

Influence Diagnostics for the Grubbs's Model

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Abstract

In this paper we consider applications of local influence (Cook, 1986) to evaluate small perturbations in the model or data set in several measuring devices, assuming Grubbs's model. Different perturbation schemes are investigated and an application is considered to two real data sets.

Key Words: Local influence, Measuring Devices, Grubbs's model.

1 Introduction

The main object of this paper is the study of local influence and diagnostic in the Grubbs's measurement model used to assess the relative quality of several measuring devices (or instruments) when measuring the same unknown quantity x in a common group of individuals or experimental units. Comparing measuring devices which varies in pricing, fastness and other features, such as efficiency, has been of growing interest in many engineering and scientific applications. Grubbs (1948, 1973, 1983) proposed a model for n items, each measured on p instruments. Outliers and detection of influent observations is an important step in the analysis of a data set. There are several ways of evaluating the influence of perturbations in the data set and in the model given the parameter estimates. Important reviews can be found in the books by Cook and Weisberg (1982) and Chatterjee and Hadi (1988) and in the paper by Cook (1986). On the other hand, there are just a few works in the literature for diagnostic and influence of observations in models with measurement errors.

Kelly (1984) considered a diagnostic procedure in the structural linear model based on the influence function. Tanaka et al. (1991) also consider the influence function introduced by Hampel for evaluating the influence of observations in the analysis of covariance structures. Zhao and Lee (1998) define leverage of one observation and Cook's distance in a simultaneous equation model. Rather than eliminating cases, the approach proposed by Cook (1986) is a general method for evaluating, under the maximum likelihood estimators, the influence of small perturbations in the model or data set. Additional results on local influence and applications in linear regression and mixed models can be found in Beckman et al. (1987), Lawrance (1988), Thomas and Cook (1990), Tsai and Wu (1992), Paula (1993), Galea et al. (1997) and Lesaffre and Verbeke (1998). Zhao and Lee (1998) and Kwan and Fung (1998) apply the local influence approach for factor analysis and simultaneous equations. Recently, Galea et al. (2002) apply the local influence method in functional and structural comparative calibration models. Thus, the main object of this paper is to apply the approach of local influence to the Grubbs's measurement models. Several perturbation schemes are considered such as case perturbation and response perturbation. In Section 2 the Grubbs's measurement model is considered and in Section 3 the main concepts of local influence are revised. In Section 4 model curvatures are considered for different perturbation schemes and in Section 5 an illustration of the methodology is presented for a real data set.

2 The Grubbs's Model

Suppose that we have at our disposal $p \geq 2$ instruments for measuring a characteristic of interest x in a group of n experimental units. Let x_i the true (unknown) value in unit i and y_{ij} the measured value obtained with instrument j in unit i , $i = 1, \dots, n$ and $j = 1, \dots, p$. A model typically considered in the literature see, Grubbs (1973, 1983), for such situation is given, in matrix notation, by

$$\begin{aligned} \mathbf{Y}_i &= \mathbf{a} + \mathbf{1}_p x_i + \boldsymbol{\epsilon}_i \\ &= \mathbf{a} + \mathbf{K} \mathbf{U}_i, \end{aligned} \tag{2.1}$$

where $\mathbf{a} = (0, \boldsymbol{\alpha}^\top)^\top = (0, \alpha_2, \dots, \alpha_p)^\top$ is $p \times 1$ vector, $\mathbf{K} = (\mathbf{1}_p, \mathbf{I}_p)$ is a $p \times (p+1)$ matrix, $\mathbf{Y}_i = (y_{i1}, \dots, y_{ip})^\top$ and $\boldsymbol{\epsilon}_i = (\epsilon_{i1}, \dots, \epsilon_{ip})^\top$ are $p \times 1$ random vectors $\mathbf{U}_i = (x_i, \boldsymbol{\epsilon}_i^\top)^\top$ is of dimension $(p+1) \times 1$, $\mathbf{1}_p$ is a $p \times 1$ vector of ones and \mathbf{I}_p denotes the identity

matrix of dimension p , $i = 1, \dots, n$. As in Bedrick (2001), to eliminate redundancy we set that $\alpha_1 = 0$. Finally, it is considered that the random vectors $\mathbf{U}_1, \dots, \mathbf{U}_n$ are independent and identically distributed $N_{p+1}(\boldsymbol{\eta}, \boldsymbol{\Psi})$, where

$$\boldsymbol{\eta} = \begin{pmatrix} \mu_x \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Psi} = \begin{pmatrix} \phi_x & 0 \\ 0 & D(\boldsymbol{\phi}) \end{pmatrix} \quad (2.2)$$

with $D(\boldsymbol{\phi}) = \text{diag}(\phi_1, \dots, \phi_p)$ and $\boldsymbol{\phi} = (\phi_1, \dots, \phi_p)^\top$. Thus, $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ are independent and identically distributed with according to the $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \boldsymbol{\alpha} + \mathbf{1}\mu_x = \boldsymbol{\mu}(\boldsymbol{\theta}) \quad \text{and} \quad \boldsymbol{\Sigma} = \phi_x \mathbf{1}_p \mathbf{1}_p^\top + D(\boldsymbol{\phi}) = \boldsymbol{\Sigma}(\boldsymbol{\theta}), \quad (2.3)$$

with $\boldsymbol{\theta} = (\mu_x, \boldsymbol{\alpha}^\top, \phi_x, \boldsymbol{\phi}^\top)^\top$. The log-likelihood function is given by

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^n l_i(\boldsymbol{\theta}), \quad (2.4)$$

where $l_i(\boldsymbol{\theta}) = (-p/2)\log(2\pi) - \frac{1}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}\|\mathbf{T}_i\|^2$, with $\|\mathbf{T}_i\|^2 = (\mathbf{Y}_i - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{Y}_i - \boldsymbol{\mu})$, $i = 1, \dots, n$.

Inference for such model is considered in Grubbs (1948, 1973, 1983), Christensen and Blackwood (1993) and recently by Bedrick (2001). Thus, the main object of this paper is to consider the approach of local influence in the Grubbs's measurement model given in (2.1). To obtain the maximum likelihood estimators we used the EM-algorithm.

3 Local Influence

Let $l(\boldsymbol{\theta})$ denote the log-likelihood function from the postulated model (here $\boldsymbol{\theta} = (\mu_x, \boldsymbol{\alpha}^\top, \phi_x, \boldsymbol{\phi}^\top)^\top$ and let $\boldsymbol{\omega}$ be a $q \times 1$ vector of perturbation restricted to some open subset of \mathbb{R}^q . The perturbations are made in the likelihood function such that it takes form $l(\boldsymbol{\theta}|\boldsymbol{\omega})$. Denoting the vector of no perturbation by $\boldsymbol{\omega}_0$, we assume $l(\boldsymbol{\theta}|\boldsymbol{\omega}_0) = l(\boldsymbol{\theta})$. To asses the influence of the perturbations on the maximum likelihood estimate of $\boldsymbol{\theta}$, one may consider the likelihood displacement

$$LD(\boldsymbol{\omega}) = 2[l(\hat{\boldsymbol{\theta}}) - l(\hat{\boldsymbol{\theta}}_\omega)],$$

where $\hat{\boldsymbol{\theta}}_\omega(\hat{\boldsymbol{\theta}})$ denotes the maximum likelihood estimator under the model $l(\boldsymbol{\theta}|\boldsymbol{\omega})(l(\boldsymbol{\theta}))$. The idea of local influence (Cook, 1986) is concerned in characterizing the behavior

of $LD(\boldsymbol{\omega})$ at $\boldsymbol{\omega}_0$. The procedure consists in selecting a unit direction \mathbf{d} , $\|\mathbf{d}\| = 1$, and then to consider the plot of $LD(\boldsymbol{\omega}_0 + a\mathbf{d})$ against a with $a \in R$. This plot is called *lifted line*. Notice that since $LD(\boldsymbol{\omega}_0) = 0$, $LD(\boldsymbol{\omega}_0 + a\mathbf{d})$ has a local minimum at $a = 0$. Each lifted line can be characterized by considering the normal curvature $C_d(\boldsymbol{\theta})$ around $a = 0$. The suggestion is to consider the direction \mathbf{d}_{\max} corresponding to the largest curvature $C_{d_{\max}}(\boldsymbol{\theta})$. The index plot of \mathbf{d}_{\max} may reveal those observations that under small perturbations exert notable influence on $LD(\boldsymbol{\omega})$. Cook (1986) showed that the normal curvature at the direction \mathbf{d} takes the form

$$C_d(\boldsymbol{\theta}) = 2|\mathbf{d}^\top \boldsymbol{\Delta}^\top \mathbf{L}^{-1} \boldsymbol{\Delta} \mathbf{d}|, \quad (3.1)$$

where $-\mathbf{L}$ is the observed Fisher information matrix for the postulated model ($\boldsymbol{\omega} = \boldsymbol{\omega}_0$) and $\boldsymbol{\Delta}$ is the $p \times q$ matrix with elements

$$\Delta_{ij} = \frac{\partial^2 l(\boldsymbol{\theta}|\boldsymbol{\omega})}{\partial \theta_i \partial \omega_j},$$

evaluated at $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0$, $i = 1, \dots, p$ and $j = 1, \dots, q$. Therefore, the maximization of (3.1) is equivalent to finding the largest absolute eigenvalue $C_{d_{\max}}$ of the matrix $\mathbf{B} = \boldsymbol{\Delta}^\top \mathbf{L}^{-1} \boldsymbol{\Delta}$ and, \mathbf{d}_{\max} is the corresponding eigenvector. In some situations, it may be of interest to assess the influence on a subset $\boldsymbol{\theta}_1$ of $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top)^\top$. For example, one may have interest on $\boldsymbol{\theta}_1 = \boldsymbol{\alpha}$ or $\boldsymbol{\theta}_1 = \boldsymbol{\phi}$. In such situations, the curvature at the direction \mathbf{d} is given by

$$C_d(\boldsymbol{\theta}_1) = 2|\mathbf{d}^\top \boldsymbol{\Delta}^\top (\mathbf{L}^{-1} - \mathbf{B}_{22}) \boldsymbol{\Delta} \mathbf{d}|, \quad (3.2)$$

where,

$$\mathbf{B}_{22} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{22}^{-1} \end{pmatrix},$$

and \mathbf{L}_{22} is obtained from the partition of \mathbf{L} according to the partition of $\boldsymbol{\theta}$. The eigenvector \mathbf{d}_{\max} corresponds to the largest absolute eigenvalue of the matrix $\mathbf{B} = \boldsymbol{\Delta}^\top (\mathbf{L}^{-1} - \mathbf{B}_{22}) \boldsymbol{\Delta}$.

Other important direction, according to Escobar and Meeker (1992) (see also Verbeke and Molenberghs, 2000) is $\mathbf{d} = \mathbf{e}_{in}$, which corresponds to the i -th position, where there is a one. In that case, the normal curvature, called the total local influence of individual i , is given by $C_i = 2|\mathbf{e}_{in}^\top \mathbf{B} \mathbf{e}_{in}| = 2|b_{ii}|$, where b_{ii} is the i th element diagonal of \mathbf{B} , $i = 1, \dots, n$. Verbeke and Molenberghs (2000) propose consider the

i -th observation influential if C_i is larger than the cutoff value $2 \sum_{i=1}^q C_i/q$. We use d_{\max} and C_i as diagnostics for local influence.

4 Curvature derivation

In this section we derive the observed information matrix and the Δ matrix for different schemes of perturbations.

4.1 The observed information matrix

From (2.4) following that the matrix of second derivatives with respect to $\boldsymbol{\theta}$ is given by:

$$\mathbf{L} = \frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} = \begin{pmatrix} L_{11} & 0 \\ & L_{22} \end{pmatrix} \quad (4.1)$$

where $\hat{\boldsymbol{\theta}}$ is the estimator of maximum likelihood of $\boldsymbol{\theta}=(\mu_x, \boldsymbol{\alpha}^\top, \phi_x, \boldsymbol{\phi}^\top)^\top=(\boldsymbol{\theta}_1^\top, \boldsymbol{\theta}_2^\top)^\top$, where $\boldsymbol{\theta}_1 = (\mu_x, \boldsymbol{\alpha}^\top)^\top$ and $\boldsymbol{\theta}_2 = (\phi_x, \boldsymbol{\phi}^\top)^\top$. The elements of this matrix are given in the appendix A.

4.2 Perturbation of Cases

We consider the model (2.1) and weights vector $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)^\top$. The log-likelihood function for perturbed model is given by

$$\ell(\boldsymbol{\theta}/\boldsymbol{\omega}) = \sum_{i=1}^n \omega_i \ell_i(\boldsymbol{\theta}), \quad (4.2)$$

where $\ell_i(\boldsymbol{\theta})$, $i = 1, \dots, n$, as defined in (2.5), with $\boldsymbol{\theta} = (\mu_x, \boldsymbol{\alpha}^\top, \phi_x, \boldsymbol{\phi}^\top)^\top$. Note that here $\boldsymbol{\omega}_0 = \mathbf{1}_n$.

The delta matrix is given by

$$\boldsymbol{\Delta} = (\boldsymbol{\Delta}_1, \dots, \boldsymbol{\Delta}_n), \quad (4.3)$$

where $\Delta_i = \frac{\partial \ell_i(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$, $i = 1, \dots, n$, with elements

$$\frac{\partial \ell_i(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma}} = -\frac{1}{2} \left[\frac{\partial \log |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\gamma}} + \frac{\partial \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\gamma}} \right], \quad \boldsymbol{\gamma} = \mu_x, \boldsymbol{\alpha}, \phi_x, \phi. \quad (4.4)$$

The components $\frac{\partial \log |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\gamma}}$ and $\frac{\partial \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\gamma}}$ are presented in Appendix B.

Note that, by (4.1), the normal curvature at the direction \mathbf{d} takes the form

$$C_d(\boldsymbol{\theta}) = C_d(\mu_x, \boldsymbol{\alpha}') + C_d(\phi_x, \phi'). \quad (4.5)$$

See Verbeke and Molenberghs (2000).

4.3 Response Perturbation

We considering here, of following perturbation schemes in the response variable

$$\mathbf{Y}_{\omega_i} = \mathbf{Y}_i + \mathbf{S}_y \omega_i, \quad i = 1, \dots, n, \quad (4.6)$$

where $\mathbf{S}_y = (s_1, \dots, s_p)^\top$ is a vector $p \times 1$, with s_j , scale factor corresponding to the j -th instrument, $j = 1, \dots, p$. In this case $\boldsymbol{\omega}_o = (0, \dots, 0)_{n \times 1}^\top$.

The log-likelihood function for the perturbed model $\ell(\boldsymbol{\theta}/\boldsymbol{\omega})$ is given by

$$\ell(\boldsymbol{\theta}/\boldsymbol{\omega}) = \sum_{i=1}^n \ell_i(\boldsymbol{\theta}/\omega_i), \quad (4.7)$$

where

$$\ell_i(\boldsymbol{\theta}/\omega_i) = (-p/2) \log(2\pi) - \frac{1}{2} \log |\boldsymbol{\Sigma}| - \frac{1}{2} \|\mathbf{T}_{\omega_i}\|^2,$$

and $\|\mathbf{T}_{\omega_i}\|^2 = (\mathbf{Y}_{\omega_i} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{Y}_{\omega_i} - \boldsymbol{\mu})$ with $\boldsymbol{\Sigma}$, $\boldsymbol{\mu}$ as in (2.3) and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)^\top$.

Then the i -th column of the Δ matrix, Δ_i , have elements given by,

$$\Delta_{\mu_x i} = \mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1} \mathbf{S}_y, \quad \Delta_{\boldsymbol{\alpha} i} = \mathbb{I}_{(p)} \boldsymbol{\Sigma}^{-1} \mathbf{S}_y, \quad \Delta_{\phi_x i} = \frac{c^{-2}}{\phi_x} (\mathbf{Y}_{\omega_i} - \boldsymbol{\mu})^\top \mathbf{M} \mathbf{S}_y \text{ and}$$

$$\begin{aligned} \Delta_{\phi_i} &= \mathbf{D}(\mathbf{Y}_{\omega_i} - \boldsymbol{\mu}) \mathbf{D}^{-2}(\phi) \mathbf{S}_y + c^{-2} \phi_x \mathbf{D}^{-2}(\phi) \mathbf{1}_p (\mathbf{Y}_{\omega_i} - \boldsymbol{\mu})^\top \mathbf{M} \mathbf{S}_y \\ &\quad - c^{-1} \phi_x \mathbf{D}^{-2}(\phi) (\mathbf{Y}_{\omega_i} - \boldsymbol{\mu}) \mathbf{1}_p^\top \mathbf{D}^{-1}(\phi) \mathbf{S}_y \\ &\quad - c^{-1} \phi_x (\mathbf{Y}_{\omega_i} - \boldsymbol{\mu})^\top \mathbf{D}^{-1}(\phi) \mathbf{1}_p \mathbf{D}^{-2}(\phi) \mathbf{S}_y, \end{aligned}$$

for $i = 1, \dots, n$, where $c = 1 + \phi_x \mathbf{1}_p^\top \mathbf{D}^{-1}(\phi) \mathbf{1}_p$, $\mathbf{M} = \phi_x \mathbf{D}^{-1}(\phi) \mathbf{1}_p \mathbf{1}_p^\top \mathbf{D}^{-1}(\phi)$

and the Δ matrix is evaluated in $\boldsymbol{\omega} = \boldsymbol{\omega}_0 = (0, \dots, 0)^\top$ and $\hat{\boldsymbol{\theta}} = (\hat{\mu}_x, \hat{\boldsymbol{\alpha}}^\top, \hat{\phi}_x, \hat{\boldsymbol{\phi}})^\top$. An other form of the response perturbation is to consider

$$\mathbf{Y}_{\omega_i} = \mathbf{Y}_i + D(\mathbf{S}_y) \boldsymbol{\omega}_i, \quad i = 1, \dots, n, \quad (4.8)$$

where $D(\mathbf{S}_y) = \text{Diag}(s_1, \dots, s_p)$ is a diagonal matrix of order p and $\boldsymbol{\omega}_i$ a $p \times 1$ perturbations vector. In this case $\boldsymbol{\omega}_0 = (0, \dots, 0)_{np \times 1}^\top$.

4.4 Perturbation of the multiplicative bias

In this section we consider the follow model perturbed:

$$\mathbf{Y}_i = \mathbf{a} + \mathbf{b}x_i + \boldsymbol{\epsilon}_i \quad (4.9)$$

where $\mathbf{a} = (0, \boldsymbol{\alpha}^\top)^\top$ and $\mathbf{b} = (1, \boldsymbol{\omega}^\top)^\top$, $i = 1, \dots, n$. Under normality $\mathbf{Y}_i \sim N_p(\boldsymbol{\mu}_\omega; \boldsymbol{\Sigma}_\omega)$, where $\boldsymbol{\mu}_\omega = \mathbf{a} + \mathbf{b}\mu_x$ and $\boldsymbol{\Sigma}_\omega = \phi_x \mathbf{b}\mathbf{b}^\top + D(\boldsymbol{\phi})$. Note that this model corresponding to the Comparative Calibration Model proposed by Barnett (1969). See also Bolfarine and Galea (1995). The log-likelihood function for the perturbed model is given by

$$\ell(\boldsymbol{\theta}/\boldsymbol{\omega}) = \sum_{i=1}^n \ell_i(\boldsymbol{\theta}/\boldsymbol{\omega}), \quad (4.10)$$

where $\ell_i(\boldsymbol{\theta}/\boldsymbol{\omega}) = (-p/2)\log(2\pi) - \frac{1}{2}\log|\boldsymbol{\Sigma}_\omega| - \frac{1}{2}\|\mathbf{T}_{\omega_i}\|^2$, with $\|\mathbf{T}_{\omega_i}\|^2 = (\mathbf{Y}_i - \boldsymbol{\mu}_\omega)^\top \boldsymbol{\Sigma}_\omega^{-1}(\mathbf{Y}_i - \boldsymbol{\mu}_\omega)$. The Δ matrix is given by

$$\Delta = \sum_{i=1}^n \frac{\partial^2 \ell_i(\boldsymbol{\theta}/\boldsymbol{\omega})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\omega}^\top},$$

where the elements of $\frac{\partial^2 \ell_i(\boldsymbol{\theta}/\boldsymbol{\omega})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\omega}^\top}$ are given in the appendix B.

5 Applications

In this section we presents two applications of the methodology discussed below.

5.1 The Grubbs Data Set

As an first illustration consider some data studied by Grubbs (1948). The data are the time of burning for fuses on projectiles as recorded by three different observers. This data set also where analysed by Jaech (1985). For this data set we analysis the local influence based in the perturbation of cases with interest in θ . The results are show in the Figure 1. The observations point as more influential are 4 and 17. The Figure 1(c) show the scatter plot of $C_i(\mu_x, \alpha)$ versus $C_i(\phi_x, \phi)$. Their respective cut-off values, indicated in the figure by the dashed line, are $2\sum_{i=1}^n C_i(\mu_x, \alpha)/n = 0.421$ and $2\sum_{i=1}^n C_i(\phi_x, \phi)/n = 0.614$. Note that the observations 2, 4, 15 and 17 are highly influential for both part of the parameter and the observation 9 is influential only for the estimation of (ϕ_x, ϕ) .

The Figure 2 show the index plot of $|dmax|$ and C_i for the response perturbation and Likelihood Displacement. Note that this graphic can be used for identify influential observations, y_{ij} , between experimental unit. The observations point as more influential are 12 and 17. Moreover, from Likelihood Displacement, figure 2(c), the observations 4 and 17 are globally influential.

The Table 1 present the maximum likelihood estimate for θ and its standard errors, if it is not considered the most influential observations, according the global/local influence method. Note that the standard errors are very affected.

5.2 Barnett Data Set

In this section we analyze one real data set given in Barnett (1969). Two instruments used for measuring the vital capacity of the human lung and operated by skilled and unskilled operators were compared on a common group of 72 patients. Figures 3a and 3b present graphics of local influence for the perturbation of case weights. The cutoff used for C_i equals $2\sum_{i=1}^n C_i/n = 0.55$ and has been indicated in the figure

by the dashed line. Patients 4, 25 and 67 are found to have a C_i value larger than 0.55 and are therefore considered to be relatively influential for the estimation of the complete parameter vector $\boldsymbol{\theta}$. Similar results are observed in the index plots of $|d_{max}|$ and C_i , Figure 4, with interest in $\boldsymbol{\alpha}$ and $\boldsymbol{\phi}$ respectively. Figures 5 and 6 present graphics of local influence for the perturbation of multiplicative bias. We can observe that the assumption of equal bias is not plausible in this case. This suggests modifying the model incorporating the possibility of bias different from one. In effect the maximum log-likelihood for the Grubbs's model is -2074.1 and for the Barnett's model the maximum log-likelihood is -2064.5 , corresponding to a likelihood ratio statistic of 19.2. This indicates that the Barnett's model fits the data significantly better than the Grubbs's model. Thus, this perturbation scheme serves as a guide in building a revised model.

Next we compare local influence and case deletion diagnostics. Following Zhao and Lee (1998), Cook's distance can be defined by

$$D_i = (\hat{\boldsymbol{\theta}}_{(i)} - \hat{\boldsymbol{\theta}})^\top (-\mathbf{L})(\hat{\boldsymbol{\theta}}_{(i)} - \hat{\boldsymbol{\theta}})/(2p + 1), \quad (5.1)$$

$i = 1, \dots, n$, where $\hat{\boldsymbol{\theta}}_{(i)}$ denotes the parameter estimates without case i . Figures 3c and 3d give the index plot of D_i and Likelihood Displacement, $LD_i = 2(l(\hat{\boldsymbol{\theta}}) - l(\hat{\boldsymbol{\theta}}_{(i)}))$, $i = 1, \dots, n$ for the Grubbs's model. Once again cases 4, 25 and 67 are prominent.

Table 2 presents the maximum likelihood estimate for $\boldsymbol{\theta}$ and its standard error, if it is not considered the three most influential observations, according to the global/local influence method.

Appendix A: The Observed Information Matrix

In this appendix we present the elements of the observed information matrix. From (2.4), it follows that

$$\frac{\partial l_i(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma}} = -\frac{1}{2} \left[\frac{\partial \log |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\gamma}} + \frac{\partial \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\gamma}} \right], \quad (\text{A.1})$$

with $\boldsymbol{\gamma} = \mu_x, \boldsymbol{\alpha}, \phi_x, \boldsymbol{\phi}$, $i = 1, \dots, n$. After some algebraic manipulations it follows that

$$\frac{\partial \log |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\gamma}} = 0, \quad \boldsymbol{\gamma} = \mu, \boldsymbol{\alpha} \quad \frac{\partial \log |\boldsymbol{\Sigma}|}{\partial \phi_x} = c^{-1} \frac{c-1}{\phi_x}, \quad \frac{\partial \log |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\phi}} = (I_p - c^{-1} \phi_x \mathbf{D}^{-1}(\boldsymbol{\phi})) \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{1}_p,$$

$$\frac{\partial \|\mathbf{T}_i\|^2}{\partial \mu_x} = -2 \mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1} \mathbf{W}_i,$$

$$\frac{\partial \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\alpha}} = -2 \mathbb{I}_{(p)} \boldsymbol{\Sigma}^{-1} \mathbf{W}_i,$$

$$\frac{\partial \|\mathbf{T}_i\|^2}{\partial \phi_x} = -\frac{c^{-2}}{\phi_x} \mathbf{W}_i^\top \mathbf{M} \mathbf{W}_i$$

$$\frac{\partial \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\phi}} = -\mathbf{D}(\mathbf{W}_i) \mathbf{D}^{-2}(\boldsymbol{\phi}) \mathbf{W}_i - c^{-2} \phi_x \mathbf{W}_i^\top \mathbf{M} \mathbf{W}_i \mathbf{D}^{-2}(\boldsymbol{\phi}) \mathbf{1}_p + 2c^{-1} \phi_x A_i \mathbf{D}^{-2}(\boldsymbol{\phi}) \mathbf{W}_i,$$

where $c = 1 + \phi_x \mathbf{1}^\top \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{1}$, $a_i = \mathbf{W}_i^\top \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{1}_p$ and $\mathbf{W}_i = \mathbf{Y}_i - \boldsymbol{\alpha} - \mathbf{b} \mu_x$.

From (A.1) it follows that the per element observed information matrix is given by

$$I_i = I_i(\boldsymbol{\theta}/\mathbf{Y}_i) = - \left(\frac{\partial^2 l_i(\boldsymbol{\theta})}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\tau}^\top} \right),$$

where

$$\frac{\partial^2 l_i}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\tau}^\top} = -\frac{1}{2} \left[\frac{\partial^2 \log |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\tau}^\top} + \frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\tau}^\top} \right]. \quad (\text{A.2})$$

with $\boldsymbol{\tau} = \mu_x, \boldsymbol{\alpha}, \phi_x, \boldsymbol{\phi}$. After some algebraic manipulations we have that

$$\frac{\partial^2 \log |\boldsymbol{\Sigma}|}{\partial \mu_x \partial \boldsymbol{\gamma}^\top} = 0, \quad \boldsymbol{\gamma} = \mu_x, \boldsymbol{\alpha}, \phi_x, \boldsymbol{\phi}, \quad \frac{\partial^2 \log |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\gamma}^\top} = 0, \quad \boldsymbol{\gamma} = \boldsymbol{\alpha}, \phi_x, \boldsymbol{\phi}$$

$$\frac{\partial^2 \log |\boldsymbol{\Sigma}|}{\partial \phi_x \partial \phi_x} = -\frac{1}{\phi_x^2} \left(\frac{c-1}{c} \right)^2, \quad \frac{\partial^2 \log |\boldsymbol{\Sigma}|}{\partial \phi_x \partial \boldsymbol{\phi}^\top} = -c^{-2} \mathbf{1}_p^\top \mathbf{D}^{-2}(\boldsymbol{\phi}),$$

$$\frac{\partial^2 \log|\boldsymbol{\Sigma}|}{\partial \boldsymbol{\phi} \partial \boldsymbol{\phi}^\top} = -\mathbf{D}^{-2}(\boldsymbol{\phi}) + 2\phi_x c^{-1} \mathbf{D}^{-3}(\boldsymbol{\phi}) - \phi_x c^{-2} \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{M} \mathbf{D}^{-1}(\boldsymbol{\phi}),$$

$$\frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \mu_x \partial \mu_x} = 2\mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}_p,$$

$$\frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \mu_x \partial \boldsymbol{\alpha}^\top} = 2\mathbf{1}_p' \boldsymbol{\Sigma}^{-1} \mathbb{I}'_{(p)},$$

$$\frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \mu_x \partial \phi_x} = 2 \frac{c^{-2}}{\phi_x} (c-1) \mathbf{1}_p^\top \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{W}_i,$$

$$\frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \mu_x \partial \boldsymbol{\phi}^\top} = 2c^{-1} \mathbf{W}_i^\top \boldsymbol{\Sigma}^{-1} \mathbf{D}^{-1}(\boldsymbol{\phi}),$$

$$\frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}^\top} = 2\mathbb{I}_{(p)} \boldsymbol{\Sigma}^{-1} \mathbb{I}_{(p)}^\top,$$

$$\frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\alpha} \partial \phi_x} = 2 \frac{c^{-2}}{\phi_x} \mathbb{I}_{(p)} \mathbf{M} \mathbf{W}_i,$$

$$\begin{aligned} \frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\phi}^\top} &= 2\mathbb{I}_{(p)} [\boldsymbol{\Sigma}^{-1} \mathbf{D}(\mathbf{W}_i) \mathbf{D}^{-1}(\boldsymbol{\phi}) + c^{-2} \phi_x \mathbf{M} \mathbf{W}_i \mathbf{1}_p^\top \mathbf{D}^{-2}(\boldsymbol{\phi}) \\ &\quad - c^{-1} \phi_x \mathbf{1}_p^\top \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{W}_i \mathbf{D}^{-2}(\boldsymbol{\phi})], \end{aligned}$$

$$\frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \phi_x \partial \phi_x} = 2 \frac{c^{-3}}{\phi_x} \mathbf{W}_i^\top \mathbf{M} \mathbf{W}_i \mathbf{1}_p^\top \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{1}_p,$$

$$\frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \phi_x \partial \boldsymbol{\phi}^\top} = -2c^{-3} \mathbf{W}_i^\top \mathbf{M} \mathbf{W}_i \mathbf{1}_p^\top \mathbf{D}^{-2}(\boldsymbol{\phi}) + 2c^{-2} \mathbf{W}_i^\top \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{1}_p \mathbf{W}_i^\top \mathbf{D}^{-2}(\boldsymbol{\phi}),$$

$$\begin{aligned} \frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\phi} \partial \boldsymbol{\phi}^\top} &= 2\mathbf{D}^2(\mathbf{W}_i) \mathbf{D}^{-3}(\boldsymbol{\phi}) - 2c^{-3} \phi_x \mathbf{W}_i^\top \mathbf{M} \mathbf{W}_i \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{M} \mathbf{D}^{-1}(\boldsymbol{\phi}) \\ &\quad + 2\phi_x c^{-2} \mathbf{D}^{-2}(\boldsymbol{\phi}) \mathbf{W}_i \mathbf{W}_i^\top \mathbf{M} \mathbf{D}^{-1}(\boldsymbol{\phi}) + 2c^{-2} \phi_x \mathbf{W}_i^\top \mathbf{M} \mathbf{W}_i \mathbf{D}^{-3}(\boldsymbol{\phi}) \\ &\quad + 2c^{-2} \phi_x \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{M} \mathbf{W}_i \mathbf{W}_i^\top \mathbf{D}^{-2}(\boldsymbol{\phi}) - 2c^{-1} \phi_x \mathbf{D}^{-2}(\boldsymbol{\phi}) \mathbf{W}_i \mathbf{W}_i^\top \mathbf{D}^{-2}(\boldsymbol{\phi}) \\ &\quad - 4c^{-1} \phi_x \mathbf{1}_p^\top \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{W}_i \mathbf{D}(\mathbf{W}_i) \mathbf{D}^{-3}(\boldsymbol{\phi}). \end{aligned}$$

Appendix B: The Delta Matrix for Perturbation of the Multiplicative Bias

The Δ matrix is given by

$$\Delta = \sum_{i=1}^n \frac{\partial^2 l_i(\boldsymbol{\theta}/\boldsymbol{\omega})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\omega}^\top},$$

where the elements of the matrix $\frac{\partial^2 l_i(\boldsymbol{\theta}/\boldsymbol{\omega})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\omega}^\top}$, are given by

$$\frac{\partial^2 l_i(\boldsymbol{\theta}/\boldsymbol{\omega})}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\omega}^\top} = -\frac{1}{2} \left[\frac{\partial^2 \log |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\omega}^\top} + \frac{\partial^2 \|\mathbf{T}_{\boldsymbol{\omega}_i}\|^2}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\omega}^\top} \right], \quad \boldsymbol{\gamma} = \mu_x, \boldsymbol{\alpha}, \phi_x, \boldsymbol{\phi},$$

with

$$\begin{aligned} \frac{\partial^2 \log |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\gamma} \partial \boldsymbol{\omega}^\top} &= 0, \quad \boldsymbol{\gamma} = \mu_x, \boldsymbol{\alpha}, \\ \frac{\partial^2 \log |\boldsymbol{\Sigma}|}{\partial \phi_x \partial \boldsymbol{\omega}^\top} &= 2c^{-2} \boldsymbol{\omega}^\top \mathbf{D}^{-1}(\boldsymbol{\psi}), \\ \frac{\partial^2 \log |\boldsymbol{\Sigma}|}{\partial \boldsymbol{\phi} \partial \boldsymbol{\omega}^\top} &= -2c^{-1} \phi_x \mathbf{D}^{-2}(\boldsymbol{\phi}) [\mathbf{D}_*(\boldsymbol{\omega}) - c^{-1} \phi_x \mathbf{D}(\mathbf{b}) \mathbf{b} \boldsymbol{\omega}^\top \mathbf{D}^{-1}(\boldsymbol{\psi})], \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \mu_x \partial \boldsymbol{\omega}^\top} &= -2c^{-1} \mathbf{A}_i, \\ \frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \phi_x \partial \boldsymbol{\omega}^\top} &= -2c^{-2} a_i(\boldsymbol{\omega}) \mathbf{A}_i, \\ \frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\omega}^\top} &= 2q_i [\mathbf{D}^{-1}(\boldsymbol{\psi}) - 2c^{-1} \phi_x \mathbf{D}^{-1}(\boldsymbol{\psi}) \boldsymbol{\omega} \boldsymbol{\omega}^\top \mathbf{D}^{-1}(\boldsymbol{\psi})] \\ &\quad + 2c^{-1} \phi_x \mathbf{D}^{-1}(\boldsymbol{\psi}) \boldsymbol{\omega} (\mathbf{Y}_{i2} - \boldsymbol{\alpha})^\top \mathbf{D}^{-1}(\boldsymbol{\psi}) \\ \frac{\partial^2 \|\mathbf{T}_i\|^2}{\partial \boldsymbol{\phi} \partial \boldsymbol{\omega}^\top} &= 2\mathbf{D}^{-2}(\boldsymbol{\phi}) [q_i(\boldsymbol{\omega}) \mathbf{D}_*(\mathbf{Y}_{i2} - \boldsymbol{\alpha} - q_i(\boldsymbol{\omega}) \boldsymbol{\omega}) + c^{-1} \phi_x \mathbf{D}(\mathbf{b}) (\mathbf{Y}_i - \mathbf{a} - \mathbf{b} q_i(\boldsymbol{\omega})) \mathbf{A}_i], \end{aligned}$$

where

$$\mathbf{D}_*(\mathbf{Y}_{i2} - \boldsymbol{\alpha} - q_i(\boldsymbol{\omega})) = \begin{pmatrix} \mathbf{0}_{(p-1) \times 1}^\top \\ \mathbf{D}(\mathbf{Y}_{i2} - \boldsymbol{\alpha} - q_i(\boldsymbol{\omega}) \boldsymbol{\omega}) \end{pmatrix}, \quad \mathbf{D}_*(\boldsymbol{\omega}) = \begin{pmatrix} \mathbf{0}_{(p-1) \times 1}^\top \\ \mathbf{D}(\boldsymbol{\omega}) \end{pmatrix}$$

, $\mathbf{A}_i = (\mathbf{Y}_{i2} - \boldsymbol{\alpha} - 2q_i(\boldsymbol{\omega}) \boldsymbol{\omega})^\top \mathbf{D}^{-1}(\boldsymbol{\psi})$, $a_i(\boldsymbol{\omega}) = (\mathbf{Y}_i - \mathbf{a} - \mathbf{b} \mu_x)_i^\top \mathbf{D}^{-1}(\boldsymbol{\phi}) \mathbf{b}$, $q_i(\boldsymbol{\omega}) = \mu_x + c^{-1} \phi_x a_i(\boldsymbol{\omega})$ and $\mathbf{Y}_{i2} = (y_{i2}, \dots, y_{ip})^\top$.

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Figure 1: Index plots for (a) $|d_{max}|$, (b) C_i and (c) $C_i(\mu_x, \alpha)$ versus $C_i(\phi_x, \phi)$ for perturbation of cases

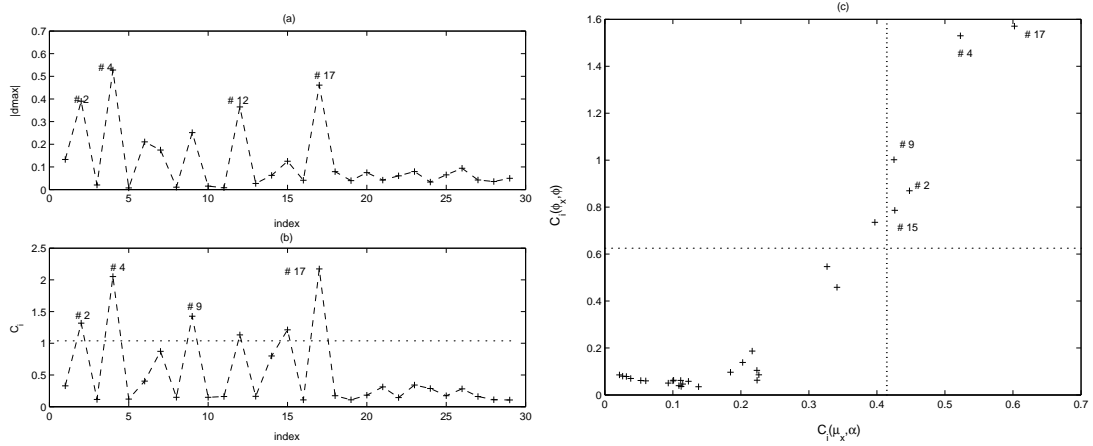


Figure 2: Index plot of (a) $|d_{max}|$, (b) C_i for perturbation of the response (c) Likelihood Displacement LD_i

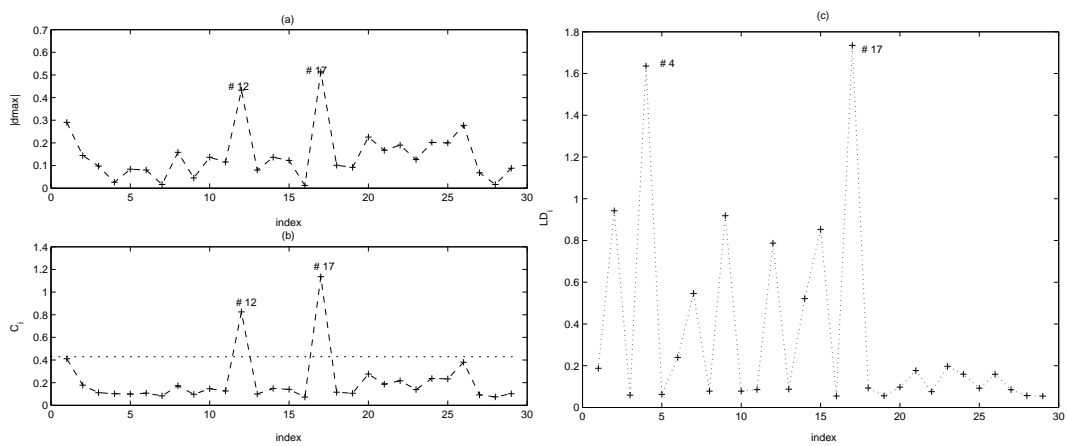


Figure 3: Index plots for (a) $|d_{max}|$ (b) C_i for perturbation of cases for θ (c) Likelihood Displacement LD_i (d) Cook's distance D_i .

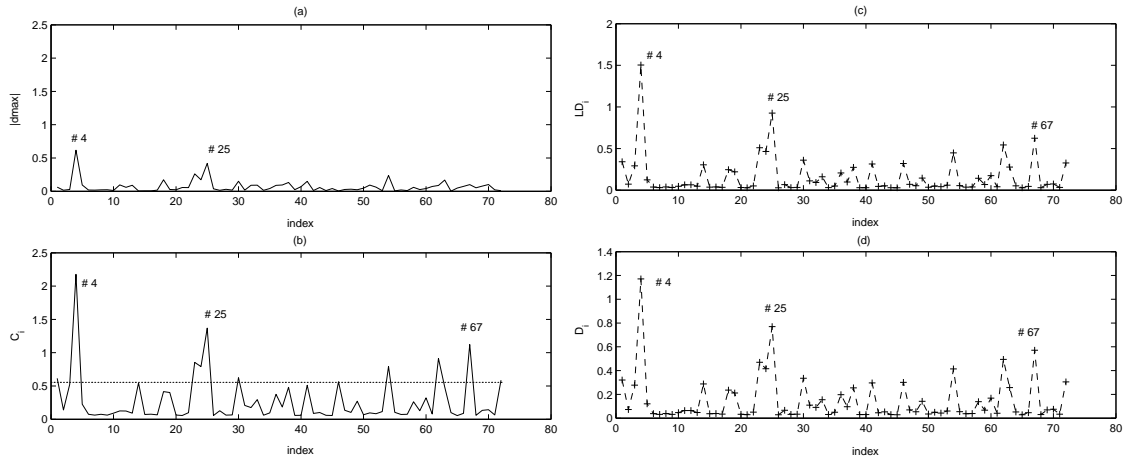


Figure 4: Index plots for $|d_{max}|$ and C_i for perturbation of cases (a) for α (b) for ϕ

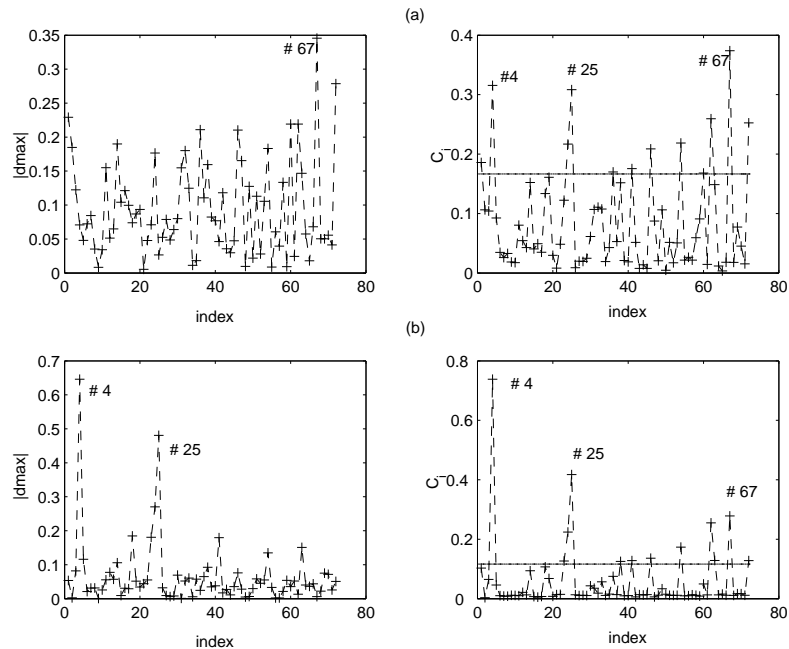


Figure 5: Index plots of (a) d_{max} and (b) C_i for perturbation of the multiplicative bias

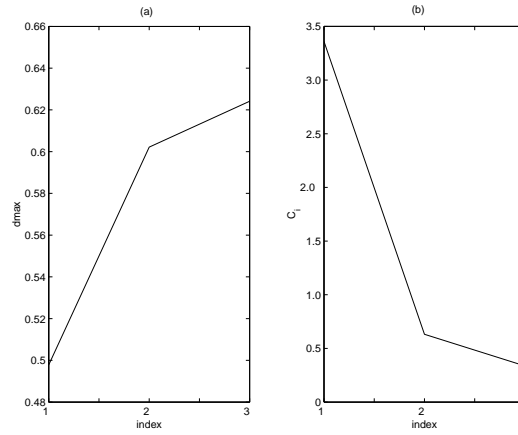


Figure 6: Index plots of (a) d_{max} and (b) C_i for perturbation of the multiplicative bias for α for ϕ

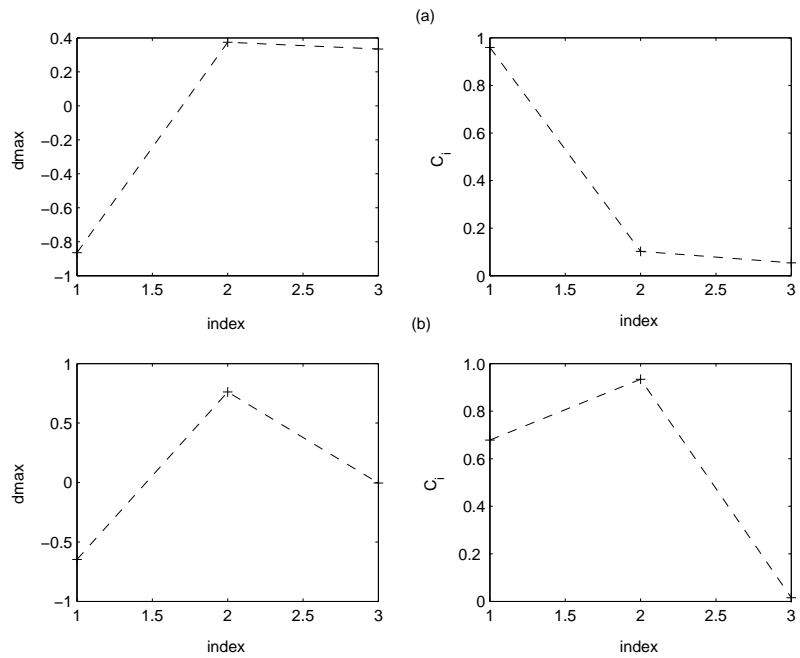


Table 1: Maximum likelihood estimative and its asymptotic standard error when some observations are removed for Grubbs Data Set

$\hat{\theta}$	None	#4	#12	#17	#4, 12	#4, 17	#12, 17	#4, 12, 17
$\hat{\mu}_x$	9.7414 (0.0387)	9.7425 (0.0401)	9.7239 (0.0360)	9.7207 (0.0342)	9.7244 (0.0374)	9.7211 (0.0355)	9.7019 (0.0298)	9.7015 (0.0310)
$\hat{\alpha}_2$	0.0238 (0.0048)	0.0218 (0.0046)	0.0243 (0.0050)	0.0246 (0.0050)	0.0222 (0.0047)	0.0226 (0.0047)	0.0252 (0.0051)	0.0231 (0.0049)
$\hat{\alpha}_3$	0.0141 (0.0032)	0.0150 (0.0032)	0.0143 (0.0033)	0.0139 (0.0033)	0.0152 (0.0033)	0.0148 (0.0033)	0.0141 (0.0034)	0.0150 (0.0034)
$\hat{\phi}_x$	0.0434 (0.0114)	0.0449 (0.0120)	0.0362 (0.0097)	0.0326 (0.0087)	0.0376 (0.0102)	0.0338 (0.0092)	0.0240 (0.0065)	0.0249 (0.0069)
$\hat{\phi}_1$	0.0001 (0.0001)	0.0001 (0.0001)	0.0001 (0.0001)	0.0001 (0.0001)	0.0001 (0.0001)	0.0001 (0.0001)	0.0001 (0.0001)	0.0001 (0.0001)
$\hat{\phi}_2$	0.0006 (0.0002)	0.0005 (0.0001)	0.0006 (0.0002)	0.0006 (0.0002)	0.0005 (0.0002)	0.0005 (0.0002)	0.0006 (0.0002)	0.0005 (0.0002)
$\hat{\phi}_3$	0.0002 (0.0001)	0.0002 (0.0001)	0.0002 (0.0001)	0.0002 (0.0001)	0.0002 (0.0001)	0.0002 (0.0001)	0.0003 (0.0001)	0.0002 (0.0001)

Table 2: Maximum likelihood estimative and its asymptotic standard errors when some observations are removed for Barnett Data Set

$\hat{\theta}$	None	#4	#25	#67	#4, 25	#4, 67	#25, 67	#4, 25, 67
$\hat{\mu}_x$	0.0225 (0.0010)	0.0225 (0.0010)	0.0225 (0.0010)	0.0224 (0.0010)	0.0225 (0.0010)	0.0224 (0.0010)	0.0224 (0.0010)	0.0224 (0.0010)
$\hat{\alpha}_2$	-0.0007 (0.0003)	-0.0008 (0.0003)	-0.0006 (0.0003)	-0.0007 (0.0003)	-0.0007 (0.0002)	-0.0009 (0.0003)	-0.0006 (0.0003)	-0.0008 (0.0002)
$\hat{\alpha}_3$	-0.0010 (0.0004)	-0.0011 (0.0004)	-0.0009 (0.0004)	-0.0011 (0.0004)	-0.0010 (0.0004)	-0.0012 (0.0003)	-0.0011 (0.0003)	-0.0012 (0.0003)
$\hat{\alpha}_4$	-0.0014 (0.0004)	-0.0015 (0.0004)	-0.0013 (0.0004)	-0.0016 (0.0004)	-0.0014 (0.0004)	-0.0016 (0.0004)	-0.0015 (0.0004)	-0.0015 (0.0004)
$\hat{\phi}_x$	6.2906 (1.0609)	6.2720 (1.0633)	6.2783 (1.0646)	6.2542 (1.0620)	6.2600 (1.0665)	6.2446 (1.0665)	6.2556 (1.0684)	6.2429 (1.0719)
$\hat{\phi}_1$	0.4998 (0.0993)	0.4170 (0.0851)	0.4419 (0.0891)	0.4805 (0.0963)	0.3602 (0.0745)	0.3982 (0.0823)	0.4269 (0.0869)	0.3439 (0.0723)
$\hat{\phi}_2$	0.1413 (0.0518)	0.1057 (0.0466)	0.1090 (0.0471)	0.1476 (0.0511)	0.0716 (0.0413)	0.1166 (0.0464)	0.1179 (0.0466)	0.0839 (0.0412)
$\hat{\phi}_3$	0.4383 (0.0898)	0.4807 (0.0949)	0.4604 (0.0920)	0.3928 (0.0827)	0.5075 (0.0974)	0.4265 (0.0867)	0.4081 (0.0839)	0.4477 (0.0885)
$\hat{\phi}_4$	0.4633 (0.0936)	0.5087 (0.0949)	0.4989 (0.0980)	0.4328 (0.0888)	0.5522 (0.1046)	0.4704 (0.0936)	0.4610 (0.0922)	0.5069 (0.0980)