Multilevel Pseudo-Wavelet Schemes: a Consistency Analysis

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Abstract

The study of this paper is devoted to the analysis of multilevel approximation schemes, in the context of multiresolution analysis. We have particular interest in expansions where the coefficients are obtained in terms of discrete convolutions of function point values with some specific wheights. In the first part we analyze aspects, such as, algorithm of construction, their accuracy and multilevel implementation for three cases: interpolation, quasi-interpolation and discrete projection.

The second part is dedicated to hybrid formulations for the discretization of nonlinear differential operators. The idea is to combine two different approximation schemes: one approximation scheme is used for functions or linear terms; another one, defined in terms of function point values, is used for nonlinear operations. Taking the bilinear advection operator as a model, we establish the consistency of the discretizations in terms of the order of the truncation error.

1 Introduction

The purpose of the present paper is two fold. Firstly, in Section 2, we shall analyze various approximation schemes in the context of biorthogonal multiresolution analysis. Besides the usual biorthogonal projections, we are also interested in approximations that can be obtained from the information of function point values. For instance, this is the case of interpolation, quasi-interpolation and discrete projection operators. Various aspects shall be analyzed, such as the algorithms for the construction of the proposed schemes, their order of accuracy and numerical aspects for multilevel implementation. To fix ideas, we shall adopt the spline biorthogonal multiresolution analyzes as model framework.

In Section 3, which is the second part of the present paper, the approximation schemes presented in the first part shall be used for the discretization of differential operators. Specially, we are interested in hybrid formulations which are suitable for the discretization of nonlinear operators. As in the traditional pseudo-spectral schemes, the idea is to combine different approximation schemes. There is one approximation scheme which is used for functions or linear operations (e.g. derivative), and there is another one, using point values, for the performance of the nonlinear operations (e.g. multiplication).

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Before properly entering into the two main parts of this paper, let us first give an overview of their contents. The formalism of the presentation aims to a unified framework for the analysis of the different approximation schemes, of the first part, as well as of their applications in the discretization of differential operators, in the second part.

The definition of an approximation scheme $\mathcal{P}^j u$ requires two basic ingredients: the approximating space V_j and the approximation strategy. Given an approximating space, several approximation strategies may be used, producing different approximation schemes. In our exposition, we shall adopt the formalism suggested by A. Harten [13] in which an approximation strategy may represented by dual applications $\{\mathcal{D}^j, \mathcal{R}^j\}$ such that $\mathcal{P}^j = \mathcal{R}^j\mathcal{D}^j$. $\mathcal{D}^j$ is a discretization operator which assigns discrete values $\mathcal{D}^j u = \mathbf{u}^j$ to a function u. Usually, the discrete values $\mathbf{u}^j$ give local information of u associated with a certain grid X^j. They can be point values or local weighted averages. Thus, typically, $\mathcal{D}^j$ is a linear mapping $\mathcal{D}^j : V \to E^j$, where V is a functional space and E^j is a discrete vector space. Conversely, there is a reconstruction operator $\mathcal{R}^j : E^j \to V_j$ which produces a function in V_j from the knowledge of discrete values $\mathbf{u}^j \in E^j$. Typically, reconstruction operators are defined in terms of an expansion

$$\mathcal{R}^j(x; \mathbf{u}^j) = \sum_k u^j(k)\phi_{j,k}(x), \quad \text{[recj]} \tag{1}$$

where the basic functions $\phi_{j,k}(x)$ form a Riesz basis for V_j.

A multiresolution analysis is a sequence of embedded approximating spaces $V_j \subset V_{j+1}$. In such context, $\phi_{j,k}(x) \in V_j$ are scaling functions that provide reconstruction operators (1) in a single scale level. The index j corresponds to the dyadic scale 2^{-j} and k indicates space localization $k2^{-j} \in X^j$. For the applications presented in this work, we shall consider shift invariant spaces V_j where $\phi_{j,k}(x) = \phi(2^j x - k),\ k \in \mathbb{Z}$, are obtained by translations and dilations of a single basic function $\phi(x)$. Given the reconstruction operator (1), we shall adopt different discretization operators $\mathcal{D}^j$ to produce different approximation schemes $\{\mathcal{D}^j, \mathcal{R}^j\}$ on V^j. For instance, discretizations

$$(\mathcal{D}^j u)(k) = 2^j \int_{\mathbb{R}} u(x)\phi^*_{j,k}(x)dx,$$

defined by local averages using dual scaling functions $\phi^*_{j,k}(x) = \phi^*(2^j x - k)$, produce biorthogonal projections $\mathcal{P}^j = \mathcal{R}^j\mathcal{D}^j$. Other schemes of interest, such as interpolation, quasi-interpolation or discrete projections, may be given in the form $\mathcal{I}^j = \mathcal{R}^j\mathcal{D}^j_c$, where $\mathcal{D}^j_c$ are defined in terms of discrete convolutions of point values with some specific weights. For instance, for some $0 \le \alpha < 1$ we shall consider discretization operators of the form

$$(\mathcal{D}^j_c u)(k) = \sum_{n\in\mathbb{Z}} \gamma(n)u((k - n + \alpha)2^{-j})$$

for interpolation and quasi-interpolation, and of the form

$$(\mathcal{D}^j_c u)(k) = \sum_{n\in\mathbb{Z}} \gamma(n)u((n + 2k)2^{-j-1})$$

for discrete projections.

A fundamental aspect of a multiresolution analysis is the possibility of multilevel decompositions in terms of direct sums

$$V_j = V_J \oplus W_J \oplus \cdots \oplus W_{j-1},$$

where J is a coarse level and W_l contains details between consecutive levels l and $l+1$. In association to such multilevel decompositions there are wavelet functions $\{\psi_{l,k}(x)\}$, which form Riesz bases for the intermediate spaces W_l. Therefore, in a multiresolution analysis framework, approximation schemes may be defined in terms of multilevel discretizations

$$\mathcal{D}^j_{MR} u = \mathbf{u}^j_{MR} = \{\mathbf{u}^J, \mathbf{d}^j, \cdots, \mathbf{d}^{j-1}\}$$

and multilevel reconstructions

$$\mathcal{R}^j_{MR}(x; \mathbf{u}^j_{MR}) = \mathcal{R}^J(x, \mathbf{u}^J) + \sum_{\ell=J}^{j-1} \sum_k d^\ell(k) \psi_{\ell,k}(x).$$

For instance, for the biorthogonal projection, the multilevel discretization is defined by local averages using dual wavelets $\psi^*_{\ell,k}(x)$ such that

$$d^\ell(k) = (\mathcal{G}^\ell u)(k) = 2^\ell \int_{\mathbb{R}} u(x) \psi^*_{\ell,k}(x) dx$$

which are known as wavelet coefficients. If the discrete values $\mathbf{u}^{\ell+1} = \mathcal{D}^{\ell+1} u$ are given, then the wavelet coefficients can be obtained by Mallat's analysis algorithm which is expressed by convolution with a high pass filter g^* followed by decimation

$$(\mathcal{G}^\ell u)(k) = \sum_k g^*(m - 2k)(\mathcal{D}^{\ell+1} u)(m) \quad \text{[gc]} \tag{2}$$

We are also interested in multilevel approximation schemes in terms of other type of multilevel discretization operators. Instead of $d^\ell(k)$, we shall use modified wavelet coefficients $\check{d}^\ell(k)$ which are produced by a modified analysis algorithm. It is based on a discretization operator $\mathcal{G}^\ell_c$ obtained by replacing $\mathcal{D}^{\ell+1}$ in formula (2) by another discretization operator $\mathcal{D}^{\ell+1}_c$ associated to some other approximation scheme. Precisely,

$$\check{d}^{j-m}(k) = \mathcal{G}^{j-m}_c(v)(k), \tag{3}$$

where v is obtained by removing from u all contributions corresponding to previously computed modified wavelet coefficients $\check{d}^{j-n}, n = 1, \cdots, m-1$. Such modified analysis algorithm was suggested by Fröhlich and Schneider [9] for the interpolation case and explored by Ware [15] for the discrete projection. As emphasized in these papers, the purpose of using the modified function v instead of simply u in formula (2) is to improve the accuracy in the aliasing error $d^\ell - \check{d}^\ell$. A crucial ingredient in the formulation of such modified analysis algorithm is the conservation property, which is satisfied both by interpolation and discrete projection operators. In the case of quasi-interpolation, which is not conservative, we shall describe the degradation in the aliasing error in coarse scales.

In numerical solution of partial differential equations (PDE), the analytical problem is replaced by a discrete model. In the applications of the present paper, special attention shall be given to the nonlinear advection operator $\mathcal{L}(u, v) = uv_x$. Using the approximation schemes defined in the first part, we shall consider discretizations of the form

$$\mathcal{L}^j(\mathbf{u}^j, \mathbf{v}^j) = \mathcal{D}^j_c[\mathcal{L}(\mathcal{R}^j(x; \mathbf{u}^j), \mathcal{R}^j(x; \mathbf{v}^j)].$$

An hybrid formulation occurs if the discretization operator $\mathcal{D}^j_c$ used after the application of $\mathcal{L}$ is different from the operator $\mathcal{D}^j$ used in the discretization of the functions $\mathbf{u}^j = \mathcal{D}^j u$ and $\mathbf{v}^j = \mathcal{D}^j v$. For instance, this is the case in pseudo-spectral discretizations of nonlinear differential operators in which $\mathcal{D}^j$ is the Fourier transform and $\mathcal{D}_c^j$ is its discrete version, which is defined in terms of point values. Pseudo-wavelets schemes have also been adopted in applications to PDE [2, 3, 9, 14], where the discretization of the nonlinear terms are usually evaluated in the physical space by means of functionals $\mathcal{D}_c^j$ defined in terms of point values.

For the schemes under study, we shall analyze the truncation error

$$TE(u,v) = \mathcal{D}^j\mathcal{L}(u,v) - \mathcal{L}^j(\mathbf{u}^j, \mathbf{v}^j)$$

by giving a precise description of the interaction between different Fourier modes. We shall prove that for the Petrov-Galerkin formulation, in which $\mathcal{D}_c^j = \mathcal{D}^j$, the superconvergence occurs, i.e., the order of accuracy M for the truncation error is higher than the maximum approximation order N allowed by the approximating spaces. For the three hybrid formulations (using interpolation, quasi-interpolation and discrete projection), the order of the truncation error is $N-1$. However, for some specific cases (e.g. splines of even order) it gets N, with an extra gain in the consistency order. These results have been partially reported in [1].

2 First Part: Approximation Schemes

[partone]

2.1 Biorthogonal Framework

[mra] For the definition of a multiresolution analysis $V_j \subset \mathbf{L}^2(\mathbb{R})$, the main ingredient is a scale relation

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} h(k)\phi(2x-k) \quad \text{[c1e1]} \tag{4}$$

which implicitly defines the basic scaling function ϕ. In the Fourier space, the scale relation is expressed by

$$\widehat{\phi}(\xi) = H(\xi/2)\widehat{\phi}(\xi/2), \quad \text{[c1e2]} \tag{5}$$

where

$$H(\xi) = \sum_{k\in\mathbb{Z}} h(k)e^{-ik\xi} \quad \text{[c1e3]} \tag{6}$$

is a low-pass filter. Two multiresolution analysis V_j and V_j^* are said to be biorthogonal provided that the biorthogonal relation holds

$$\int_{\mathbb{R}} \phi^*(x)\phi(x-k)dx = \delta_k.$$

Approximations of functions u are found in V_j by means of the biorthogonal projection operator $\mathcal{P}^j = \mathcal{R}^j\mathcal{D}^j$, where

$$\mathcal{R}^j(x;\mathbf{u}^j) = \sum_{k\in\mathbb{Z}} u^j(k)\phi_{j,k}(x), \quad \text{[projection]} \tag{7}$$

and

$$\mathcal{D}^j u(k) := 2^j \int_{\mathbb{R}} u(x)\phi^*_{j,k}(x)dx. \quad \text{[cjk]} \tag{8}$$

It can also be represented in a multilevel setting $\mathcal{P}^j = \mathcal{R}^j_{MR}\mathcal{D}^j_{MR}$. The multilevel reconstruction has the form

$$\begin{aligned}\mathcal{R}^j_{MR}(x;\mathbf{u}^j_{MR}) &= \sum_{k\in\Gamma_J} u^J(k)\phi_{J,k}(x) + \sum_{l=J}^{j-1}\sum_{k\in\Lambda_l} d^l(k)\psi_{l,k}(x) \\ &= \mathcal{P}^J u(x) + \sum_{l=J}^{j-1} \mathcal{Q}^l u(x). \quad \textbf{[e20]}\end{aligned} \tag{9}$$

The mother wavelets are obtained by the scale relations

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} g(k)\phi(2x-k) \quad \text{and} \quad \psi^*(x) = 2\sum_{k\in\mathbb{Z}} g^*(k)\phi^*(2x-k),$$

where $g(k) = (-1)^{k+1}h^*(1-k)$ and $g^*(k) = (-1)^{k+1}h(1-k)$ are high-pass filter coefficients, and the following biorthogonal relations hold

$$\int_{\mathbb{R}} \psi^*(x)\psi(x-k)dx = \delta_k, \quad \textbf{[c1e9]} \tag{10}$$

$$\int_{\mathbb{R}} \phi^*(x)\psi(x-k)dx = \int_{\mathbb{R}} \psi^*(x)\phi(x-k)dx = 0. \quad \textbf{[c1e10]} \tag{11}$$

The multilevel discretization are obtained by the functionals

$$\mathcal{D}^j_{MR} = \{\mathcal{D}^J, \mathcal{G}^J, \cdots \mathcal{G}^{j-1}\}$$

such that

$$\mathcal{G}^l u(k) = d^l(k) = 2^l \int_{\mathbb{R}} \psi^*_{l,k}(x)u(x)dx. \quad \textbf{[wcof]} \tag{12}$$

The transformation relating the information at the finest level $\mathbf{u}^j$ and its multilevel representation $\mathbf{u}^j_{MR} = \{\mathbf{u}^J, \mathbf{d}^j, \cdots, \mathbf{d}^{j-1}\}$ is known as Analysis Algorithm and it is defined by the recursive application of the formulas

$$u^{j-1}(k) = 2\sum_{s\in\mathbb{Z}} h^*(s-2k)u^j(s), \quad \textbf{[mallat1]} \tag{13}$$

$$d^{j-1}(k) = 2\sum_{s\in\mathbb{Z}} g^*(s-2k)u^j(s). \quad \textbf{[mallat2]} \tag{14}$$

On the other hand, the Synthesis Algorithm recovers the finest level information by multilevel representation

$$u^j(k) = \sum_{s\in\mathbb{Z}} h(h-2s)u^{j-1}(s) + \sum_{s\in\mathbb{Z}} g(k-2s)d^{j-1}(s). \tag{15}$$

- **Accuracy**

It is well known that the best order of accuracy in shift-invariant approximating spaces is characterized by the Strang-Fix condition. A function $\phi(x)$ satisfies the Strang-Fix condition of order p if $\widehat{\phi}(0) \neq 0$ and $\widehat{\phi}(\xi)$ have zeros of order $p+1$ at $\xi = 2k\pi$, $k \in \mathbb{Z}\setminus\{0\}$. In such case, all the polynomials up to degree p can be locally reproduced by linear combinations of the basic functions $\phi_{j,k}(x)$. If ϕ and ϕ^* are integrable scaling functions of compact support, and ϕ satisfies the Strang-Fix condition of order p, then the biorthogonal projection $\mathcal{P}^j f$ in V_j satisfies the error estimation [3]

$$\|f - \mathcal{P}^j f\|_{\mathbf{H}^s} \lesssim 2^{-j(p+1-s)} ||f||_{\mathbf{H}^{p+1}}, \quad \text{[c1eq333]} \tag{16}$$

for $0 \le s \le \min\{r, p+1\}$, where r is degree of regularity of ϕ, so that $\phi \in \mathbf{H}^r(\mathbb{R})$. The following estimations also hold

$$|d^l(k)| \lesssim 2^{-l(p+1)} \|u\|_{\mathbf{H}^{p+1}(\mathrm{Supp}\psi^*_{l,k})}, \quad \text{[c1e18]} \tag{17}$$

$$\|\mathcal{Q}^j f\|_{\mathbf{H}^s} \lesssim 2^{-j(p+1-s)} ||f||_{\mathbf{H}^{p+1}}, \quad \text{[c1eq334]} \tag{18}$$

where $_{\mathrm{Supp}}\psi^*_{l,k}$ represent the support of function $\psi^*_{l,k}$.

2.1.1 Spline Biorthogonal Family

[secb] We have particular interest in the family of biorthogonal multiresolution analysis introduced by Cohen, Daubechies and Feauveau [4]. Let N^* and N be positive integers of same parity, i.e., $N^* + N = M$ is an even integer. The function $\phi^* = \phi_{N^*}$ is chosen as B-spline of order N^*. For even $N^* = 2l^*$ the corresponding scaling filter is

$$H^*(\xi) = \left(\cos\frac{\xi}{2}\right)^{N^*}.$$

If $N = 2l$, then scaling functions $\phi(x) = \phi_{N^*,N}(x)$ may be found with scaling filters

$$H(\xi) = \left(\cos\frac{\xi}{2}\right)^{N} \sum_{k=0}^{l+l^*-1} \begin{pmatrix} l + l^* - 1 + k \\ k \end{pmatrix} \left(\sin\frac{\xi}{2}\right)^{2k}.$$

Similarly, for the odd $N^* = 2l^* + 1$, and $N = 2l + 1$, the corresponding filters are

$$H^*(\xi) = e^{-i\xi/2}\left(\cos\frac{\xi}{2}\right)^{N^*}$$

and

$$H(\xi) = e^{-i\xi/2}\left(\cos\frac{\xi}{2}\right)^{N} \sum_{k=0}^{l+l^*} \begin{pmatrix} l + l^* + k \\ k \end{pmatrix} \left(\sin\frac{\xi}{2}\right)^{2k}.$$

In this case, all functions have compact support. The function ϕ^* is a C^{N^*-2} piecewise polynomials of degree $N^* - 1$, and ϕ has increasing regularity with increasing N. The functions, ϕ^* and ϕ, are symmetric functions centered at $x = 0$, for even N^* and N, and centered at $x = \frac{1}{2}$, for odd N^* and N. They satisfy Strang-Fix conditions of order $N^* - 1$ and $N - 1$, respectively.

In the extreme case $N^* = 0$, $\phi^*(x) = \delta(x)$ is the Dirac distribution and $\theta_M(x) = \phi_{0,M}$ corresponds to the interpolation scaling functions defined by Delauries and Dubuc [8]. It can be shown that

$$\theta_M(x) = \int_{\mathbb{R}} \phi_{N^*}(y)\phi_{N,N^*}(y+x)dy,$$

independently of the choices of N, N^* such that $M = N + N^*$ [12].

2.2 Other Approximation Schemes

[sea] Following Harten's formalism [13], we shall present a class of approximation schemes in the spline biorthogonal framework. Different schemes shall be distinguished by different form in which the discretization operators are defined. Hence, in the one-level setting, we shall always assume that the reconstruction operator has the form

$$\mathcal{R}^j(x;\mathbf{u}^j) = \sum_{k\in\mathbb{Z}} u^j(k)\phi(2^j x - k).$$

We say that the approximation scheme $\{\mathcal{D}^j, \mathcal{R}^j\}$ is conservative if $\mathcal{R}^j$ is a right-inverse operator of $\mathcal{D}^j$, i.e., $\mathcal{D}^j\mathcal{R}^j(\cdot;\mathbf{u}^j) = \mathbf{u}^j$, $\forall \mathbf{u}^j \in E^j$. This means that

$$(\mathcal{D}^j\phi(2^j\cdot - k))(s) = \delta_{k-s}.$$

The biorthogonal projection is an example of a conservative approximation scheme. For this case, the conservation property is equivalent to the biorthogonal relation.

Now we turn our attention to discretization operators defined in terms of discrete convolutions of point values with some specific weights. We consider three cases: interpolation, quasi-interpolation and discrete projection.

- **Interpolation Scheme**

For some $0 \leq \alpha < 1$ consider the discretization operator

$$(\mathcal{D}^j_c u)(k) = \sum_{n\in\mathbb{Z}} \gamma(n) u((k-n+\alpha)2^{-j}). \quad \text{[c2e26]} \tag{19}$$

The coefficients $\gamma = \gamma_\alpha$ are obtained in such a way that the operator

$$\mathcal{I}^j u(x) = \sum_{k\in\mathbb{Z}} (\mathcal{D}^j_c u)(k)\phi(2^j x - k) \quad \text{[c2e101]} \tag{20}$$

interpolates u at the nodes $x^j_k = (k+\alpha)2^{-j}$. Therefore, it is necessary that $(\mathcal{D}^j_c\phi(2^j\cdot - l))(k) = \delta_{k-l}$, what shows that the interpolation scheme is conservative. The interpolation constraint is equivalent to the following relation

$$\widetilde{\gamma}(\xi)\widetilde{\phi}_\alpha(\xi) \equiv 1. \quad \text{[c2e24]} \tag{21}$$

where

$$\widetilde{\phi}_\alpha(\xi) = \sum_{m\in\mathbb{Z}} \phi(m+\alpha)e^{-im\xi}, \quad \widetilde{\gamma}(\xi) = \sum_{m\in\mathbb{Z}} \gamma(m)e^{-im\xi}. \quad \text{[c2e91]} \tag{22}$$

It is possible to find coefficients $\gamma(k)$ such that relation (21) is satisfied provided that the following interpolation condition holds

$$\widetilde{\phi}_\alpha(\xi) \neq 0, \ \forall\xi. \quad \text{[ci2]} \tag{23}$$

It is well known that the B-splines functions $\phi_{N*}(x)$ satisfy the interpolation condition with $\alpha = 0$ for even N^*, and $\alpha = 1/2$ for odd N^*. Numerical experiments suggest that equivalent results are valid for the dual functions $\phi(x) = \phi_{N,N^*}(x)$. For example, Figure 1 shows the function $\widetilde{\phi}_\alpha(\xi)$ corresponding $N^* = 1$ and $N = 3$. Note that, for $\alpha = 0$, $\widetilde{\phi}_0(\pi) = \widetilde{\phi}_0(-\pi) = 0$ a fact that contradicts the interpolation condition. On the other

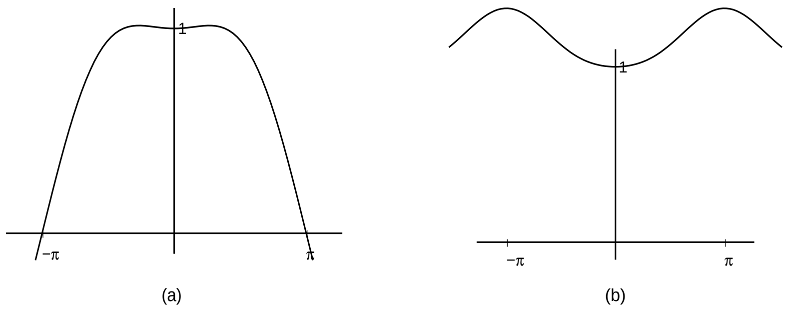


Figure 1: (a)-$\widetilde{\phi}_0$ e (b)- $\widetilde{\phi}_{1/2}$ for $N^* = 1$ and $N = 3$

[c2f1]

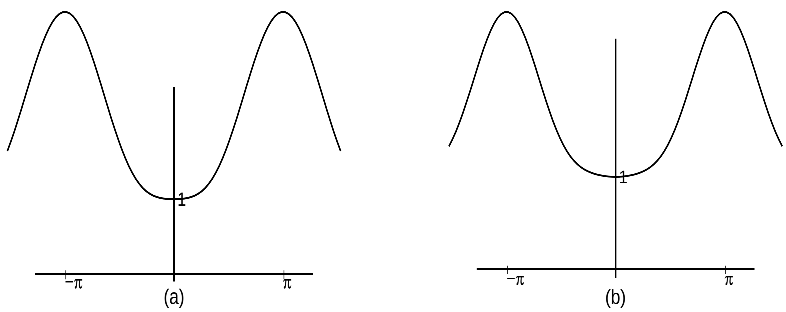


Figure 2: $\widetilde{\phi}_0$ for (a)- $N^* = 2, N = 4$ and (b)- $N^* = 2, N = 6$

[c2f4]

hand, $\widetilde{\phi}_{1/2}(\xi) \neq 0$ for $-\pi \leq \xi \leq \pi$. The same type of behavior is verified for other scaling functions with odd N. For even N, Figure 2 shows the graph of $\widetilde{\phi}_0(\xi)$ for $N^* = 2$ and $N = 4, 6$. In both cases, we have $\widetilde{\phi}_0(\xi) \neq 0$ for $-\pi \leq \xi \leq \pi$.

Assuming that the interpolation condition is verified, the coefficients $\gamma(k)$ can be obtained in terms of the Fourier's coefficients of the function $1/\widetilde{\phi}_\alpha(\xi)$. However, excepting the case $N^* = 0$, where ϕ_{N,N^*} is an interpolation function of compact support, the interpolation constraint can only be achieved with infinitely many coefficients $\gamma(k) \neq 0$. Therefore, the implementation of $\mathcal{D}_c^j$ in physical space requires truncated filter coefficients [9] and the scheme becomes non-conservative. Examples of some interpolating coefficients are presented in Table 1.

- **Quasi-Interpolation Scheme**

The interpolation constraint may be replaced by a less restrictive condition to obtain a scheme that requests only a finite number of non zero coefficients. In this case, the discretization operator has the same form (19), as in the interpolation case. However the coefficients $\gamma(k)$, are chosen in such a way that the operator $\mathcal{I}^j = \mathcal{R}^j\mathcal{D}^j$ is a quasi-interpolation of order n. That is, $\mathcal{I}^j q(x) = q(x)$ for every polynomial $q(x)$ of degree up to n. The quasi-interpolation condition may be translated into a relation between discrete moments of γ and the moments of the function ϕ.

Table 1: Interpolation coefficients for $|\gamma(k)| \geq 10^{-6}$ and $k \geq 0$

[c2t1]

(N^*, N)	k									
	0	1	2	3	4	5	6	7	8	9
(1,3)	$\frac{1181}{1339}$	$\frac{24}{379}$	$-\frac{1}{288}$	$-\frac{1}{1204}$	$-\frac{1}{35183}$	$\frac{1}{180840}$				
(1,5)	$\frac{491}{569}$	$\frac{227}{2872}$	$-\frac{7}{648}$	$-\frac{1}{6887}$	$\frac{1}{2246}$	$\frac{1}{81644}$	$-\frac{1}{114680}$			
(2,4)	$\frac{718}{1165}$	$\frac{172}{911}$	$\frac{37}{2060}$	$-\frac{9}{974}$	$-\frac{7}{1441}$	$-\frac{1}{1004}$	$\frac{1}{23306}$	$\frac{1}{10133}$	$\frac{1}{30947}$	$\frac{1}{284026}$
(2,6)	$\frac{427}{613}$	$\frac{151}{895}$	$-\frac{11}{1164}$	$-\frac{7}{761}$	$\frac{1}{1596}$	$\frac{1}{1028}$	$\frac{1}{9261}$	$-\frac{1}{19095}$	$-\frac{1}{80053}$	$\frac{1}{347092}$

The k-moments of a function, ϕ are defined by

$$\mathcal{M}_\phi^k = \int_{\mathbb{R}} x^k \phi(x) dx.$$

The scaling functions are normalized in such a way that $\mathcal{M}_\phi^0 = \mathcal{M}_{\phi^*}^0 = 1$. Using the scale relation, the moments can be calculated recursively

$$\begin{aligned} \mathcal{M}_\phi^0 &= 1 \\ \mathcal{M}_\phi^k &= \frac{1}{2^k - 1} \sum_{l=1}^{k} \begin{pmatrix} k \\ l \end{pmatrix} \mu_H^l \mathcal{M}_\phi^{k-l}, \ k = 1, 2, \ldots. \quad \text{[mo1]} \end{aligned} \tag{24}$$

where μ_H^l are the discrete moments of filter H defined by

$$\mu_H^l = \sum_{s \in \mathbb{Z}} s^i h(s). \quad \text{[moH]} \tag{25}$$

Lemma 2.1 Let p be the order of Strang-Fix condition of the function $\phi(x)$, and suppose that $0 \leq n \leq p$. The operator $\mathcal{I}^j u(x)$ is a quasi-interpolation operator of order n, if and only if the following moment relations are satisfied

$$\sum_{l=0}^{m} \begin{pmatrix} m \\ l \end{pmatrix} \mu_\gamma^l \mathcal{M}_\phi^{m-l} = \alpha^m, \ 0 \leq m \leq n, \quad \text{[c2e4]} \tag{26}$$

where

$$\mu_\gamma^l = \sum_{k \in \mathbb{Z}} k^l \gamma(k), \quad \text{[c2e5]} \tag{27}$$

[c2l9]

Proof: By definition of quasi-interpolation operator, the coefficients $\gamma(k)$ must to be so that, for $0 \leq m \leq n$,

$$\begin{aligned} x^m &= \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \gamma(n)(k - n + \alpha)^m \phi(x - k) \\ &= \sum_{s \in \mathbb{Z}} (s + \alpha)^m \sum_{n \in \mathbb{Z}} \gamma(n) \phi(x - s - n) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s\in\mathbb{Z}} (s+\alpha)^m \Phi(x-s) \\
&= \sum_{s\in\mathbb{Z}} f_\alpha(s;x), \quad \text{[c2e54]} \qquad (28)
\end{aligned}
$$

where $\Phi(x) = \sum_{n\in\mathbb{Z}} \gamma(n)\phi(x-n)$ and $f_\alpha(y;x) = (y+\alpha)^m\Phi(x-y)$. The Fourier transform of Φ satisfies $\widehat{\Phi}(\xi) = \widetilde{\gamma}(\xi)\widehat{\phi}(\xi)$, from which we conclude that $\Phi(x)$ also satisfies the Strang-Fix condition with order p. Considering that

$$
\begin{aligned}
\widehat{f}_\alpha(\xi;x) &= \int_{\mathbb{R}} e^{-i\xi y}(y+\alpha)^m\Phi(x-y)dy \\
&= e^{-ix\xi}\sum_{s=0}^{m}\begin{pmatrix} m \\ s \end{pmatrix}(x+\alpha)^{m-s}(-1)^s\int_{\mathbb{R}} e^{i\xi y}(y)^s\Phi(y)dy \\
&= e^{-ix\xi}\sum_{s=0}^{m}\begin{pmatrix} m \\ s \end{pmatrix}(x+\alpha)^{m-s}\frac{(-1)^s}{(-i)^s}\frac{d^s\widehat{\Phi}}{d\xi^s}(-\xi),
\end{aligned}
$$

then $\widehat{f}_\alpha(2\pi k;x) = 0$ for $k\in\mathbb{Z}\setminus\{0\}$. For $k=0$ we have

$$
\begin{aligned}
\widehat{f}_\alpha(0;x) &= \sum_{s=0}^{m}\begin{pmatrix} m \\ s \end{pmatrix}(x+\alpha)^{m-s}\frac{(-1)^s}{(-i)^s}\frac{d^s\widehat{\Phi}}{d\xi^s}(0) \\
&= \sum_{s=0}^{m}\begin{pmatrix} m \\ s \end{pmatrix}(x+\alpha)^{m-s}(-1)^s\mathcal{M}_\Phi^s.
\end{aligned}
$$

Applying the Poisson summation formula and the equation (28), the moments $\mathcal{M}_\Phi^s$ satisfy the relations

$$
x^m = \widehat{f}_\alpha(0;x) = \sum_{s=0}^{m}\begin{pmatrix} m \\ s \end{pmatrix}(x+\alpha)^{m-s}(-1)^s\mathcal{M}_\Phi^s.
$$

This is only possible if

$$
\mathcal{M}_\Phi^m = \alpha^m,
$$

and the statement of the Lemma follows by considering $\xi = 0$ in the expression

$$
\frac{d^m\widehat{\Phi}}{d\xi^m}(\xi) = \sum_{s=0}^{m}\begin{pmatrix} m \\ s \end{pmatrix}\frac{d^s\widetilde{\gamma}}{d\xi^s}(\xi)\frac{d^{m-s}\widehat{\phi}}{d\xi^{m-s}}(\xi).
$$

■

Next, we shall describe another useful relation characterizing a quasi-interpolation scheme.

Lemma 2.2 Let $\phi(x)$ be a function that satisfies the Strang-Fix condition with order p. Suppose that $\gamma(k)$ are the coefficients of a quasi-interpolation scheme $\mathcal{I}^j u(x)$ of order $n\leq p$. Then the following relation holds

$$
\widetilde{\gamma}(\xi)\widetilde{\phi}_\alpha(\xi) = 1 + \mathcal{O}(\xi)^{n+1}, \quad \text{[quasi]} \qquad (29)
$$

where $\widetilde{\gamma}(\xi)$ and $\widetilde{\phi}_\alpha(\xi)$ are given by (22). [c2l2]

Proof: From the definition of the functions $\widetilde{\gamma}(\xi)$ and $\widetilde{\phi}_\alpha(\xi)$ it follows that

$$\begin{aligned}
\widetilde{\gamma}(\xi)\widetilde{\phi}_\alpha(\xi) &= \widetilde{\gamma}(\xi)\sum_{k\in\mathbb{Z}}\phi(k+\alpha)e^{-i\xi k} \\
&= \widetilde{\gamma}(\xi)\sum_{k\in\mathbb{Z}}\widehat{\phi}(\xi+2k\pi)e^{i\alpha(\xi+2k\pi)} \\
&= \sum_{k\in\mathbb{Z}}\widetilde{\gamma}(\xi+2k\pi)\widehat{\phi}(\xi+2k\pi)e^{i\alpha(\xi+2k\pi)} \\
&= \sum_{k\in\mathbb{Z}}\widehat{\Phi}(\xi+2k\pi)e^{i\alpha(\xi+2k\pi)} \\
&= \widehat{\Phi}(\xi)e^{i\alpha\xi}+\sum_{k\neq 0}\widehat{\Phi}(\xi+2k\pi)e^{i\alpha(\xi+2k\pi)} \\
&= f(\xi)+\sum_{k\neq 0}f(\xi+2k\pi)
\end{aligned}$$

where we use the fact that $\widetilde{\gamma}(\xi)$ is a 2π-periodic function and $f(\xi)=e^{i\alpha\xi}\widehat{\Phi}(\xi)$. We note that $\widehat{\Phi}(\xi)=\widetilde{\gamma}(\xi)\widehat{\phi}(\xi)$, from which we conclude that Φ satisfies the Strang-Fix condition with same order of ϕ. Therefore $f(0)=1$ and $f(\xi)$ has zeros of order $p+1$ at $\xi=2k\pi$, $k\in\mathbb{Z}\setminus\{0\}$. For $1\leq n\leq p$

$$\begin{aligned}
\frac{d^n f}{d\xi^n}(0) &= \sum_{s=0}^{n}\binom{n}{s}(i\alpha)^{n-s}\frac{d^s\widehat{\Phi}}{d\xi^s}(0) \\
&= \sum_{s=0}^{m}\binom{n}{s}(-1)^s\alpha^{n-s}\mathcal{M}_\Phi^s \\
&= \alpha^n\sum_{s=0}^{n}\binom{n}{s}(-1)^s = 0
\end{aligned}$$

and the result of Lemma holds ■

Since the Strang-Fix condition determines the degree of the polynomials that can be represented in V_j, then the order of a quasi-interpolation scheme is bounded by the order of the Strang-Fix condition of $\phi(x)$. Therefore, the largest order of quasi-interpolation scheme in terms of $\phi(x)=\phi_{N,N^*}(x)$, is $N-1$ (Lemma 2.1). The coefficients $\gamma(k)$, can be obtained by solving the linear systems (26) and (27). For that, it is necessary to know the moments of the function $\phi(x)$, which can be calculated by the recursive procedure (24). Knowing the moments $\mathcal{M}_\phi^m$, $0\leq m\leq N-1$, the moments μ_γ^l can be determined by solving the linear system (26), which is upper triangular, with 1´s on the main diagonal. Therefore, the coefficients $\gamma(k)$ should be obtained by relations (27), which are of Vandermonde type. Theses equations present infinite solutions depending on the range of indices k for which $\gamma(k)$ are nonzero. Considering the support $|k|\leq\lfloor(N-1)/2\rfloor$, where $\lfloor\cdot\rfloor$ represents the integer part of the number, the coefficients $\gamma(k)$ are symmetric around $k=0$ and they are uniquely determined. Table 2 shows the coefficient $\gamma(k)\neq 0$, $k\geq 0$ for the families $(N^*,N)=(1,3),(1,5),(2,4),(2,6),(3,5),(3,7)$

- **Discrete Projection Scheme**

In opposition to biorthogonal projection and interpolation operator, quasi-interpolation schemes are not usually conservative. The conservation property is an important fact in multiscale representations, as shall be described in Section 2.3.

Table 2: Quasi-Interpolation coefficients for $\gamma(k) \neq 0$

[c2t5] (N^*, N)	k			
	0	1	2	3
(1,3)	$\frac{11}{12}$	$\frac{1}{24}$		
(1,5)	$\frac{863}{960}$	$\frac{77}{1440}$	$\frac{-17}{5760}$	
(2,4)	$\frac{5}{6}$	$\frac{1}{12}$		
(2,6)	$\frac{97}{120}$	$\frac{1}{10}$	$\frac{-1}{240}$	
(3,5)	$\frac{233}{320}$	$\frac{67}{480}$	$\frac{-7}{1920}$	
(3,7)	$\frac{173863}{241920}$	$\frac{47309}{322560}$	$\frac{-209}{32256}$	$\frac{457}{967680}$

The concept of discrete projection was introduced by Ware [15] with the idea of having a conservative quasi-interpolation scheme, where the discretization is performed with finitely many non-zero coefficients. In this case, oversampling is needed

$$(\mathcal{D}_c^j u)(k) = \sum_{n \in \mathbb{Z}} \gamma(n) u((n+2k)2^{-j-1}).$$

The coefficients $\gamma(k)$ are obtained so that the operator $\mathcal{I}^j u(x) = \mathcal{R}^j(x; \mathcal{D}_c^j u)$ is a projection, which means that the discretization operator must satisfy

$$(\mathcal{D}_c^j \phi(2^j \cdot -l))(k) = \delta_{l-k}, \quad \text{[ce2202]} \tag{30}$$

producing a conservative scheme.

Lemma 2.3 The filter coefficients $\gamma(k)$ for a discrete projection are characterized by the relation

$$1 = \overline{\widetilde{\gamma}_e(\xi)}\widetilde{\phi}_0(\xi) + \overline{\widetilde{\gamma}_o(\xi)}\widetilde{\phi}_{1/2}(\xi), \quad \text{[bezout]} \tag{31}$$

where

$$\widetilde{\gamma}_e(\xi) = \sum_{k \in \mathbb{Z}} \gamma(2k) e^{-ik\xi} \quad \text{and} \quad \widetilde{\gamma}_o(\xi) = \sum_{k \in \mathbb{Z}} \gamma(2k+1) e^{-ik\xi}. \quad \text{[c2e41]} \tag{32}$$

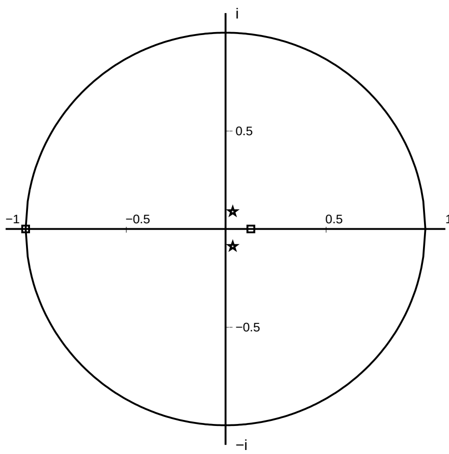
Proof: Formula (30) can be expressed as

$$\begin{aligned} 1 &= \sum_{k \in \mathbb{Z}} e^{-ik\xi} \sum_{n \in \mathbb{Z}} \gamma(n) \phi(n/2 + k) \\ &= \sum_{n \in \mathbb{Z}} \gamma(2n) \sum_{k \in \mathbb{Z}} e^{-ik\xi} \phi(n+k) + \sum_{n \in \mathbb{Z}} \gamma(2n+1) \sum_{k \in \mathbb{Z}} e^{-ik\xi} \phi(n + 1/2 + k) \\ &= \overline{\widetilde{\gamma}_e(\xi)}\widetilde{\phi}_0(\xi) + \overline{\widetilde{\gamma}_o(\xi)}\widetilde{\phi}_{1/2}(\xi), \end{aligned}$$

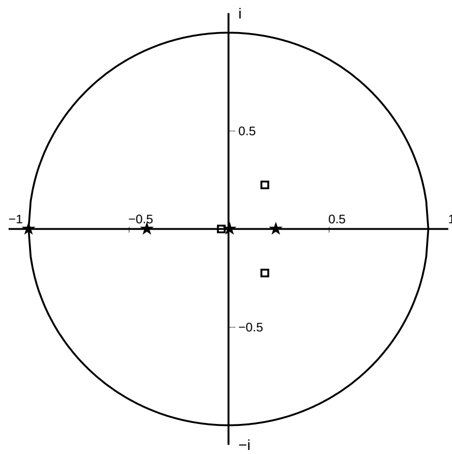
which proves the Lemma. ∎

Note that the role of the relation (31) for the discrete projection is similar to role of the relations (21) and (29) for interpolation and quasi-interpolation cases.

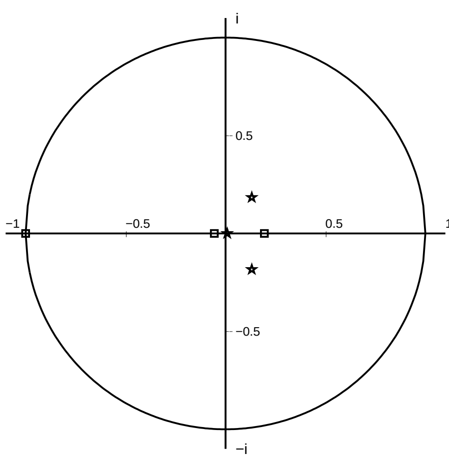
Bezout´s Theorem [7] guarantees the existence of a solution for equation (31), with finitely many nonzero coefficients, if the symbols $\widetilde{\phi}_0(\xi)$ and $\widetilde{\phi}_{1/2}(\xi)$ do not have common zeros. This property is known to be valid for the B-splines functions [6]. We have tested this property for some dual scaling functions $\phi(x) = \phi_{N,N^*}(x)$, and the results show that $\widetilde{\phi}_0(\xi)$ and $\widetilde{\phi}_{1/2}(\xi)$ do not have common zeros. However, we could not figure out yet whether this remains true or not in all the cases. Figure 3 displays zeros of $\widetilde{\phi}_0$ and $\widetilde{\phi}_{1/2}(\xi)$ for the cases $(N^*, N) = (1,3), (2,4), (3,3), (2,6)$.



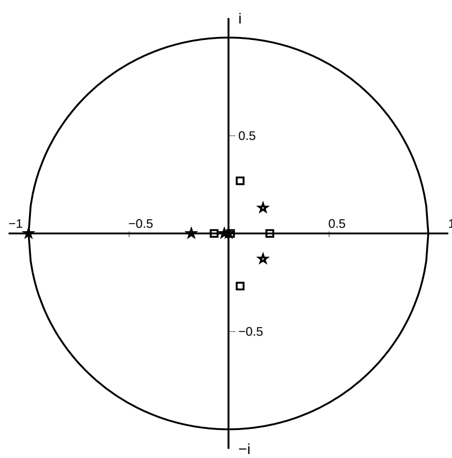
$N^* = 1$ e $N = 3$



$N^* = 2$ e $N = 4$



$N^* = 3$ e $N = 3$



$N^* = 2$ e $N = 6$

Figure 3: Zeros of ϕ_0 ($\square$) and $\phi_{1/2}$ ($\star$).

[c2f7]

As well as for the quasi-interpolation operator, the system (30), which defines the coefficients for the discrete projection, can have infinitely many solutions, depending on the range of indices k for which $\gamma(k) \neq 0$. However, fixing this range domain in $2\,|\mathrm{Supp}(\phi)| - 3$ and considering that the coefficients are symmetric around $k = 0$, the system (30) has a unique solution. Table 3 shows the coefficients $\gamma(k)$ obtained for some cases.

Table 3: Discrete Projection coefficients $\gamma(k) \neq 0$

(N^*, N)	k							
	0	1	2	3	4	5	6	7
(1,3)	$\frac{7}{9}$	$\frac{55}{576}$	$\frac{1}{72}$	$\frac{1}{576}$				
(1,5)	$\frac{1151}{1624}$	$\frac{107}{792}$	$\frac{142}{12067}$	$-\frac{19}{25832}$	$-\frac{27}{63649}$	$-\frac{4}{54441}$	$-\frac{7}{934226}$	$\frac{18}{107334566}$
(2,4)	$\frac{535}{1346}$	$\frac{2605}{10013}$	$\frac{231}{4475}$	$-\frac{94}{16799}$	$-\frac{33}{7450}$	$-\frac{9}{17374}$	$\frac{5}{205914}$	

[c2t6]

2.2.1 Discretization and Global Approximation Errors

[sec23] In the approximation schemes $\{\mathcal{D}_c^j, \mathcal{R}^j\}$ corresponding to interpolation, quasi-interpolation and discrete operator, the discretization operators $\mathcal{D}_c^j$ are an alternative form of the biorthogonal one. The next statement gives an estimation of the discretization error $\mathcal{E}^j u = \mathcal{D}^j u - \mathcal{D}_c^j u$.

Theorem 2.4 [erdis] Let ϕ and ϕ^* be integrable scaling functions with compact support, and suppose that ϕ satisfies the Strang-Fix condition with order p. If $\mathcal{D}_c^j$ is the discretization operator associated to interpolation, quasi-interpolation of order p or discrete projection, then, for $u \in \mathbf{H}^{n+1}, n \leq p$, the discretization error satisfies the estimation

$$|(\mathcal{D}_c^j u)(s) - (\mathcal{D}^j u)(s)| \lesssim 2^{-j(n+1)} \|u\|_{\mathbf{H}^{n+1}}, \ \forall s \in \mathbb{Z}. \quad \text{[discrer]} \tag{33}$$

<u>Proof:</u> In all the cases, the error $\mathcal{E}^j q = \mathcal{D}^j q - \mathcal{D}_c^j q$ is cancelled for polynomials $q \in \mathbb{P}_n, n \leq p$. For the interpolation and quasi-interpolation case we have

$$\begin{aligned} |\mathcal{E}^j u(s)| &= |\mathcal{E}^j (u-q)(s)| \\ &\leq \left| 2^j \int_{\mathbb{R}} (u-q)(x) \phi^*(2^j x - s) dx \right| + \left| \sum_{k \in \mathbb{Z}} \gamma(k)(u-q)((s-k+\alpha)2^{-j}) \right| \\ &\lesssim \max_{x \in \Omega_{j,s}} |u-q|, \end{aligned}$$

where $\Omega_{j,s} = \mathrm{Supp}(\phi^*(2^j x - s)) \cup \{\bigcup_k I_{j,k}; \gamma(s-k) \neq 0\}$, with $I_{j,k} = [2^{-j}k, 2^{-j}(k+1))$. The error estimation (33) is obtained by Whitney´s Theorem [3], which establishes that

$$\inf_{q \in \mathbb{P}_n} \max_{x \in \Omega_{j,s}} |u-q| \lesssim 2^{-j(n+1)} \|u\|_{\mathbf{H}^{n+1}(\Omega_{j,s})}.$$

For the discrete projection case, the proof is similar. ■

The global error approximation of biorthogonal projection, $\mathcal{P}^j = \mathcal{R}^j \mathcal{D}^j$, satisfies the estimation (16). The approximation schemes considered in the present paper are included in a broader class of schemes treated in [10, 11]. Global error estimates can be obtained, provided some basic hypothesis are verified, as stated in the following Lemma.

Lemma 2.5 [gamma] Let $\phi(x)$ be a scaling function of compact support satisfying a Strang-Fix condition of order p. In association with the operators of interpolation or quasi-interpolation of order p, define the function $\widetilde{\nu}(\xi) = \widetilde{\gamma}(\xi)e^{-i\alpha\xi}$. Similarly, in the case of discrete projection, let $\widetilde{\nu}(\xi) = \widetilde{\gamma}(\xi/2)$. Then the assymptotic relation is verified

$$\widetilde{\nu}(\xi)\widehat{\phi}_\alpha(\xi) = 1 + \mathcal{O}(\xi)^{p+1}.$$

Proof: For interpolation case we consider the equation (21) and we have

$$\begin{aligned} 1 &= \widetilde{\gamma}(\xi)\widetilde{\phi}_\alpha(\xi) \\ &= \widetilde{\gamma}(\xi)\sum_{k\in\mathbb{Z}} \phi(k+\alpha)e^{-ik\xi} \\ &= \widetilde{\gamma}(\xi)\sum_{k\in\mathbb{Z}} \widehat{\phi}(\xi+2k\pi)e^{-i\alpha(\xi+2k\pi)} \\ &= \widetilde{\gamma}(\xi)\widehat{\phi}(\xi)e^{-i\alpha\xi} + \widetilde{\gamma}(\xi)\sum_{k\neq 0} \widehat{\phi}(\xi+2k\pi)e^{-i\alpha(\xi+2k\pi)}. \end{aligned}$$

Being the order of the Strang-Fix condition of ϕ equal to p, it follows that

$$\widehat{\phi}(\xi)\widetilde{\gamma}(\xi)e^{-i\alpha\xi} = 1 + \mathcal{O}(\xi^{p+1}).$$

For the quasi-interpolation case, instead of (21), we consider equation (29), with $n = p$, and the proof proceeds analogously as in the interpolation case.

For the discrete projection, we take equation (31), which implies that

$$\begin{aligned} 1 &= \overline{\widetilde{\gamma}_e(\xi)}\sum_{k\in\mathbb{Z}} \phi(k)e^{-i\xi k} + \overline{\widetilde{\gamma}_o(\xi)}\sum_{k\in\mathbb{Z}} \phi(k+1/2)e^{-i\xi k} \\ &= \overline{\widetilde{\gamma}_e(\xi)}\sum_{k\in\mathbb{Z}} \widehat{\phi}(\xi+2k\pi) + \overline{\widetilde{\gamma}_o(\xi)}\sum_{k\in\mathbb{Z}} \widehat{\phi}(\xi+2k\pi)e^{i/2(\xi+2k\pi)} \\ &= \sum_{k\in\mathbb{Z}} \widehat{\phi}(\xi+2k\pi)\left(\overline{\widetilde{\gamma}_e(\xi)} + \overline{\widetilde{\gamma}_o(\xi)}e^{i(\xi/2+k\pi)}\right) \\ &= \sum_{k\in\mathbb{Z}} \widehat{\phi}(\xi+2k\pi)\left(\sum_{s\in\mathbb{Z}} \gamma(2s)e^{is\xi} + e^{ik\pi}\sum_{s\in\mathbb{Z}} \gamma(2s+1)e^{\frac{i(2s+1)\xi}{2}}\right) \\ &= \sum_{k\in\mathbb{Z}} \widehat{\phi}(\xi+2k\pi)\left(\sum_{s\in\mathbb{Z}} \gamma(2s)e^{i2s(\xi/2+k\pi)} + \sum_{s\in\mathbb{Z}} \gamma(2s+1)e^{i(2s+1)(\xi/2+k\pi)}\right) \\ &= \sum_{k\in\mathbb{Z}} \widehat{\phi}(\xi+2k\pi)\overline{\widetilde{\gamma}(\xi/2+k\pi)}. \end{aligned}$$

From the Strang-Fix condition of ϕ, we have

$$\widetilde{\gamma}(\xi/2)\widehat{\phi}(\xi) = 1 + \mathcal{O}(\xi^{p+1}). \quad \text{[eqdp3]} \tag{34}$$

■

Having in mind the statement of Lemma 2.5, the application of the results in [10, 11] implies the following global error estimates.

Theorem 2.6 Let $\phi(x)$ be a scaling function of compact support satisfying a Strang-Fix condition of order p. If $\mathcal{I}^j u(x)$ is an operator of interpolation, quasi-interpolation of order p or discrete projection associated to ϕ, then the following approximation error estimate holds

$$\|u - \mathcal{I}^j u\|_{\mathbf{H}^s} \le C2^{-j(N-s)}||u||_{\mathbf{H}^N},$$

for $0 \le s \le \min\{N, r\}$, where r is the regularity degree such that $\phi \in \mathbf{H}^r$.

2.3 Multilevel Approximation Schemes

[anaME]

As described in Section 2.1, the biorthogonal projection $\mathcal{P}^j$ may be expressed in two ways. There is the one-level representation $\mathcal{P}^j = \mathcal{R}^j\mathcal{D}^j$, where the discretization and reconstruction operators are expressed in terms of scaling functions, and there is the multilevel representation $\mathcal{P}^j = \mathcal{R}^j_{ME}\mathcal{D}^j_{ME}$, , where the discretization operator $\mathcal{D}^j_{ME}$ and reconstruction operator $\mathcal{R}^j_{ME}$ are expressed in terms of multilevel wavelet bases.

In Section 2.2, other types of one-level approximation schemes $\mathcal{I}^j = \mathcal{R}^j\mathcal{D}^j_c$ have been considered for discretizations $\mathcal{D}^j_c$ defined in terms of discrete convolutions with function point values. In the multilevel context, there is also interest in considering approximation schemes $\{\mathcal{D}^j_{c,ME}, \mathcal{R}^j_{ME}\}$ where the discretization operators

$$\mathcal{D}^j_{c,ME} = \{\mathcal{D}^J_c, \mathcal{G}^J_c, \dots, \mathcal{G}^{j-1}_c\}$$

are also functionals defined by discrete convolutions. In this sense, one idea could be to define $\mathcal{G}^l_c u$ by the substitution of $u^{l+1} = \mathcal{D}^{l+1}u$ in formula (12) by some of the alternative discretizations $\mathcal{D}^{l+1}_c u$. That is,

$$(\mathcal{G}^l_c u)(k) = \sum_{m\in\mathbb{Z}} g^*(m-2k)(\mathcal{D}^{l+1}_c u)(m). \quad \text{[gdis]} \tag{35}$$

If $\{\mathcal{D}^j_c, \mathcal{R}^j\}$ is a conservative scheme then, it holds

$$(\mathcal{G}^l_c \phi(2^l\cdot - m))(k) = 0, \quad (\mathcal{G}^l_c \psi(2^l\cdot - m))(k) = \delta_{m-k}, \quad \text{[c2e6]} \tag{36}$$

which means that the scheme $\{\mathcal{D}^j_{c,ME}, \mathcal{R}^j_{ME}\}$ is also conservative. In this sense, let $\mathcal{G}^l_c u$ be the discretization defined by (35), where $\mathcal{D}^{l+1}_c u$ is associated with interpolation or discrete projection operator, which are conservative schemes. If

$$u(x) = \sum_{k\in\mathbb{Z}} u^J(k)\phi(2^J x - k) + \sum_{l\geq J}\sum_{k\in\mathbb{Z}} d^l(k)\psi(2^l x - k),$$

then, by the conservation property (36), it follows that

$$(\mathcal{G}^\lambda_c u)(s) = d^\lambda(s) + \sum_{l\geq\lambda+1}\sum_{k\in\mathbb{Z}} d^l(k)(\mathcal{G}^\lambda_c\psi(2^l\cdot - k))(s). \quad \text{[c2e3]} \tag{37}$$

This equation shows that the aliasing error $\mathcal{G}^\lambda u - \mathcal{G}^\lambda_c u$ can be expanded in terms of contributions from superior levels $l \geq \lambda + 1$.

By Lemma 2.4 it follows that

$$|(\mathcal{G}^\lambda_c u)(s) - (\mathcal{G}^\lambda u)(s)| \lesssim 2^{-(\lambda+1)(p+1)}||u||_{H^{p+1}}, \quad \text{[c2e47]} \tag{38}$$

which is not reasonable for less refined scale levels.

2.3.1 An Alternative Multilevel Discretization

Having in mind the degradation of the aliasing error (38) at coarse scale levels, we shall describe a procedure to improve this estimate. It was suggested by Fröhlich and Schneider [9] for interpolation case and explored by Ware [15] for the discrete projection.

Define

$$v(x) = u(x) - \sum_{k\in\mathbb{Z}} d^\lambda(k)\psi(2^\lambda x - k),$$

and let

$$(\mathcal{G}_c^\lambda v)(s) = d^\lambda(s) + \sum_{l\geq\lambda+2}\sum_{k\in\mathbb{Z}} d^l(k)(\mathcal{G}_c^\lambda\psi(2^l\cdot - k))(s).$$

If $\check{d}^\lambda = \mathcal{G}_c^\lambda v$ is used as an approximation for $d^\lambda = \mathcal{G}^\lambda u$, then the error $\check{d}^\lambda - d^\lambda$ only depends on the wavelet coefficients of u on levels $l \geq \lambda + 2$, one order higher than in formula (37). Consequently, the accuracy order for the estimation (38) is improved. This argument can be applied to obtain approximations of d^λ in unrefined levels, which are more precise than the ones given by $\mathcal{G}_c^\lambda u$. Precisely, if $\check{d}^{j-1} = \mathcal{G}_c^{j-1}u$, then, for $m = 2, 3, \ldots$, define $\check{d}^{j-m}(s) = (\mathcal{G}_c^{j-m}v)(s)$, where $v(x)$ is the modified function

$$v(x) = u(x) - \sum_{n=1}^{m-1}\sum_{k\in\mathbb{Z}} \check{d}^{j-n}(k)\psi(2^{j-n}x - k).$$

Therefore, we obtain the multilevel discretization

$$\check{\mathcal{D}}_{c,ME}^j u = \check{\mathbf{u}}_{ME}^j = \{\check{\mathbf{u}}^J, \check{\mathbf{d}}^J, \ldots \check{\mathbf{d}}^{j-1}\},$$

which is the result of the Modified Analysis Algorithm 2.1.

Algorithm 2.1 Modified Analysis

[alg1]

Require: $u(x), \;\; x \in X^j$

  for $l = j - 1 : (-1) : J$ **do**

    $\check{\mathbf{d}}^l \leftarrow \mathcal{G}_c^l(u)$

    $u(x) \leftarrow u(x) - \sum_{k\in\mathbb{Z}} \check{d}^l(k)\psi(2^l x - k), \;\; x \in X^l$

  end for

  $\check{\mathbf{u}}^J \leftarrow \mathcal{D}_c^J(u)$

Ensure: $\check{\mathbf{u}}_{ME}^j = \{\check{\mathbf{u}}^J, \check{\mathbf{d}}^J, \ldots \check{\mathbf{d}}^{j-1}\}$

The inverse transform is obtained by the Modified Synthesis Algorithm 2.2. As the functionals $\mathcal{D}_c^l$ and $\mathcal{G}_c^l$ are defined in terms of a finite number of non zero coefficients, in both algorithms, the total number of operations is of the order $\sum_{l=J}^j \#X^l$.

In Ware [15] an estimation for the aliasing error is given in the case of the discrete projection. The proof can be easily extended to consider the case interpolation.

Theorem 2.7 [alternativo] Let $\mathcal{D}_c^j$ be the discretization associated with the interpolation or to the discrete projection. If $\check{\mathbf{u}}_{ME}^j = \{\check{\mathbf{u}}^J, \check{\mathbf{d}}^J, \ldots \check{\mathbf{d}}^{j-1}\}$ is the multiresolution analysis of u generated by the corresponding to Algorithm 2.1. If $u \in \mathbf{H}^{p+1}$, and $\alpha > 0$, then we have

$$||\mathbf{u}_{ME}^j - \check{\mathbf{u}}_{ME}^j||_\infty \lesssim 2^{-j(p+1-\alpha)}||u||_{\mathbf{H}^{p+1}},$$

where p is the Strang-Fix condition order of function ϕ.

Algorithm 2.2 Modified Synthesis

[sintese]

Require: $\check{\mathbf{u}}^j_{ME} = \{\check{\mathbf{u}}^J, \check{\mathbf{d}}^J, \ldots \check{\mathbf{d}}^{j-1}\}$

$$u(x) \leftarrow \sum_{k\in\mathbb{Z}} \check{u}^J(k)\phi(2^J x - k), \;\; x \in X^J$$

for $l = J : 1 : j-1$, **do**

$$u(x) \leftarrow u(x) + \sum_{k\in\mathbb{Z}} \check{d}^l(k)\psi(2^l x - k), \quad x \in X^l$$

$$u(x) \leftarrow \sum_{k\in\mathbb{Z}} \check{u}^l(k)\phi(2^l x - k) + \sum_{k\in\mathbb{Z}} \check{d}^l(k)\psi(2^l x - k), \quad x \in X^{l+1}/X^l$$

$\check{\mathbf{u}}^{l+1} \leftarrow \mathcal{D}_c^{l+1} u$

end for

Ensure: $u(x)$, $x \in X^j$.

2.3.2 The Importance of the Conservation Property

[scon] The conservation property allows us to obtain the expression (37) for the aliasing error which is fundamental for a good performance of algorithm 2.1. If the scheme is not conservative, the relation (36) is not valid and an equation similar to (37) cannot be obtained. This is the case of the non conservative quasi-interpolation operators.

In order to verify these facts, we shall apply the modified Analysis Algorithm 2.1 corresponding to the quasi-interpolation scheme to the Fourier modes. Let $u(x) = e^{-i\eta x}$. Thus

$$\begin{aligned}(\mathcal{D}_c^{j-m} u)(s) &= e^{-i2^m z s}\overline{\widetilde{\gamma}(2^m z)},\\ (\mathcal{G}_c^{j-m} u)(s) &= e^{-i2^m z s} S_1(2^{m-1} z),\end{aligned}$$

where $z = 2^{-j}\eta$, $S_1(z) = G^*(z)\overline{\widetilde{\gamma}(z)}$ with $G^*(z) = \sum_{n\in\mathbb{Z}} g^*(n)e^{-inz}$.

The function u is modified on each iteration m and the coefficients $\check{d}^{j-m}(s)$ are applied to this new function to get

$$\check{d}^{j-m}(s) = (\mathcal{G}_c^{j-m} u)(s) - \sum_{n=1}^{m-1}\sum_{k\in\mathbb{Z}} \check{d}^{j-n}(k)(\mathcal{G}_c^{j-m}\psi(2^{j-n}\cdot - k))(s). \;\; \text{[c2e29]} \qquad (39)$$

In general, for $\lambda + 1 \leq \kappa$ we have

$$\mathcal{G}_c^{\lambda}(\psi(2^\kappa \cdot - k))(s) = \sum_{l\in\mathbb{Z}} g^*(l-2s)\sum_{n\in\mathbb{Z}}\gamma(n)\psi(2^{\kappa-\lambda-1}(l-n+\alpha)-k). \;\; \text{[c2e30]} \qquad (40)$$

Considering $m = 1$ in (39) it follows that

$$\check{d}^{j-1}(s) = \mathcal{G}_c^{j-1}(u)(s) = e^{-i2zs}S_1(z).$$

Through the results obtained in (39) and (40), for $m = 2$ it holds

$$\begin{aligned}\check{d}^{j-2}(s) &= \mathcal{G}_c^{j-2}(u)(s) - \sum_{k\in\mathbb{Z}} \check{d}^{j-1}(k)(\mathcal{G}_c^{j-2}\psi(2^{j-1}\cdot - k))(s)\\ &= e^{-i4zs}S_1(2z)\left[1 - S_1(z)\overline{\widetilde{\psi}(2z)}\right]\\ &= e^{-i4sk}S_2(z),\end{aligned}$$

where e $\widetilde{\psi}(z) = \sum_{k\in\mathbb{Z}} e^{-isz}\psi(k)$. Applying this procedure for $m = 3, 4, \ldots$, we obtain

$$\check{d}^{j-m}(s) = e^{-i2^m zs} S_m(z),$$

where $S_m(z)$ are calculate by interactive formula

$$S_m(z) = S_1(2^{m-1}z)\left\{1 - \left[\sum_{n=1}^{m-1} S_n(z)\overline{\widetilde{\psi}(2^n z)}\right]\right\}, \quad \text{with} \quad S_1 = G^*(z)\overline{\widetilde{\gamma}(z)}.$$

For comparison, consider also the case of discrete projection. Now, the iterative formula is given by

$$S_m(z) = S_1(2^{m-1}z)\left\{1 - \left[\sum_{n=1}^{m-2} S_n(z)\overline{\widetilde{\psi}(2^n z)}\right]\right\} - S_{m-1}(z)T(2^{m-1}z), \;\; m > 2, \;\; \text{[eq1]} \quad (41)$$

where

$$\begin{aligned} S_1(z) &= G^*(z)\overline{\widetilde{\gamma}(z/2)}, \\ S_2(z) &= S_1(2z) - S_1(z)T(2z), \\ T(z) &= \widetilde{G}^*(z)\left[\widetilde{\gamma}_e(z)\overline{\widetilde{\psi}(z)} + \widetilde{\gamma}_o(z)\overline{\widetilde{\psi}_{1/2}(z)}\right]. \end{aligned}$$

The exact wavelet coefficients are $d^{j-m}(k) = e^{-i2^{j-m}kz}\widehat{\psi}^*(2^m z)$. Therefore, the aliasing error, in both cases, satisfies

$$\begin{aligned} e^m(k) &= |\check{d}^{j-m}(k) - d^{j-m}(k)| \\ &= |e^{-i2^m zk}| \; |G^*(2^m z)\widehat{\phi}^*(2^m z) - S_m(z)| \\ &\sim c(m)z^{2N}. \end{aligned}$$

Numerical results show that the term $c(m)$ maintains a growth of the type $c_{DP}(m) = 2^{Nm}$, for the discrete projection case, and $c_{QI}(m) = 2^{2Nm}$, for quasi-interpolation case. Figure 4 shows this result for the family $(2,4)$ and $(2,6)$. In both cases, the lines marked with $\circ$ and $+$ correspond to the degradation factor for the aliasing error associated to the simple algorithm $\check{d}_c^{j-m} = \mathcal{G}_c^{j-m}u$ for quasi-interpolation and discrete projection, respectively, which are quite similar in both cases. On the other hand, the lines marked with $\square$ and $\lozenge$ correspond to their modified versions. As expected, it is noticeable the decrease in the degradation factor for the modified conservative discrete projection case. We also note that in the quasi-interpolation case, not only the modified algorithm is not able to improve the aliasing error but it even gets worst.

3 Second Part: Discretization of Differential Operators

[parttwo] This section is dedicated to discretizations $\mathcal{L}^j$ for the operator $\mathcal{L}(u,v) = uv_x$. Using the concepts of discretization and reconstruction operators, a general formulation is described for several strategies. This methodology was proposed by Cullen and Morton [5] in order to generalize the concept of truncation error used in different schemes. Precisely, for a differential operator $\mathcal{L}(u,v)$ we shall consider a general form of discretization $\mathcal{L}^j(\mathbf{u}^j, \mathbf{v}^j)$ given by

$$\mathcal{L}(u,v) \sim \mathcal{L}^j(\mathbf{u}^j, \mathbf{v}^j) = \mathcal{D}_c^j\left[\mathcal{L}(\mathcal{R}^j(x;\mathbf{u}^j), \mathcal{R}^j(x;\mathbf{v}^j))\right], \;\; \text{[c2e8]} \quad (42)$$

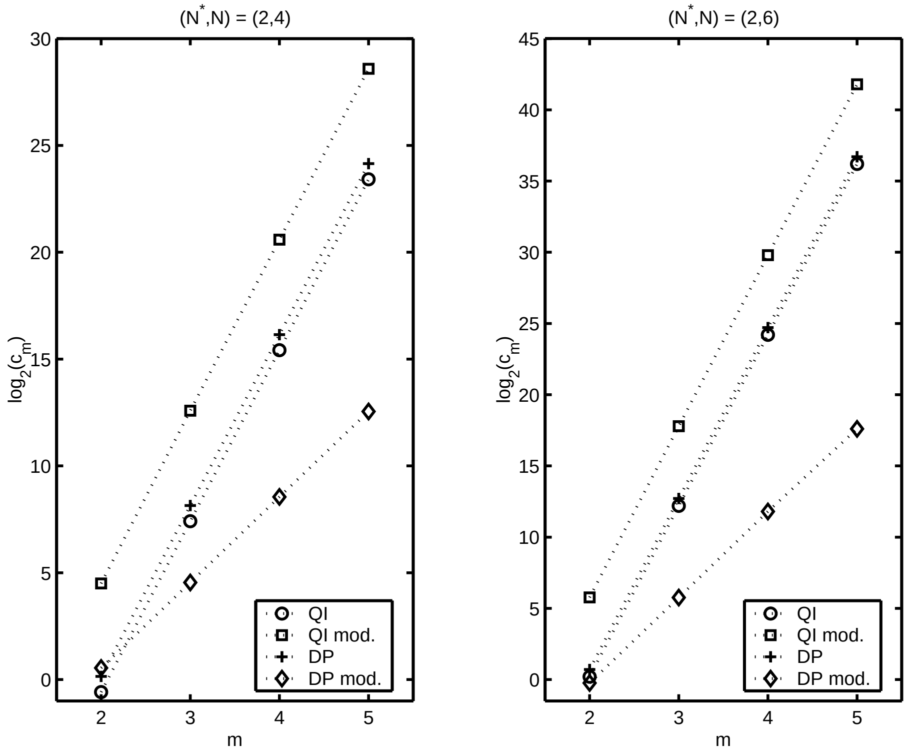


Figure 4: Degradation Factor for the Aliasing Error

[fig3]

where $\mathbf{u}^j = \mathcal{D}^j u$ and $\mathbf{v}^j = \mathcal{D}^j v$. If $\mathcal{D}_c^j = \mathcal{D}^j$ we have a Petrov-Galerkin scheme, which usually does not give an efficient strategy for the evaluation of nonlinear terms. Instead, hybrid formulations use two different discretizations ($\mathcal{D}_c^j \neq \mathcal{D}^j$). The operator $\mathcal{D}^j$ is used in discretization of u and v and the operator $\mathcal{D}_c^j$ is used after differentiation and multiplication. For the consistence analysis, we shall consider the truncation error given by

$$TE(u,v) \quad = \quad \mathcal{D}^j \mathcal{L}(u,v) - \mathcal{L}^j(\mathbf{u}^j, \mathbf{v}^j). \text{ [c2e10]} \tag{43}$$

The purpose of this analysis is to establish the consistency of the discretization $\mathcal{L}^j$ in terms the truncation error order. We shall focus on nonlinear interactions of Fourier modes $u(x) = e^{-i\eta x}$ and $v(x) = e^{-i\zeta x}$, where $\eta, \zeta \in \mathbb{R}$. To fix ideas, in what follows, the exposition shall be restricted to approximation schemes in the context of biorthogonal multiresolution analyses defined by splines $\phi^* = \phi_N$ and their duals $\phi = \phi_{N,N^*}$. However, the same analysis can be performed for other similar contexts.

3.1 Petrov-Galerkin Formulation

Given the discrete values $\mathbf{u}^j$ and $\mathbf{v}^j$, the calculation of $\mathcal{L}^j(\mathbf{u}^j, \mathbf{v}^j)$ is given by

$$\begin{aligned} \mathcal{L}^j(\mathbf{u}^j, \mathbf{v}^j)(s) &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} u^j(m) v^j(n) \int_{\mathbb{R}} \phi^*(y)\phi(y+s-m)\frac{d\phi}{dy}(y+s-n)dy \\ &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} u^j(m) v^j(n) \lambda(s-m, s-n), \text{ [c2e19]} \end{aligned} \tag{44}$$

where

$$\lambda(m,n)=\lambda_{N,N^*}(m,n)=\int_{\mathbb{R}}\phi^*(y)\phi(y+m)\frac{d\phi}{dy}(y+n)dy. \quad \text{[c2e17]} \tag{45}$$

The values of the coefficients $\lambda(m,n)$ depend on different choices of N and N^*. Considering the Fourier modes $u(x)=e^{-i\eta x}$ and $v(x)=e^{-i\xi x}$ it proceeds that

$$u^j(m)=e^{-i2^{-j}\eta m}\widehat{\phi}^*(2^{-j}\eta) \quad \text{and} \quad v^j(n)=e^{-i2^{-j}\xi n}\widehat{\phi}^*(2^{-j}\xi).$$

Therefore the discrete operator is given by

$$\mathcal{L}^j(u^j,v^j)(s)=e^{-is(w+z)}\widehat{\phi}^*(w)\widehat{\phi}^*(z)\overline{\widetilde{\lambda}(w,z)}, \quad \text{[eq01]} \tag{46}$$

where $w=2^{-j}\eta$, $z=2^{-j}\xi$ and

$$\widetilde{\lambda}(w,z)=\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}e^{-imw}e^{-inz}\lambda(m,n).$$

On the other hand,

$$(\mathcal{D}^j\mathcal{L}(u,v))(s)=-i\xi e^{-is(w+z)}\widehat{\phi^*}(w+z). \quad \text{[eq11]} \tag{47}$$

Substituting (47) and (46) in truncation error equation (43), we obtain

$$(TE)\,(s)=-i\xi e^{-is(w+z)}\left[\widehat{\phi}^*(w+z)-\frac{i}{z}\widehat{\phi}^*(w)\widehat{\phi}^*(z)\overline{\widetilde{\lambda}(w,z)}\right]=-i\xi e^{-is(w+z)}\Lambda(w,z).$$

The order of the truncation error depends on the behavior of the symbol

$$\Lambda(w,z)=\widehat{\phi}^*(w+z)-\frac{i}{z}\widehat{\phi}^*(w)\widehat{\phi}^*(z)\overline{\widetilde{\lambda}(w,z)}. \quad \text{[eq09]} \tag{48}$$

As shall be proved in Theorem 3.2 of Section 3.3,

$$\Lambda(w,z)\sim\sum_{m=0}^{M-1}\mathcal{O}(w^m z^{M-m}). \quad \text{[est]} \tag{49}$$

In spite of the fact that the asymptotic order depends only on M, the asymptotic constants also depend on the choice of N and N^*. We shall consider some numerical evidences of this fact. To calculate the coefficients $\lambda(m,n)$, some properties are required. By the scale relation (45) we obtain that the coefficients $\lambda(m,n)$ satisfy the eigenvector problem

$$\lambda(m,n)=8\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\alpha(2m-k,2n-l)\lambda(n,m), \quad \text{[eq14]} \tag{50}$$

where $\alpha(k,l)$ depends on filter coefficients of H and H^* as following

$$\alpha(k,l)=\sum_{s\in\mathbb{Z}}h^*(s)h(s+k)h(s+l).$$

The statements of the next lemma are also useful for the calculations.

Lemma 3.1 The coefficients $\lambda(m,n)$, defined in (45), satisfy

(i) $\forall m,n\in\mathbb{Z},\ \lambda(-m,-n)=-\lambda(m,n)$.

(ii) $\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} m\lambda(m,n) = 0.$

(iii) $\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} n\lambda(m,n) = -1.$

[c2l1]

The main ingredients for the proof are the symmetry of ϕ and the fact that the set $\{\phi(\cdot - k); k \in \mathbb{Z}\}$ is a partition of the unit, i.e., $\sum_{k\in\mathbb{Z}} \phi(x-k) = 1, \ \forall x \in \mathbb{R}$.

We report some numerical experiments that confirm the estimation (49) and give the asymptotic constants of symbol Λ for families $(N^*, N) = (1,3), (1,5)$. The values of $\lambda(p,q)$ are obtained by solving the eigenvalue problem (50) for $N = 3, 5$. All the cases present 2-dimension eigenspaces, but only one eigenvector satisfies the normalization criterion, described in Lemma 3.1. In this way, we obtain the following results:

- $N^* = 1$, $N = 3$
  In this case, $\lambda(n,m) \neq 0$ for $-2 \leq n \leq 2$ and $-2 \leq m \leq 2$. Table 4 shows the values of $\lambda(m,n) \neq 0$ for $m \geq 0$. For $m < 0$ we observe that $\lambda(m,n) = -\lambda(-m,-n)$.

Table 4: Values of $\lambda(m,n)$ for $N^* = 1$ e $N = 3$

m	n				
	-2	-1	0	1	2
0	$-\frac{11}{180}$	$\frac{28}{45}$			
1	$\frac{1}{90}$	$-\frac{31}{270}$	$\frac{7}{30}$	$-\frac{1}{6}$	$\frac{1}{27}$
2	$\frac{1}{4320}$	$\frac{1}{240}$	$-\frac{1}{80}$	$\frac{5}{432}$	$-\frac{1}{288}$

[c2t3]

The first terms of $\Lambda(w,z)$ is given by

$$\Lambda(w,z) \quad \sim \quad \frac{1}{30}z^4 + \frac{1}{30}z^3 w + \frac{1}{80}z^2w^2 - \frac{1}{80}zw^3.$$

This result can be compared to one obtained by Cullen-Morton for Galerkin scheme, using the hat function. They obtained

$$\Lambda(w,z) \quad \sim \quad \frac{1}{180}z^4 + \frac{1}{90}z^3 w + \frac{7}{720}z^2w^2 - \frac{1}{360}zw^3.$$

Note that the constants are smaller than for the Petrov-Galerkin scheme of the same order.

- $N^* = 1$, $N = 5$
  In this case $\lambda(m,n) \neq 0$ for $-4 \leq m \leq 4$ e $-4 \leq n \leq 4$. Table 5 shows the values of $\lambda(m,n) \neq 0$ for $m \geq 0$, and we obtain

$$\Lambda(w,z) \quad \sim \quad \frac{4}{511}z^6 + \frac{4}{511}z^5 w + \frac{5}{2016}z^4w^2 - \frac{5}{2016}zw^5.$$

Table 5: Values of $\lambda(m,n)$ for $N^*=1$ e $N=5$

[tab35]

m	n								
	-4	-3	-2	-1	0	1	2	3	4
0	$\frac{20}{89267}$	$\frac{314}{29267}$	$-\frac{395}{3003}$	$\frac{611}{837}$					
1	$-\frac{10}{190967}$	$-\frac{40}{13383}$	$\frac{369}{11429}$	$-\frac{768}{4751}$	$\frac{466}{1715}$	$-\frac{2281}{12630}$	$\frac{265}{5374}$	$-\frac{46}{5879}$	$-\frac{8}{39923}$
2	$\frac{1}{151618}$	$\frac{11}{37150}$	$-\frac{201}{44792}$	$\frac{245}{9993}$	$-\frac{181}{4285}$	$\frac{40}{1387}$	$-\frac{171}{20428}$	$\frac{76}{54335}$	$\frac{3}{73456}$
3		$\frac{8}{212103}$	$\frac{1}{35896}$	$-\frac{17}{20471}$	$\frac{3}{1721}$	$-\frac{45}{31771}$	$\frac{30}{54131}$	$-\frac{19}{170011}$	$-\frac{1}{223264}$
4			$\frac{1}{678343}$	$-\frac{1}{128891}$	$\frac{1}{74371}$	$-\frac{2}{195791}$	$\frac{1}{265548}$		

[c2t4]

3.2 Hybrid Schemes

[sechi] In general, Galerkin or Petrov-Galerkin schemes are not efficient strategies for non-linear operators. To overcome this difficulty, pseudo-spectral schemes appear in numerical analysis of nonlinear evolution equations, where the linear part is calculated in the Fourier space, and the nonlinear terms are evaluated in the physical domain. In wavelet analysis, similar pseudo-wavelet methods have been adopted. [2, 3, 9, 14]. In these formulations, the evaluation of the nonlinear terms in physical domain uses functionals defined in terms of point values like, interpolation and quasi-interpolation. Our purpose is to give a common formulation for these schemes in order to have a unified framework for the analysis of the truncation error.

- **Interpolation and quasi-interpolation schemes**

In both cases, the discretization operator has the form

$$(\mathcal{D}_c^j u)(k) = \sum_{n\in\mathbb{Z}} \gamma(n) u((k-n+\alpha)2^{-j}), \tag{51}$$

for some $0 \leq \alpha < 1$. The discretization of $\mathcal{L}(u,v)$, defined in (42), is given by

$$\mathcal{L}^j(u^j,v^j)(s) = 2^j \sum_{k\in\mathbb{Z}} \gamma(k) \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} u^j(m) v^j(n) \phi(s-k+\alpha-m) \frac{d\phi}{dx}(s-k+\alpha-n). \quad \text{[c2e23]} \tag{52}$$

Applying the discretization operator on the Fourier modes $u(x)=e^{-i\eta x}$ and $v(x)=e^{-i\zeta x}$ we have $u^j(m) = e^{-iwm}\ \widehat{\phi}^*(w)$ and $v^j(n) = e^{-izn}\widehat{\phi}^*(z)$, where $w = 2^{-j}\eta$ and $z = 2^{-j}\zeta$. So, we have

$$\mathcal{L}^j(u^j,v^j)(s) = 2^j \widehat{\phi}^*(z)\widehat{\phi}^*(w) e^{-is(z+w)} \overline{\widetilde{\gamma}(z+w)}\ \ \overline{\widetilde{\phi}_\alpha(w)}\ \ \overline{\widetilde{\beta}_\alpha(z)}, \tag{53}$$

where

$$\widetilde{\gamma}(\xi) \quad = \quad \sum_{k\in\mathbb{Z}} e^{-i\xi k}\gamma(k), \tag{54}$$

$$\widetilde{\phi}_\alpha(\xi) \quad = \quad \sum_{m\in\mathbb{Z}} e^{-i\xi m}\phi(m+\alpha), \quad \textbf{[c2e25]} \tag{55}$$

$$\widetilde{\beta}_\alpha(\xi) \quad = \quad \sum_{n\in\mathbb{Z}} e^{-i\xi n}\frac{d\phi}{dx}(n+\alpha). \quad \textbf{[c2e27]} \tag{56}$$

Therefore, the truncation error has the following form

$$(TE)\,(s) = -i\zeta e^{-is(w+z)}\left[\widehat{\phi}^*(w+z) - \frac{i}{z}\widehat{\phi}^*(w)\widehat{\phi}^*(z)\overline{\widetilde{\phi}_\alpha(w)}\ \ \overline{\widetilde{\beta}_\alpha(z)}\ \ \overline{\widetilde{\gamma}(w+z)}\right]. \quad \textbf{[c2e28]} \tag{57}$$

- **Discrete projection scheme**

In this case, the discretization operator is given by

$$(\mathcal{D}^j_c u)(k) = \sum_{n\in\mathbb{Z}} \gamma(n)u((n+2k)2^{-j-1}).$$

Applying to the Fourier modes $u(x) = e^{-i\eta x}$ and $v(x) = e^{-i\zeta x}$, we obtain

$$\begin{aligned}
\mathcal{L}^j(u^j,v^j)(s) &= 2^j\widehat{\phi}^*(z)\widehat{\phi}^*(w)e^{-is(w+z)}\left[\widetilde{\gamma}_e(w+z)\overline{\widetilde{\phi}_0(w)}\ \overline{\widetilde{\beta}_0(w)} + \widetilde{\gamma}_o(w+z)\overline{\widetilde{\phi}_{1/2}(w)}\ \overline{\widetilde{\beta}_{1/2}(w)}\right] \\
&= 2^j\widehat{\phi}^*(z)\widehat{\phi}^*(w)e^{-is(w+z)}\overline{\widetilde{\Gamma}(w,z)},
\end{aligned} \tag{58}$$

where, $\widetilde{\gamma}_e(\xi)$ and $\widetilde{\gamma}_o(\xi)$ are defined in (32) and

$$\widetilde{\Gamma}(w,z) = \overline{\widetilde{\gamma}_e(w+z)}\ \widetilde{\phi}_0(w)\widetilde{\beta}_0(z) + \overline{\widetilde{\gamma}_o(w+z)}\ \widetilde{\phi}_{1/2}\widetilde{\beta}_{1/2}(z).$$

The truncation error for the discrete projection is given by

$$(TE)\,(s) = -i\zeta e^{-is(w+z)}\left[\widehat{\phi^*}(w+z) - \frac{i}{z}\widehat{\phi^*}(w)\widehat{\phi^*}(z)\overline{\widetilde{\Gamma}(w,z)}\right]. \tag{59}$$

As shall be proved in Theorem 3.3, in all three hybrid formulations, the symbol $\Lambda(w,z)$ satisfies the asymptotic behavior

$$\Lambda(w,z) \sim \sum_{j=0}^{N}\mathcal{O}(w)^j\mathcal{O}(z)^{N-j},$$

for even N, and

$$\Lambda(w,z) \sim \mathcal{O}(z)^{N-1} + \sum_{j=0}^{N+1}\mathcal{O}(w)^j\mathcal{O}(z)^{N+1-j},$$

for odd N. In Table 6 numerical results are reported for the cases $(N^*,N) = (1,5),(2,4)$, where we give the asymptotic constants of the symbol Λ.

3.3 Truncation Error Analysis

[ter]

In all the considered formulations, the truncation errors for the Fourier's modes are given by the general form

Table 6: Asymptotic terms of Λ.

Scheme	(1,5)	(2,4)
I	$\frac{23}{180}z^4$	$\frac{691}{3150}z^4 + \frac{184}{315}z^3w + \frac{92}{105}z^2w^2 + \frac{184}{315}zw^3$
QI	$\frac{23}{180}z^4$	$\frac{83}{1200}z^4 - \frac{1}{60}z^3w - \frac{1}{40}z^2w^2 - \frac{1}{60}zw^3 - \frac{757}{5040}w^4$
DP	$-\frac{46}{315}z^4$	$\frac{83}{7996}z^4 + \frac{123}{4450}z^3w + \frac{109}{2629}z^2w^2 + \frac{123}{4450}zw^3$

[c2t7]

$$(TE)(s) = -i\zeta e^{-is(w+z)}\Lambda(w,z),$$

where

$$\Lambda(w,z) = \left[\widehat{\phi}^*(w+z) - \frac{i}{z}\widehat{\phi}^*(w)\widehat{\phi}^*(z)\overline{\widetilde{\lambda}(w,z)}\right]$$

in the Petrov-Galerkin formulation,

$$\Lambda(w,z) = \widehat{\phi}^*(w+z) - \frac{i}{z}\widehat{\phi}^*(w)\widehat{\phi}^*(z)\overline{\widetilde{\phi}_\alpha(w)}\quad\overline{\widetilde{\beta}_\alpha(z)}\quad\overline{\widetilde{\gamma}(w+z)},$$

in the case of interpolation and quasi-interpolation operator, and

$$\Lambda(w,z) = \widehat{\phi^*}(w+z) - \frac{i}{z}\widehat{\phi^*}(w)\widehat{\phi^*}(z)\widetilde{\Gamma}(w,z)$$

in the case of discrete projection operator. Thus, the truncation error order depends on the behaviour of the symbol $\Lambda(w,z)$.

Theorem 3.2 Suppose that $N \geq N^*$. For the Petrov-Galerkin formulation, the symbol $\Lambda(w,z)$, defined in (48), satisfies

$$\Lambda(w,z) \sim \sum_{m=0}^{M-1}\mathcal{O}(w^m z^{M-m}).$$

[teo2]

Proof: Applying the Poisson summation formula on $\widetilde{\lambda}(w,z)$ we have

$$\widetilde{\lambda}(w,z) = i\sum_{m,n\in\mathbb{Z}}(z+2n\pi)\widehat{\phi}(w+2m\pi)\ \widehat{\phi}(z+2n\pi)\overline{\widehat{\phi}^*(w+z+2(m+n)\pi)}. \quad \text{[c2eq229]} \tag{60}$$

We consider the representation

$$\phi(x) = \varphi(x-\alpha), \quad \phi^*(x) = \varphi^*(x-\alpha), \quad \text{[eq59]} \tag{61}$$

where $\alpha = 0$ for even N, N^* and $\alpha = 1/2$ for odd N, N^*. The symmetric properties of ϕ e ϕ^* assure that φ and φ^* are symmetricaly centered on zero and they satisfy the Strang-Fix condition of the same order as ϕ and ϕ^*. The Fourier transform $\widehat{\varphi}(\xi)$ and $\widehat{\varphi}^*(\xi)$ are real function that satisfy

$$\widehat{\phi}(\xi) = e^{-i\xi\alpha}\widehat{\varphi}(\xi), \quad \widehat{\phi}^*(\xi) = e^{-i\xi\alpha}\widehat{\varphi}^*(\xi).$$

Consequently,

$$\widetilde{\lambda}(w,z) = i \sum_{m,n\in\mathbb{Z}} (z+2n\pi)\widehat{\varphi}(w+2m\pi)\ \widehat{\varphi}(z+2n\pi)\widehat{\varphi}^*(w+z+2(m+n)\pi).$$

We split the summation above into four terms

$$\widetilde{\lambda}(w,z) = \overbrace{\sum_{m=-n} [\ \ ]}^{\text{(I)}} + \overbrace{\sum_{\substack{m=0\\ n\neq 0}} [\ \ ]}^{\text{(II)}} + \overbrace{\sum_{\substack{m\neq 0\\ n=0}} [\ \ ]}^{\text{(III)}} + \overbrace{\sum_{\substack{m\neq 0, n\neq 0\\ m\neq -n}} [\ \ ]}^{\text{(IV)}}.$$

The main ingredients for the analysis of each term are the Strang-Fix condition of φ^* and φ, the biorthogonal relation and symmetry of φ^* and φ.

I-Term: In this case, we have

$$\text{(I)} = -i\widehat{\varphi}^*(w+z)\left[z\widehat{\varphi}(z)\ \widehat{\varphi}(w) + \sum_{m\neq 0}(z-2m\pi)\widehat{\varphi}(z-2m\pi)\ \widehat{\varphi}(w+2m\pi)\right].$$

For $m \geq 1$, let $f_m(w,z) = (z-2m\pi)q_m(w,z) + (z+2m\pi)q_{-m}(w,z)$, where $q_m(w,z) = \widehat{\varphi}(w+2m\pi)\ \widehat{\varphi}(z-2m\pi)$, in such a way that

$$(I) = -i\widehat{\varphi}^*(w+z)\left[z\widehat{\varphi}(z)\ \widehat{\varphi}(w) + \sum_{m\geq 1} f_m(w,z)\right].$$

Taking into consideration that

$$\frac{\partial^k q_m}{\partial w^l \partial z^{k-l}}(0,0) = \frac{d^l\widehat{\varphi}}{dw^l}(2m\pi)\frac{d^{k-l}\widehat{\varphi}}{dz^{k-l}}(-2m\pi),$$

and the Strang-Fix condition of φ, we conclude that the partial derivative of the functions $q_m(w,z)$ are zero at $(0,0)$ for $0 \leq l \leq N-1$ or $0 \leq k-l \leq N-1$. Particularly, it holds for all partial derivative of order $k \leq 2N-1$ or for all superior orders if $l = k$. Consequently, it also proceeds that

$$\frac{\partial^k f_m}{\partial w^l \partial z^{k-l}}(0,0) = 0,$$

where $0 \leq k \leq 2N-1$ or $l = k$. Besides, $f_m(w,z)$ are anti-symmetric around the point $(0,0)$, i.e., $f_m(-w,-z) = -f_m(w,z)$. Therefore, all partial derivative of $f_m(w,z)$ with even order are zero at $(0,0)$. Considering the above results, and the fact that

$$\overline{\widehat{\varphi}^*(\xi)}\widehat{\varphi}(\xi) = 1 + \mathcal{O}(\xi^N), \quad \text{[erb]} \tag{62}$$

we conclude that

$$\frac{-i}{z}\widehat{\varphi}^*(w)\widehat{\varphi}^*(z)(I) = -\widehat{\varphi}^*(w+z) + \sum_{m=0}^{2N-1}\mathcal{O}(w^m z^{2N-m}). \quad \text{[I]} \tag{63}$$

II-Term: This term is given by

$$(\text{II}) \ = -i\widehat{\varphi}(w)\sum_{n\neq 0}(z+2\pi m)\widehat{\varphi}(z+2\pi n)\ \widehat{\varphi}^*(z+w+2\pi n).$$

In this case, we consider $q_m(w,z)=\widehat{\varphi}(z+2m\pi)\widehat{\varphi}^*(z+w+2m\pi)$ and $f_m(w,z)$ as in the previous case, obtaining that

$$(II)=-i\widehat{\varphi}(w)\ \sum_{m\geq 1} f_m(w,z).$$

Again, by Strang-Fix condition of both φ^* and φ it follows that

$$\frac{\partial^k q_m}{\partial w^l\partial z^{k-l}}(0,0)=\sum_{m=0}^{k-l}\left(\begin{array}{c}k-l\\ m\end{array}\right)\frac{d^m\widehat{\varphi}}{dz^m}(2m\pi)\frac{\partial^{k-m}\widehat{\varphi}^*}{\partial w^l\partial z^{k-l-m}}(2m\pi)=0,$$

for $k\leq M-1$ or superior order if $l=k$. By anti-symmetry of $f_m(w,z)$ and (62) it follows that

$$\frac{-i}{z}\widehat{\varphi}^*(w)\widehat{\varphi}^*(z)(II)=\sum_{m=0}^{M-1}\mathcal{O}(w^m z^{M-m}).\ \text{[II]} \tag{64}$$

III-Term: In this case, we have

$$(\text{III})=-iz\ \widehat{\varphi}(z)\sum_{m\neq 0}\widehat{\varphi}(w+2m\pi)\widehat{\varphi}^*(z+w+2m\pi).$$

Now, we consider $f_m(w,z)=\widehat{\varphi}(w+2m\pi)\widehat{\varphi}^*(z+w+2m\pi)$, so that

$$(III)=-iz\ \widehat{\varphi}(z)\sum_{m\neq 0} f_m(w,z).$$

As in the previous cases, it proceeds that for $k\leq M-1$

$$\frac{\partial^n f_k}{\partial w^l\partial z^l}(0,0)=0.$$

Therefore

$$\frac{-i}{z}\widehat{\varphi}^*(w)\widehat{\varphi}^*(z)(III)=\sum_{m=0}^{M-1}\mathcal{O}(w^m z^{M-m}).\ \text{[III]} \tag{65}$$

VI-Term: We have

$$(\text{IV})=-i\sum_{\substack{m,n\neq 0\\ m\neq -n}}(z+2n\pi)\widehat{\varphi}(z+2n\pi)\ \widehat{\varphi}(w+2m\pi)\ \widehat{\varphi}^*(z+w+2(n+m)\pi).$$

In this case, we consider $f_{m,n}(w,z)=(z+2m\pi)\ q_{m,n}(w,z)\ +\ (z-2m\pi)\ q_{-m,-n}(w,z)$, where $q_{m,n}(w,z)\ =\ \widehat{\varphi}(w+2n\pi)\widehat{\varphi}(z+2m\pi)\widehat{\varphi}^*(z+w+2(m+n)\pi)$, so that

$$(IV)=-i\sum_{\substack{m,n>1\\ m\neq n}} f_{m,n}(w,z).$$

By the Strang-Fix condition of φ and φ^* and anti-symmetric property of $f_{m,n}(w,z)$, it follows that

$$\frac{\partial^k f_{m,n}}{\partial w^l \partial z^{k-l}}(0,0) = 0,$$

for $0 \leq k \leq N^* + 2N$. Therefore

$$\frac{-i}{z}\widehat{\varphi}^*(w)\widehat{\varphi}^*(z)(IV) = \sum_{m=0}^{N^*+2N-1} \mathcal{O}(w^m z^{N^*+2N-m}), \quad \text{[IV]} \tag{66}$$

which is a higher order term than previous ones. Substituting the contribution of terms (63), (64), (65) e (66) into (48), we conclude the proof. ■

Theorem 3.3 In all three hybrid formulations corresponding to interpolation, quasi-interpolation of order $N-1$ and discrete projetion, the symbol Λ satisfies the asymptotic behavior

(a) For even N

$$\Lambda(w,z) \sim \sum_{j=0}^{N} \mathcal{O}(w)^j \mathcal{O}(z)^{N-j}.$$

(b) For odd N

$$\Lambda(w,z) \sim \mathcal{O}(z)^{N-1} + \sum_{j=0}^{N+1} \mathcal{O}(w)^j \mathcal{O}(z)^{N+1-j}.$$

[c2teo1]

Proof: For the interpolation case we have

$$\Lambda(w,z) = \widehat{\phi}^*(w+z) - \frac{i}{z}\widehat{\phi}^*(w)\widehat{\phi}^*(z)\overline{\widetilde{\phi}_\alpha(w)} \quad \overline{\widetilde{\beta}_\alpha(z)} \quad \overline{\widetilde{\gamma}(w+z)}.$$

The interpolation condition $\widetilde{\gamma}(\xi)\widetilde{\phi}_\alpha(\xi) \equiv 1$ is satisfied with $\alpha = 0$, for even N, and $\alpha = 1/2$, for odd N. Therefore, we consider the representation (61) and it proceeds that

$$\Lambda(w,z) = \frac{e^{-i\alpha(w+z)}}{\overline{\widetilde{\phi}_\alpha(w+z)}}\left[\widehat{\varphi^*}(w+z)\overline{\widetilde{\phi}_\alpha(w+z)} - \frac{i}{z}\widehat{\varphi^*}(w)\widehat{\varphi^*}(z)\overline{\widetilde{\phi}_\alpha(w)} \quad \overline{\widetilde{\beta}_\alpha(z)}\right],$$

where we have used the inerpolation condition and the fact that $\widetilde{\phi}_\alpha(w)$ is a function bounded away from zero. The estimation of $\Lambda(w,z)$ depends on the terms into brackets. Again, the main ingredients for this analysis are: the Strang-Fix condition of φ^* and φ, the biorthogonal relation and symmetry of φ^* and φ. The symbol $\Lambda(w,z)$ can be represented by

$$\Lambda(w,z) = \frac{e^{-i\alpha(w+z)}}{\overline{\widetilde{\phi}_\alpha(w+z)}}\left[\widetilde{q}(w+z) - \frac{i}{z}\widetilde{q}(w)\widetilde{f}(z)\right]. \quad \text{[eqq1]} \tag{67}$$

where $\widetilde{q}(w) = \widehat{\varphi}^*(w)\overline{\widetilde{\phi}_\alpha(w)}$ and $\widetilde{f}(z) = \widehat{\varphi}^*(z)\overline{\widetilde{\beta}_\alpha(z)}$. Applying Poisson summation formula, it follows that

$$\widetilde{\phi}_\alpha(w) = \sum_{k\in\mathbb{Z}} \widehat{\phi}(w+2\pi k)e^{i\alpha(w+2k\pi)} = \sum_{k\in\mathbb{Z}} \widehat{\varphi}(w+2\pi k).$$

Therefore, we obtain

$$\widetilde{q}(w) = \widehat{\varphi}^*(w)\widehat{\varphi}(w) + \widehat{\varphi}^*(w)\sum_{k\geq 1} q_k(w),$$

where $q_k(w) = \widehat{\varphi}(w + 2k\pi) + \widehat{\varphi}(w - 2\pi k)$. By the Strang-Fix condition and symmetry of φ, and recalling (62), it follows that

$$\frac{d^n q_k}{dw^n}(0) = 0,$$

for $0 \leq n \leq N-1$ and all odd n. Therefore

$$\widetilde{q}(w) = \begin{cases} 1 + \mathcal{O}(w)^N & \text{for even } N\ , \\ \\ 1 + \mathcal{O}(w)^{N+1} & \text{for odd } N\ . \end{cases} \qquad \text{[eb2]} \tag{68}$$

Applying the same procedure to $\widetilde{q}(w+z)$, we have

$$\widetilde{q}(w+z) = \begin{cases} 1 + \displaystyle\sum_{m=0}^{N} \mathcal{O}(w^m z^{N-m}), & \text{for even } N, \\ \\ 1 + \displaystyle\sum_{m=0}^{N+1} \mathcal{O}(w^m z^{N+1-m}), & \text{for odd } N. \end{cases} \qquad \text{[eb3]} \tag{69}$$

Poisson summation formula applied to $\widetilde{\beta}_\alpha(z)$ implies that

$$\widetilde{\beta}_\alpha(z) = i\sum_{k\in\mathbb{Z}}(z + 2\pi k)\widehat{\phi}(z + 2\pi k)e^{i\alpha(z+2\pi k)} = i\sum_{k\in\mathbb{Z}}(z + 2\pi k)\widehat{\varphi}(z + 2\pi k).$$

Therefore, we have

$$\widetilde{f}(z) = z\widehat{\varphi}^*(z)\widehat{\varphi}(z) + \widehat{\varphi}^*(z)\sum_{k>0} f_k(z),$$

where $f_k(z) = (z + 2\pi k)\widehat{\varphi}(z + 2\pi k) + (z - 2\pi k)\widehat{\varphi}(z - 2\pi k)$. By the Strang-Fix condition and symmetry of ϕ, and recalling (62), it follows that

$$\frac{d^n f_k}{dz^n}(0) = 0,$$

for $0 \leq n \leq N-1$ and all even n Consequently

$$-\frac{i}{z}\widetilde{f}(z) = \begin{cases} -1 + \mathcal{O}(z)^{N+N^*} + \mathcal{O}(z)^N, & \text{for even } N, \\ \\ -1 + \mathcal{O}(z)^{N+N^*} + \mathcal{O}(z)^{N-1}, & \text{for odd } N. \end{cases} \qquad \text{[eb1]} \tag{70}$$

Therefore, substituting (69), (68) e (70) into the symbol equation (67) we conclude the statement of Theorem for the interpolation case.

The quasi-interpolation differs from the interpolation case for the fact that, instead of the interpolation constraint, it holds that $\widetilde{\gamma}(\xi)\widetilde{\phi}_\alpha(\xi) = 1 + \mathcal{O}(\xi^N)$ (see Lemma 29 with $n = p = N-1$). Therefore, following the same steps of the proof for the interpolation, the statement of Theorem also holds for the quasi-interpolation case.

For the discrete projection, we have

$$\Lambda(w,z) = \widehat{\phi^*}(w+z) - \frac{i}{z}\widehat{\phi^*}(w)\widehat{\phi^*}(z)\widetilde{\Gamma}(w,z),$$

where

$$\widetilde{\Gamma}(w,z) = \overline{\widetilde{\gamma}_e(w+z)}\ \widetilde{\phi}_0(w)\widetilde{\beta}_0(z) + \overline{\widetilde{\gamma}_o(w+z)}\ \widetilde{\phi}_{1/2}(w)\widetilde{\beta}_{1/2}(z),$$

and the functions $\widetilde{\gamma}_e$ and $\widetilde{\gamma}_o$ are defined by (32). Combining (34) with (62), it proceeds that

$$\left(\overline{\widehat{\phi}^*(\xi)} - \overline{\widetilde{\gamma}(\xi/2)}\right)\widehat{\phi}(\xi) = \mathcal{O}(\xi^N). \text{ [re1]} \tag{71}$$

Therefore,

$$\widetilde{\gamma}(\xi/2) = \widehat{\phi}^*(\xi) + \mathcal{O}(\xi^N). \text{ [c2eq255]} \tag{72}$$

The term $\widetilde{\Gamma}$ can be expressed by

$$\begin{aligned} \widetilde{\Gamma}(w,z) &= \overline{\widetilde{\gamma}_e(w+z)}\ \widetilde{\phi}_0(w)\widetilde{\beta}_0(z) + \overline{\widetilde{\gamma}_o(w+z)}\ \widetilde{\phi}_{1/2}(w)\widetilde{\beta}_{1/2}(z) \\ &= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}(z+2n\pi)\widehat{\phi}(z+2n\pi)\widehat{\phi}(w+2n\pi)\overline{\widetilde{\gamma}((w+z)/2+(n+k)\pi)}. \text{ [c2eq261]} \end{aligned} \tag{73}$$

Note that if this expression $\widetilde{\gamma}((w+z)/2+(n+k)\pi)$ is replaced by $\widehat{\phi}^*(w+z+2(n+k)\pi)$ we obtain the formula (60) associated to the symbol $\widetilde{\lambda}(w,z)$ of the Petrov-Galerkin formulation. Therefore, having in mind the result in (72), the proof for the discret projection proceeds as in the case of the Petrov-Galerkin formulation. ■

4 Conclusion

In the context of biorthogonal multirresolution analysis, we have considered different approximation schemes $\{\mathcal{D}_c^j, \mathcal{R}^j\}$ where, instead of the usual biorthogonal discretization, alternative discretizations $\mathcal{D}_c^j$ are used. Three cases have been analyzed: interpolation, quasi-interpolation and discrete projection. In all the cases, $\mathcal{D}_c^j$ are functionals defined in terms of discrete convolutions with function point values. We have also applied these schemes in the definition of hybrid discretizations of the nonlinear advection operator. These hybrid schemes may present a disadvantage in relation to a Petrov-Galerkin scheme, regarding the truncation error order. While Petrov-Galerkin scheme presents superconvergence order $N+N^*$, where $N-1$ is the order of thr Strang-Fix condition of the trial functions and N^*-1 is the one of the test functions, in the hybrid schemes the consistency order is $N-1$ (in some cases, due to symmetric properties, an improvement up to N may be obtained. However, hybrid schemes may have the advantage of an easy numerical implementation. On this aspect, some considerations are in order:

- Quasi-interpolation and discrete projection can be defined with a finite number of non zero coefficients. On the other hand, excepting some special cases, interpolation constraint (21) can only be achieved with infinitely many nonzero coefficients.
- For multilevel representations, a modified analysis algorithm is recommended to improve the precision of wavelet coefficients in less refined levels. For this, it is essential to have a conservative approximation scheme, which is not the case for quasi-interpolation.

After these considerations, the discrete projection scheme seems to be a better option for applications that involve the calculation of nonlinear terms in the multilevel context.

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