 1 is just equality (2) holding in the case of odd dimension. Item 2 is condition (3). $\triangle$

We now consider the realization of $h_{\min}$ as a minimal Morse flow on some compact manifold M respecting the homological boundary information. The following proposition asserts that in the odd case, $\chi(M, \partial M)$ neither depends on M nor on the choice of singularities that realize the minimal flow and depends only on homological boundary information.

In the even case, it is still true that $\chi(M, \partial M)$ depends on homological boundary information, however it will depend on the minimal flow, i.e. on the choice of singularities realizing $h_{\min}$. We present the range of values taken by $\chi(M, \partial M)$ for all minimal flows on M realizing $h_{\min}$. Our formula not only generalizes in the minimal setting item 2 of Proposition 3.1.1 since it is true for any even dimension but also is more precise.

Proposition 3.1.2. *Let $(M, \partial M)$ be abstractly given in terms of positive integers e^+, e^- and integers corresponding to the differences (of Betti numbers) $B_j^+ - B_j^-$ where $j = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ (n being the dimension). Recall that $\mathcal{H} = \{(h_1, \ldots, h_{n-1}) : \sum_{i=1}^{n-1} h_i = h_{\min} : \text{Poincaré-Hopf inequalities} \text{ are satisfied}\}$ and denote by $\chi_{\min}(M, \partial M)$ the set of values of $\chi(M, \partial M)$ taken over $\mathcal{H}$. Then*

1. if $n = 2i + 1$, $\chi_{\min}(M, \partial M)$ is the singleton given by

$$\chi_{\min}(M, \partial M) = \left\{ \sum_{j=1}^{i-1}(B_j^+ - B_j^-) + \frac{B_i^+ - B_i^-}{2} + (e^+ - e^-) \right\}$$

2. if n is (any) even dimension, then

$$\chi_{\min}(M, \partial M) = \{h_{\min} - 2p, \ p = 0, \ldots, h_{\min}\}$$

Proof Item 1 is as in the proposition above or can be proved by direct computation (use Proposition 2.1.2). Item 2 follows by direct computation. First of all, remark that, in general

$$\mid \chi(M, \partial M) \mid = \mid \sum_{j=0}^{n}(-1)^j h_j \mid \leq \sum_{j=0}^{n} h_j$$

and in the minimal case the bound is $h_{\min}$. We show that the bound is taken by a particular vector of $\mathcal{H}$. Recall the decomposition of the vectors of $\mathcal{H}$ in the even case (Propositions 2.2.2 and 2.3.2) and choose $\underline{v}_j(k_j)$ in the following way:

if j is odd then	if $B_j^+ \geq B_j^-$ then	$n - j - 1$ is even
		choose $\underline{v}_j(k_j = 0)$ (all the singularities are in h_{n-j-1})
	else if $B_j^+ \leq B_j^-$ then	$j + 1$ is even
		choose $\underline{v}_j(k_j = \mid B_j^+ - B_j^- \mid)$ (all the singularities are in h_{j+1})
if j is even then	if $B_j^+ \geq B_j^-$ then	choose $\underline{v}_j(k_j = \mid B_j^+ - B_j^- \mid)$ (all the singularities are in h_j)
	else if $B_j^+ \leq B_j^-$ then	$n - j$ is even
		choose $\underline{v}_j(k_j = 0)$ (all the singularities are in h_{n-j}).

With these choices, we have a minimal flow on M and the corresponding Euler characteristic is $\chi(M, \partial M) = h_{\min}$. Each time we change the value of one of the k_js by one (hence considering another minimal flow), we decrease the value of $\chi(M, \partial M)$ by 2 and we are done since we can do it exactly $h_{\min}$ times. $\triangle$

3.2 Lyapunov Graphs of Morse Type

Given the abstract data e^+, e^- and the differences $B_j^+ - B_j^-$ we can associate a family $\mathcal{F}$ of directed semi-graphs with one vertex with e^+ incoming and e^- outgoing edges. An element in this family has its edges labelled with specific Betti numbers such that the differences are satisfied. Any choice of labelling is admissible as long as it satisfies the differences.

Starting from the same abstract data, we have shown in this paper how to compute the minimal number of singularities $h_{\min}$ (Propositions 2.1.1, 2.2.1 and 2.3.1). We have also determined the set $\mathcal{H}$ of vectors realizing $h_{\min}$ (Propositions 2.1.2, 2.2.2 and 2.3.2). Hence we can label the vertex of

any semi-graph of $\mathcal{F}$ with any $\underline{h} \in \mathcal{H}$ thus obtaining an abstract Lyapunov semi-graph as defined in Section 1.

We have also shown here that the set $\mathcal{H}^{cd}(\underline{h})$ of $(h_1^c, h_1^d, \ldots, h_{n-1}^c, h_{n-1}^d)$ realizing $\underline{h}$ and the abstract data is a singleton. Now, with these data we have an abstract Lyapunov semi-graph of Morse type, $L(\underline{h})$, which is unique up to permutation of the labels of the vertices. All this is equivalent to saying that for each $\underline{h}$ we have a unique family $\mathcal{F}_{L(\underline{h})}$ of abstract Lyapunov semi-graphs of Morse type. Given $\underline{h}$, each element of the family $\mathcal{F}_{L(\underline{h})}$ is again determined by fixing the labels of the edges (satisfying the differences $B_j^+ - B_j^-$ given a priori).

The process of obtaining all the possible abstract Lyapunov semi-graphs of Morse type from an abstract Lyapunov semi-graph is called *continuation* and developed for the first time in [BMRez]. Uniqueness has been proved in this paper by using the results of [BMRez] on the number of continuations of a given abstract Lyapunov semi-graph.

Last, as well as the vectors of $\mathcal{H}$ are linked by the fact of realizing $h_{\min}$, the corresponding families of Lyapunov semi-graphs of Morse type are linked by the fact that one can obtain one semi-graph from the other by replacing the label of one vertex by the type of singularity having the same algebraic effect on the Betti numbers, as shown in the example below.

Furthermore, following [CrRez], define a *null pair* of types of singularities as the pairs of singularities having the opposite algebraic effect on the same Betti numbers and with consecutive indices, i.e. h_j of type j-d and h_{j+1} of type j-c. Define also a *dual pair* of types of singularities as the pairs of singularities having the opposite algebraic effect on the same Betti numbers, with complementary indices, i.e. h_j of type j-d and h_{n-j} of type $(n-j-1)$-c. Formulas for $h_{\min}$ show that there is no waste of singularities, hence the labels of the corresponding families of abstract Lyapunov semi-graphs of Morse type contain neither null pairs nor dual pairs. In other words, such abstract Lyapunov semi-graphs possess lowest topological complexity.

3.3 Example

Consider the same homological boundary information as in the example of Subsection 2.1.4.

$$\{e^+ = 2, e^- = 3, B_1^+ - B_1^- = -2, B_2^+ - B_2^- = -2\}$$

Then from Proposition 3.1.2 we have $\chi_{\min}(M, \partial M) = \{-4\}$.

Now we fix the labels of the edges satisfying our initial data, for instance as in the example of Figure 3: this directed semi-graph is an element of the family $\mathcal{F}$.
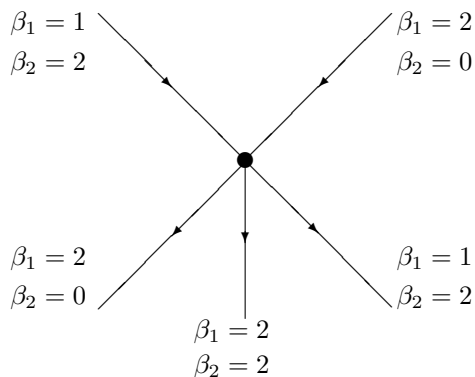

Figure 3: Labels respecting $\{e^+ = 2, e^- = 3, B_1^+ - B_1^- = -2, B_2^+ - B_2^- = -2\}$

Using results of the example of Subsection 2.1.4 applied to this specific directed semi-graph we have three abstract Lyapunov semi-graphs, one for each $\underline{h}$ realizing $h_{\min}$ (Figure 3.3).
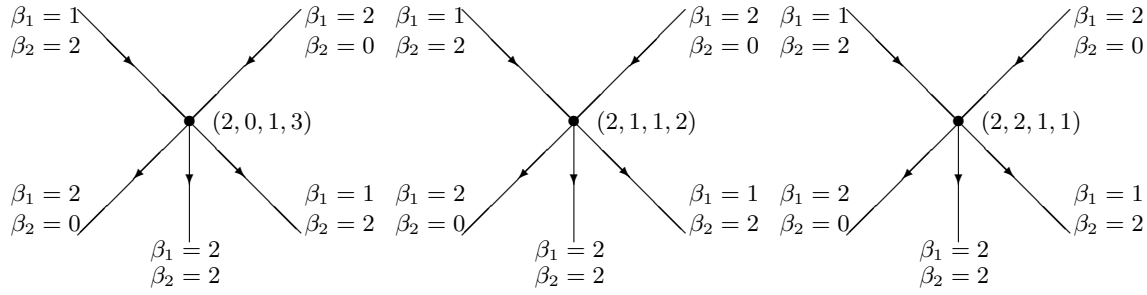
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Figure 4: The three abstract Lyapunov semi-graphs

Hence we have three abstract Lyapunov semi-graphs of Morse type (Figure 3.3), respectively in $\mathcal{F}_{L(2,0,1,3)}$, $\mathcal{F}_{L(2,1,1,2)}$ and $\mathcal{F}_{L(2,2,1,1)}$. Observe that we can obtain one abstract Lyapunov semi-graph of Morse type from another one by replacing singularities of type 3-c with singularities of type 1-c, both having the algebraic effect of decreasing β_1. Furthermore, note that in these Lyapunov linear semi-graphs for each j, β_j is strictly decreasing or increasing as one walks on the graph following the opposite orientation of the directed edges. We can easily see that this implies that these Lyapunov linear semi-graphs possess neither dual pairs nor null pairs. Hence these semi-graphs possess the lowest topological complexity.
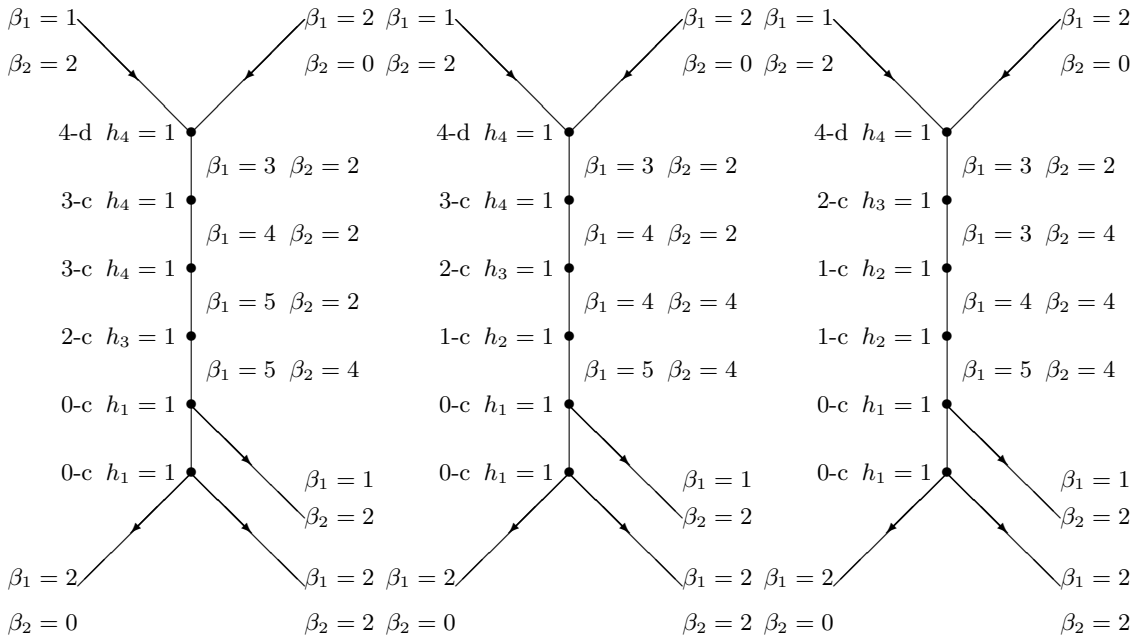

Figure 5: The three abstract Lyapunov semi-graphs of Morse type

It is worth mentioning once more that any odd-dimensional compact manifold M realizing the Lyapunov semi-graphs of Morse type above has the same Euler characteristic as was shown in Subsection 3.1. However, if we had even dimensional Lyapunov semi-graphs of Morse type, each graph would determine an Euler characteristic which is the same for any compact manifold realizing it. As in Subsection 3.1 the range of the Euler characteristics is determined in this case.

References

[BMRez] M. A. Bertolim, M. P. Mello and K. A. de Rezende *Lyapunov graph continuation.* Ergodic Theory and Dynamical Systems, (2003), 23, pp. 1-58.

[Co] C. Conley. *Isolated invariant sets and the Morse index.* CBMS Regional Conference Series in Mathematics, 38. American Mathematical Society, Providence, R.I., 1978.

[CrRez] R. N. Cruz and K. A. de Rezende. *Gradient-like flows on high-dimensional manifolds.* Ergodic Theory Dynam. Systems 19 (1999), no. 2, 339–362.

[Rez] K. A. de Rezende. *Smale flows on the three-sphere.* Transactions of the American Mathematical Society 303 (1987), no. 1, 283–310.

[F] J. Franks. *Nonsingular Smale flows on S^3.* Topology 24 (1985), no. 3, 265–282.

[Rei] Reineck, James F. *Continuation to the minimal number of critical points in gradient flows.* Duke Math. J. 68 (1992), no. 1, 185–194.