The CPLD condition of Qi and Wei implies the quasinormality constraint qualification

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Abstract

The Constant Positive Linear Dependence (CPLD) condition for feasible points of nonlinear programming problems was introduced by Qi and Wei and used for the analysis of SQP methods. In the paper where the CPLD was introduced, the authors conjectured that this condition could be a constraint qualification. This conjecture is proved in the present paper. Moreover, it will be shown that the CPLD condition implies the quasinormality constraint qualification, but the reciprocal is not true. Relations with other constraint qualifications will be given.

Key words: Nonlinear Programming, Constraint Qualifications, CPLD condition, Quasi-normality.

1 Introduction

A constraint qualification is a property of feasible points of nonlinear programming problems that, when satisfied by a local minimizer, guarantees that the KKT conditions take place at that point. See, for example, [1]. When a constraint qualification is fulfilled it is possible to think in terms of Lagrange multipliers and, consequently, efficient algorithms based on duality ideas can be defined.

From the theoretical point of view it is interesting to find weak constraint qualifications. On the other hand, it is desirable that the fulfillment of a constraint qualification should be easy to verify.

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The most widely used constraint qualification is the linear independence of the gradients of the active constraints at the given point. However, many weaker constraint qualification properties exist. For many optimization algorithms convergence theorems can be proved that say that if a feasible limit point satisfies some constraint qualification then the KKT conditions hold at this point. Clearly, the weaker is the constraint qualification used at such a theorem, the stronger the convergence theorem turns out to be.

The CPLD condition was introduced by Qi and Wei in [2] and used to analyze SQP algorithms. Theorem 4.2 of [2] says that if a limit point of a general SQP method satisfies some conditions that include CPLD, then this point is KKT. However, they did not prove that the CPLD condition is a constraint qualification. This question was presented as an open problem in Section 2 of [2]. Observe that, if a solution of the nonlinear programming problem existed satisfying CPLD (and the other hypotheses of Theorem 4.2 of [2]) but not satisfying KKT, this solution would be impossible to find by the analyzed algorithm. Therefore, it is important to prove that CPLD is a constraint qualification, so that the existence of such minimizers is impossible.

Hestenes ([3], page 296) introduced a very general constraint qualification called quasinormality. See, also, ([1], page 337). We will prove that CPLD implies quasinormality and, thus, CPLD is also a constraint qualification. Moreover, we will show that quasinormality is strictly weaker than CPLD.

This paper is organized as follows. In Section 2 we state the main definitions. In Section 3 we prove that CPLD implies quasinormality. In Section 4 we state the relationships of CPLD with other constraint qualifications. Conclusions are given in Section 5.

2 Definitions

Let $h : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ be continuosly differentiable. Define the *feasible set* X as

$$X = \{x \in \mathbb{R}^n \mid h(x) = 0,\ g(x) \leq 0\}.$$

For all $x \in X$, define the set of indices of the active inequality constraints as

$$I(x) = \{j \in \{1, \ldots, p\} \mid g_j(x) = 0\}$$

We say that $x \in X$ is a *MF-nonregular* if there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, $\mu_j \geq 0\ \forall\ j \in I(x)$, $\sum_{i=1}^m |\lambda_i| + \sum_{j=1}^p \mu_j \neq 0$ such that

$$\sum_{i=1}^m \lambda_i \nabla h_i(x) + \sum_{j \in I(x)} \mu_j \nabla g_j(x) = 0.$$

An MF-nonregular point is exactly a point that does not satisfy the Mangasarian-Fromovitz (MF) constraint qualification. See [4, 5]. Feasible points that are not MF-nonregular will be called *MF-regular*.

We say that $x \in X$ satisfies the *CPLD condition* (see [2]) if it is MF-regular or, being MF-nonregular with scalars λ_i, μ_j, the set of vectors

$$\{\nabla h_i(y) \mid \lambda_i \neq 0, i = 1, \ldots, m\} \cup \{\nabla g_j(y) \mid \mu_j > 0, j \in I(x)\} \tag{1}$$

is linearly dependent for all y in some neighborhood (not restricted to feasible points) of x.

In other words, the CPLD condition says that linear dependence of the vectors (1) takes place whenever the Mangasarian-Fromovitz constraint qualification [4, 5] is not fulfilled at x. So, the Mangasarian-Fromovitz constraint qualification implies the CPLD condition. The reciprocal is not true. Consider, for example, the feasible set defined by $h_1(x) = x_1 = 0$ and $h_2(x) = x_1 = 0$.

We say that $x \in X$ with active constraints $I(x)$ satisfies the *sequential B-property* related to the scalars $\bar{\lambda}_1, \ldots, \bar{\lambda}_m, \bar{\mu}_j, j \in I(x)$ if there exists a sequence of (not necessarily feasible) points $y_k \in I\!R^n$ such that $\lim_{k\to\infty} y_k = x$ and, for all $i \in \{1, \ldots, m\}$ such that $\bar{\lambda}_i \neq 0$ and all $j \in I(x)$ such that $\bar{\mu}_j \neq 0$,

$$\bar{\lambda}_i h_i(y_k) > 0 \text{ and } \bar{\mu}_j g_j(y_k) > 0$$

for all $k \in \{0, 1, 2, \ldots\}$.

We say that $x \in X$ satisfies the *quasinormality constraint qualification* [1, 3] if it is MF-regular or, being MF-nonregular with scalars $\lambda_1, \ldots, \lambda_m, \mu_j, j \in I(x)$, it does not satisfy the sequential B-property related to $\lambda_1, \ldots, \lambda_m, \mu_j, j \in I(x)$. In other words, the quasinormality constraint qualification says that the sequential B-property cannot be true if the Mangasarian-Fromovitz constraint qualification is not fulfilled at x.

Our strategy for proving that CPLD implies quasinormality will consist in showing that the linear dependence of the vectors (1) implies that the sequential B-property cannot be true.

3 Main results

The proof that CPLD implies quasinormality needs two preparatory results of multidimensional Calculus. These results are given in Lemmas 1 and 2.

Lemma 1. *Let $D \subset I\!R^n$ be an open set, $x \in D$, $F : D \to I\!R^n$, $F = (f_1, \ldots, f_n) \in C^1(D)$, $F'(x)$ nonsingular. (By the Inverse Function Theorem F is an homeomorphism between an open neighborhood D_1 of x and $F(D_1)$ and $F^{-1} : F(D_1) \to D_1$ is well defined.)*

Assume that $q \in \{1, \ldots, n-1\}$, $f : D \to I\!R$, $f \in C^1(D)$ is such that $\nabla f(y)$ is a linear combination of $\nabla f_1(y), \ldots, \nabla f_q(y)$ for all $y \in D$.

Define, for all $u \in F(D_1)$,

$$\varphi(u) = f[F^{-1}(u)]. \tag{2}$$

Then, for all $u \in F(D_1), j = q + 1, \ldots, n$,

$$\frac{\partial \varphi}{\partial u_j}(u) = 0. \tag{3}$$

Proof. Apply the Chain Rule to (2). □

Lemma 2. *Let $f, f_1, \dots, f_q : D \subset I\!R^n \to I\!R$ continuously differentiable, $x \in D$, D an open set.*

Assume that $\nabla f_1(x), \dots, \nabla f_q(x)$ are linearly independent and that $\nabla f(y)$ is a linear combination of $\nabla f_1(y), \dots, \nabla f_q(y)$ for all $y \in D$. In particular,

$$\nabla f(x) = \sum_{i=1}^{q} \lambda_i \nabla f_i(x). \tag{4}$$

Then, there exists $D_2 \subset I\!R^q$, an open neighborhood of $(f_1(x), \dots, f_q(x))$, and a function $\varphi : D_2 \to I\!R$, $\varphi \in C^1(D_2)$, such that, for all $y \in D_1$, we have that $(f_1(y), \dots, f_q(y)) \in D_2$ and

$$f(y) = \varphi(f_1(y), \dots, f_q(y)).$$

Moreover, for all $i = 1, \dots, q$,

$$\lambda_i = \frac{\partial \varphi}{\partial u_i}(f_1(x), \dots, f_q(x)). \tag{5}$$

Proof . Define $f_{q+1}, \dots, f_n$ in such a way that the hypotheses of Lemma 1 hold. Then, apply (3) and, finally, use the Chain Rule and the linear independence of $\nabla f_1(x), \dots, \nabla f_q(x)$ to deduce (5). □

Let us prove now the main result of this paper.

Theorem 1. *Assume that X, h, g are as defined in Section 2 and that $x \in X$ satisfies the CPLD condition. Then, x satisfies the quasinormality constraint qualification.*

Proof. Assume that $x \in X$ satisfies the CPLD condition. If x is MF-regular we are done. Suppose that x is MF-nonregular. Then, there exist $\lambda_1, \dots, \lambda_m \in I\!R, \mu_j \ge 0 \ \forall j \in I(x)$ such that

$$\sum_{i=1}^{m} |\lambda_i| + \sum_{j \in I(x)} \mu_j > 0 \tag{6}$$

and

$$\sum_{i=1}^{m} \lambda_i \nabla h_i(x) + \sum_{j \in I(x)} \mu_j \nabla g_j(x) = 0. \tag{7}$$

Define

$$I_+(x) = \{i \in \{1, \dots, m\} \mid \lambda_i > 0\},$$
$$I_-(x) = \{i \in \{1, \dots, m\} \mid \lambda_i < 0\},$$
$$I_0(x) = \{j \in I(x) \mid \mu_j > 0\}.$$

Then, by (7),

$$\sum_{i \in I_+(x)} \lambda_i \nabla h_i(x) + \sum_{i \in I_-(x)} \lambda_i \nabla h_i(x) + \sum_{j \in I(x)} \mu_j \nabla g_j(x) = 0. \tag{8}$$

By (6), $I_+(x) \cup I_-(x) \cup I_0(x) \neq \emptyset$.

Assume, first, that $I_+(x) \neq \emptyset$. Let $i_1 \in I_+(x)$. Then, by (8),

$$\lambda_{i_1} \nabla h_{i_1}(x) = -\sum_{i \in I_+(x)-\{i_1\}} \lambda_i \nabla h_i(x) - \sum_{i \in I_-(x)} \lambda_i \nabla h_i(x) - \sum_{j \in I_o(x)} \mu_j \nabla g_j(x). \tag{9}$$

We consider two possibilities:

(a) $\nabla h_{i_1}(x) = 0$;

(b) $\nabla h_{i_1}(x) \neq 0$.

If (a) holds, then, by the CPLD condition, $\nabla h_{i_1}(y) = 0$ for all y in a neighborhood of x. So, $h_{i_1}(y) = 0$ for all y in that neighborhood. Therefore, a sequence $y_k \to x$ such that $h_{i_1}(y_k) > 0$ for all k cannot exist. Therefore, the sequential B-property related to $\lambda_1, \ldots, \lambda_m, \mu_j, j \in I(x)$ cannot hold. Thus, quasinormality is fulfilled.

Assume now that (b) holds. Then, by the Caratheodory's Theorem for Cones (see, for example, [1], page 689), there exist

$$I_{++} \subset I_+(x) - \{i_1\},\ I_{--} \subset I_-(x),\ I_{oo} \subset I_o(x)$$

such that the vectors

$$\{\nabla h_i(x)\}_{i \in I_{++}},\ \{\nabla h_i(x)\}_{i \in I_{--}},\ \{\nabla g_j(x)\}_{j \in I_{oo}} \tag{10}$$

are linearly independent and

$$\lambda_{i_1} \nabla h_{i_1}(x) = -\sum_{i \in I_{++}} \bar\lambda_i \nabla h_i(x) - \sum_{i \in I_{--}} \bar\lambda_i \nabla h_i(x) - \sum_{j \in I_{oo}} \bar\mu_j \nabla g_j(x) \tag{11}$$

with

$$\bar\lambda_i > 0\ \forall\ i \in I_{++},$$

$$\bar\lambda_i < 0\ \forall\ i \in I_{--}$$

and

$$\bar\mu_j > 0\ \forall\ j \in I_{oo}.$$

By the linear independence of the vectors (10) and continuity arguments, the vectors

$$\{\nabla h_i(y)\}_{i \in I_{++}},\ \{\nabla h_i(y)\}_{i \in I_{--}},\ \{\nabla g_j(y)\}_{j \in I_{oo}} \tag{12}$$

are linearly independent for all y in a neighborhood of x. But, by the CPLD condition, the vectors

$$\lambda_{i_1} \nabla h_{i_1}(y), \{\nabla h_i(y)\}_{i \in I_{++}}, \{\nabla h_i(y)\}_{i \in I_{--}}, \{\nabla g_j(y)\}_{j \in I_{oo}}$$

are linearly dependent in a neighborhood of x. Therefore, $\lambda_{i_1} \nabla h_{i_1}(y)$ must be a linear combination of the vectors (12) for all y in a neighborhood of x.

For simplicity, and without loss of generality, let us write:

$$I_{++} = \{1, \ldots, m_1\},$$

$$I_{--} = \{m_1 + 1, \dots, m_2\},$$

$$I_{oo} = \{1, \dots, p_1\},$$

$$m_2 + p_1 = q.$$

By Lemma 2, there exists a smooth function φ defined in a neighborhood of $(0, \dots, 0) \in \mathbb{R}^q$ such that, for all y in a neighborhood of x,

$$\lambda_{i_1} h_{i_1}(y) = \varphi(h_1(y), \dots, h_{m_2}(y), g_1(y), \dots, g_{p_1}(y)). \tag{13}$$

Now, suppose that $\{y_k\}$ is a sequence that converges to x such that

$$h_i(y_k) > 0, i = 1, \dots, m_1,$$

$$h_i(y_k) < 0, i = m_1 + 1, \dots, m_2,$$

$$g_j(y_k) > 0, j = 1, \dots, p_1.$$

Then, by (5), (11) and (13), for k large enough we must have that $\lambda_{i_1} h_{i_1}(y_k) \leq 0$. This implies that the sequential B-property cannot hold.

The proofs for the cases $I_-(x) \neq \emptyset$ and $I_o(x) \neq \emptyset$ are entirely analogous to this case. Therefore, CPLD implies quasinormality as we wanted to prove. □

4 Relations with other constraint qualifications

If a local minimizer of a nonlinear programming problem satisfies CPLD, then, by Theorem 1, it satisfies the quasinormality constraint qualification and, so, it satisfies the KKT conditions. Therefore, the CPLD condition is a constraint qualification. In this section we prove some relations of CPLD with other constraint qualifications. The most obvious question is whether quasinormality implies CPLD. The following counterexample shows that this is not true.

Counterexample 1. *Quasi-normality does not imply CPLD.*

Take $n = 2, m = 2, p = 0,$

$$h_1(x_1, x_2) = x_2 e^{x_1}, \ h_2(x_1, x_2) = x_2, x_* = (0, 0).$$

We have that $\nabla h_1(x_*) = \nabla h_2(x_*)$. So, $\lambda_1 \nabla h_1(x_*) + \lambda_2 \nabla h_2(x_*) = 0$ implies that $\lambda_1 = -\lambda_2$. The sequential B-property does not hold because the sign of $h_1(x)$ is the same as the sign of $h_2(x)$ for all x. Therefore, x_* is quasinormal. However, at every neighborhood of x_* there are points where ∇h_1 and ∇h_2 are linearly independent. So, the CPLD condition is not satisfied at x_*.

The *Constant Rank constraint qualification* (CRCQ) was introduced by Janin [6]. We say that $x \in X$ satisfies the CRCQ if the linear dependence of a subset of gradients of active (equality or inequality) constraints at x implies that those gradients are linearly dependent for all y on some neighborhood of x. Several well known constraint qualifications with practical relevance imply CRCQ. For example, if the constraints are defined by affine functions, the CRCQ is fulfilled. Moreover, if $x \in X$ satisfies the CRCQ and each equality constraint $h_i(x) = 0$ is replaced by two inequality constraints ($h_i(x) \leq 0$ and $-h_i(x) \leq 0$) the CRCQ still holds with the new description of X. Note that the Mangasarian-Fromovitz constraint qualification does not enjoy this property.

Clearly, CRCQ implies CPLD. Since the Mangasarian-Fromovitz constraint qualification also implies CPLD, it is interesting to ask whether the CPLD condition is strictly weaker than CRCQ and Mangasarian-Fromovitz together. The following counterexample shows that, in fact, there are exist points that satisfy CPLD but do not satisfy neither CRCQ nor Mangasarian-Fromovitz.

Counterexample 2. *CPLD does not imply CRCQ∨MF.*

Take $n = 2, m = 0, p = 4$, $x_* = (0, 0)$,

$$\begin{aligned} g_1(x_1, x_2) &= x_1 \\ g_2(x_1, x_2) &= x_1 + x_2^2 \\ g_3(x_1, x_2) &= x_1 + x_2 \\ g_4(x_1, x_2) &= -x_1 - x_2. \end{aligned}$$

Since $\nabla g_3(x_*) + \nabla g_4(x_*) = 0$, x_* does not satisfy the Mangasarian-Fromovitz constraint qualification.

The gradients $\nabla g_1(x_*), \nabla g_2(x_*)$ are linearly dependent but in every neighborhood of x_* there are points where $\nabla g_1(x), \nabla g_2(x)$ are linearly independent. Therefore, x_* does not satisfy CRCQ.

Let us show that x_* satisfies CPLD. Assume that $\mu_j \geq 0, j = 1, \ldots, 4, \sum_{j=1}^4 \mu_j > 0$, and

$$\mu_1 \nabla g_1(x_*) + \mu_2 \nabla g_2(x_*) + \mu_3 \nabla g_3(x_*) + \mu_4 \nabla g_4(x_*) = 0. \tag{14}$$

If $\mu_2 = 0$, (14) implies that $\nabla g_1(x_*), \nabla g_3(x_*), \nabla g_4(x_*)$ are linearly dependent. So, since g_1, g_3, g_4 are linear, $\nabla g_1(x), \nabla g_3(x), \nabla g_4(x)$ are linearly dependent for all $x \in I\!R^2$.

Suppose that $\mu_2 > 0$. It is easy to see that this is impossible unless at least two of the multipliers μ_1, μ_3, μ_4 are positive. But, since three vectors in $I\!R^2$ are always linearly dependent it turns out that the CPLD condition holds.

We say that $x \in X$ with active constraints $I(x)$ satisfies the *sequential P-property* related to the scalars $\bar{\lambda}_1, \ldots, \bar{\lambda}_m, \bar{\mu}_j, j \in I(x)$ if there exists a sequence of (not necessarily feasible) points $y_k \in I\!R^n$ such that $\lim_{k \to \infty} y_k = x$ and

$$\sum_{i=1}^m \bar{\lambda}_i h_i(y_k) + \sum_{j \in I(x)} \bar{\mu}_j g_j(y_k) > 0. \tag{15}$$

We say that $x \in X$ satisfies the *pseudonormality constraint qualification* [7] if it is MF-regular or, being MF-nonregular with scalars $\lambda_1, \ldots, \lambda_m, \mu_j, j \in I(x)$, it does not satisfy the sequential P-property related to $\lambda_1, \ldots, \lambda_m, \mu_j, j \in I(x)$. In other words, the pseudonormality constraint qualification says that the sequential P-property cannot be true if the Mangasarian-Fromovitz constraint qualification is not fulfilled at x.

Clearly, the sequential B-property implies the sequential P-property. So, pseudonormality implies quasinormality. In the following counterexample we see that CPLD does not imply pseudonormality.

Counterexample 3. *CPLD does not imply pseudonormality.*

Take $n = 1, m = 0, p = 2$, $x_* = 0$,

$$g_1(x_1) = -x_1$$
$$g_2(x_1) = x_1 + x_1^2.$$

The point x_* is not MF-regular because $\nabla g_1(x_*) + \nabla g_2(x_*) = 0$. But $g_1(x_1) + g_2(x_1) = x_1^2 > 0$ for all $x_1 \neq x_*$, therefore the sequential P-property holds. Therefore, x_* is not pseudonormal.

Now, $\nabla g_1(x_*) \neq 0 \neq \nabla g_2(x_*)$ and $\nabla g_1(x), \nabla g_2(x)$ are linearly dependent for all $x \in I\!R$. Therefore, x_* satisfies the CRCQ. So, x_* also satisfies CPLD.

Counterexample 1 shows a situation where quasinormality takes place but CPLD does not. It is easy to see that, in this example, the point x_* is not pseudonormal. In fact, take $\lambda_1 = 1, \lambda_2 = -1$ and consider the sequence $y_k = (1/k, 1/k)$, for which it is trivial to see that (15) is fulfilled. However, the following counterexample shows that pseudonormality does not imply CPLD.

Counterexample 4. *Pseudo-normality does not imply CPLD.*

Take $n = 2, m = 0, p = 2$, $x_* = (0, 0)$,

$$g_1(x_1, x_2) = -x_1$$
$$g_2(x_1, x_2) = x_1 - x_1^2 x_2^2.$$

Then,

$$\nabla g_1(x_1, x_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \nabla g_2(x_1, x_2) = \begin{pmatrix} 1 - 2x_1 x_2^2 \\ -2x_1^2 x_2 \end{pmatrix}.$$

So, for all $\mu_1 = \mu_2 > 0$, we have:

$$\mu_1 \nabla g_1(0, 0) + \mu_2 \nabla g_1(0, 0) = \mu_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0. \tag{16}$$

However, if $\mu_1 = \mu_2 > 0$,

$$\mu_1 g_1(x_1, x_2) + \mu_2 g_2(x_1, x_2) = -\mu_1 x_1^2 x_2^2 < 0 \quad \forall (x_1, x_2) \neq (0, 0).$$

So, x_* is pseudonormal.

On the other hand, the gradients $\nabla g_1(x), \nabla g_2(x)$ are linearly independent if $x_1 \neq 0 \neq x_2$. This implies that the CPLD condition does not hold.

5 Conclusions

The CPLD condition seems to be an useful tool for the analysis of convergence of SQP methods. Qi and Wei [2] used this concept for studying convergence properties of a feasible SQP algorithm due to Panier and Tits [8]. Very likely, their methodology can be applied to other optimization algorithms. This condition is reasonably general, in the sense that is implied by the classical Mangasarian-Fromovitz constraint qualification, by the linearity of the constraints, by the constant-rank constraint qualification and by the linear independence of the gradients of active constraints. These features make it appropriate to be included as one of the standard constraint qualifications in text books.

Here we proved that the CPLD is, in fact, a constraint qualification, as conjectured in [2]. However, it is not as general as the quasinormality constraint qualification which, on the other hand, does not seem to be appropriate for the analysis of SQP-like algorithms.

Convergence results of algorithms associated to weak constraint qualifications are stronger than convergence results associated to strong constraint qualifications, as linear independence of gradients. The discovery of the status of different constraint qualifications opens different theoretical and practical questions related to minimization methods. On one hand, it is interesting to ask whether the stronger convergence properties are satisfied. On the other hand, it is interesting to verify whether the satisfaction of stronger convergence results has practical consequences.

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