Stochastic Exponential in Lie Groups and its Applications

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Abstract

The aim of this article is to develop new simple proofs for the basic formulas of stochastic analysis in Lie groups, in particular the stochastic exponential and logarithm. We present applications to direct proofs of the (multiplicative) Doob-Meyer decomposition, Girsanov theorem for semimartingales in Lie groups and solution of stochastic Lax equations.

Key words: Lie groups, semimartingales, Doob-Meyer decomposition, Girsanov theorem, Lax equation.

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1 Introduction

Let G be a Lie group with the corresponding Lie algebra $\mathcal{G}$. We denote by ω the Maurer-Cartan form in G, i.e. if $v \in T_gG$, then $\omega_g(v) = L_{g^{-1}*}(v)$. It corresponds to the unique $\mathcal{G}$-valued left invariant 1–form in G. We recall that in the case of $G = (\mathbb{R}_{>0}, \cdot)$ the Maurer-Cartan form is $\omega_g = \frac{1}{g}dg$, and in the case of the general linear group $GL(n, \mathbb{R})$ the Maurer-Cartan form ω is $g^{-1}dg = (x_{ij})^{-1}(dx_{ij})$ where (x_{ij}) are the coordinate functions on $GL(n, \mathbb{R})$.

The aim of this article is to develop new proofs for a set of formulas which are basic in the construction of stochastic analysis in Lie groups, in particular

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we start with basic properties of the stochastic exponential and logarithm. These formulas will lead naturally to a Doob-Meyer decomposition and an extension of the Girsanov theorem for semimartingales in Lie groups.

Let $\theta_{X_t} \in T^*_{X_t}M$ be an adapted stochastic 1-form along X_t, an M-valued semimartingale. The integral of the form θ along X was proposed by Ikeda and Manabe [6] (see also Emery [4] or Meyer [10]). Locally this integral can be described as: let $(U, x^1, \ldots, x^n)$ be a local system of coordinates in M. With respect to this chart the 1-form θ can be written as $\theta_x = \theta^1(x)\, dx^1 + \ldots \theta^n(x)\, dx^n$, where $\theta^i(x)$, $i = 1, 2, \ldots n$, are (C^∞, say) functions in M. Then, the Stratonovich integral of θ along X_t is defined by:

$$\int \theta \circ dX_t = \sum_{i=1}^{n} \int \theta^i(X_t) \ \circ dX_t^i.$$

Let M_t be a semimartingale in the Lie algebra $\mathcal{G}$. We recall that the (left) stochastic exponential $\epsilon(M)$ of M_t is the stochastic process X_t which is solution of the Stratonovich left invariant equation on G:

$$\begin{cases} dX_t = L_{X_t *} \ \circ dM_t, \\ X_0 = e. \end{cases}$$

An interesting geometric characterization of the exponential $\epsilon(M)$ is the fact that it corresponds to the stochastic development of $M_t \in T_eG$ to the group G with respect to the left invariant connection ∇^L, i.e. $\nabla^L_X Y = 0$ for all $X, Y \in \mathcal{G}$.

The logarithm of a process X_t on G (with $X_0 = e$) is the following semimartingale in the Lie algebra:

$$(\log X)_t = \int_0^t \omega \circ dX_s.$$

where ω is the Maurer-Cartan form in G. One easily checks that the logarithm is the inverse of the stochastic exponential ϵ.

In the next section we present a simpler and more direct proof of the stochastic Campbell-Hausdorff formula (cf. Hakim-Dowek and Lépingle [5]). In the last section we apply these formulas to obtain direct proofs of the (multiplicative) Doob-Meyer, Girsanov theorems in Lie groups and solve stochastic Lax equations.

2 Main results

Initially we recall some rather straightforward results for semimartingales in Lie groups. We start with a characterization of ∇^L–martingales in G.

Theorem 2.1 *A process X_t on G is a ∇^L–martingale if and only if $X_t = X_0\epsilon(M)$ for some local martingale M in $\mathcal{G}$.*

Proof:

See Hakim-Dowek and Lépingle [5].

□

Next lemma concerns a pull-back of the Maurer-Cartan forms by homomorphisms of Lie groups, this formula will be useful later on.

Lemma 2.1 *Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then the pull-back $\varphi^*\omega_H$ satisfies, for $v \in T_gG$:*

$$(\varphi^*\omega_H)v = \varphi_*(\omega_G(v))$$

Proof:

Once $\varphi(L_{g^{-1}}(h)) = L_{\varphi(g)^{-1}}(\varphi(h))$, chain rule implies that

$$L_{\varphi(g)^{-1}*}(\varphi_*(v)) = \varphi_*(L_{g^{-1}*}(v)).$$

□

We shall denote by $I_g : G \to G$ the adjoint in the group G given by $h \mapsto ghg^{-1}$. The map I_g is an automorphism of G and its derivative corresponds to the isomorphism of the Lie algebra called adjoint in $\mathcal{G}$ denoted by $Ad(g) = I_{g*} : \mathcal{G} \to \mathcal{G}$. We have the following well known relation of the adjoint of the Maurer-Cartan form and the pull-back by the right action (see e.g. Kobayashi and Nomizu [9]):

Proposition 2.2 *The pull-back by the right action satisfies*

$$R_g^*\ \omega = Ad(g^{-1})\omega.$$

The pull-back of the canonical form by multiplication and inverse is given by:

Proposition 2.3 *Let $m : G \times G \to G$ be the multiplication and $i : G \to G$ be the inverse in the group. Then the pull-backs satisfy:*

a) *$m^*\omega = Ad^{-1}(\pi_2)(\pi_1^*\omega) + \pi_2^*\omega$;*

b) *$i^*\omega = -Ad\ \omega$.*

Proof:

Let $w = (u, v) \in T_{(g,h)}G \times G \simeq T_g G \times T_h G$. Then

$$\begin{aligned}
m^*\omega(w) &= \omega(m_* w) = \omega(R_{h*}u + L_{g*}v) \\
&= L_{(gh)^{-1}*}(R_{h*}u + L_{g*}v) \\
&= L_{h^{-1}*}R_{h*}L_{g^{-1}*}u + L_{h^{-1}*}L_{g^{-1}*}L_{g*}v \\
&= Ad(h^{-1})\omega(u) + \omega(v).
\end{aligned}$$

For the inverse function, consider the diagonal map $\Delta : G \to G \times G$ given by $\Delta(g) = (g, g)$. We have that $m \circ (Id \times i) \circ \Delta = e$, then the pull-back $(m \circ (Id \times i) \circ \Delta)^*\omega = 0$ which implies, using the formula of item (a), that

$$Ad\omega + i^*\omega = 0$$

□

Next lemma presents the main formulas which are useful in calculations with the logarithm.

Lemma 2.2 *Given semimartingales X and Y in G, we have the following formulas:*

a) *If $\varphi : G \to H$ is a homomorphism then*

$$\varphi_*(\log X) = \log(\varphi(X));$$

b) $\log(XY) = \int Ad(Y^{-1}) \circ d(\log X) + \log Y$*;*

c) $\log(X^{-1}) = \int Ad(X) \circ d(\log X)$.

Proof:

For the first formula, note that

$$\begin{aligned}
\log(\varphi X) &= \int \varphi^*\omega_H \circ dX \\
&= \int \varphi_*\omega_G \circ dX \\
&= \varphi_* \log X.
\end{aligned}$$

The second identity follows from the calculation:

$$
\begin{aligned}
\log(XY) &= \int \omega \circ dm(X,Y) \\
&= \int m^*\omega \circ d(X,Y) \\
&= \int \left(Ad^{-1}(\pi_2)\pi_1^*\omega + \pi_2^*\omega\right) \circ d(X,Y) \\
&= \int Ad(Y^{-1}) \circ d(\int \omega \circ dX) + \int \omega \circ dY \\
&= \int Ad(Y^{-1}) \circ d\log X + \log Y.
\end{aligned}
$$

Finally, for the last formula we have that

$$
\begin{aligned}
\log(X^{-1}) &= \int i^*\omega \circ dX \\
&= \int -Ad\ \omega \ \circ dX \\
&= -\int Ad(X) \circ d(\int \omega \ \circ dX) \\
&= -\int Ad(X) \circ d(\log X).
\end{aligned}
$$

□

We have now an easy way to prove the formulae below:

Theorem 2.4 *We have the following stochastic Campbell-Hausdorff formula:*

a) $\epsilon(M+N) = \epsilon\left(\int Ad(\epsilon(N)) \circ dM\right)\epsilon(N)$;

b) $\epsilon(M)^{-1} = \epsilon\left(-\int Ad(\epsilon(M)) \circ dM\right)$.

Proof:

For the first formula we just have to check that:

$$
\begin{aligned}
&\log\left(\epsilon\left(\int Ad(\epsilon(N) \ \circ dM\right)\epsilon(N)\right) \\
=\ & \int Ad(\epsilon(N)^{-1}) \ \circ d\log\left(\epsilon\left(\int Ad(\epsilon(N)) \ \circ dM\right)\right) + \log(\epsilon(N)) \\
=\ & M+N.
\end{aligned}
$$

And for the second formula:

$$
\begin{aligned}
& \log\left(\epsilon\left(-\int Ad(\epsilon(M)\ \circ dM\right)\right) \\
= \ & -\int Ad(\epsilon(M))\ \circ dM \\
= \ & -\int Ad(\epsilon(M))\ \circ d\log(\epsilon(M)) \\
= \ & \log(\epsilon(M)^{-1}).
\end{aligned}
$$

□

3 Applications

Our first application of these formulas is a multiplicative version of the Doob-Meyer decomposition. It was originally established by R. L. Karandikar in the case of group of matrices [8] and by M. Hakim-Dowek, D. Lepingle [5] (See also [2], [3]) in the general case.

Theorem 3.1 (Doob-Meyer decomposition in Lie groups) *Let $X = X_0\epsilon(M)$ be a semimartingale in G with $M = N + A$, where N is a local martingale and A is a process of finite variation in $\mathcal{G}$. Then we have that*

$$X = X_0 Y Z = X_0 Z' Y'$$

where Y, Y' are ∇^L–martingales and Z, Z' are processes of finite variation in G. Moreover, they are given by $Y = \int Ad(\epsilon(A)) \circ dN$, $Y' = \epsilon(N)$, $Z = \epsilon(A)$ and $Z' = \epsilon(\int Ad(\epsilon(N)\ dA)$.

Proof:
Apply the stochastic Campbell-Hausdorff formula (Theorem 2.4) to the classical Doob-Meyer decomposition $M = N + A$. The processes Y and Y' are ∇^L–martingales by Theorem 2.1.

□

We call the decomposition of the above theorem $X = X_0 Y Z$ ($X = X_0 Z' Y'$) the left (right) multiplicative Doob-Meyer decomposition of X. Now, we show a multiplicative version of the Girsanov theorem.

Theorem 3.2 (Girsanov-Meyer theorem in Lie groups) *Let P and Q be equivalent probability laws on the filtered space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0})$ with Radon-Nikodyn derivative $A_t = \mathbf{E}_P(\frac{dQ}{dP} \mid \mathcal{F}_t)$. Let X be a semimartingale in G with* *left multiplicative Doob-Meyer decomposition X_0YZ with respect to P. Then X has left multiplicative Doob-Meyer decomposition X_0VW with respect to Q where*

$$V = \epsilon(\int Ad\epsilon(\log Z + \int \frac{1}{A} d[A, B])d(B - \int \frac{1}{A} d[A, B]))$$

and

$$W = \epsilon(\log Z + \int \frac{1}{A} d[A, B])$$

where B is the semimartingale $B_t = \log YZ - \log Z$.

Proof:

Apply the classical Girsanov-Meyer theorem (see e.g. [12, Thm. 20, p. 109]) to $\log(YZ)$ and the stochastic exponential.

$\square$

3.1 Stochastic Lax Equation

Lax equations have been well known for quite a long time by its applications in integrable systems, see e.g. among many other authors, Perelomov [11]. In this application we use the formulae presented before to show an explicit solution for the stochastic Lax equation. Firstly, we recall that given a $\mathcal{G}$-valued semimartingale M, with $M_0 = 0$, an equation of the form

$$\begin{cases} dX_t = [X_t, \ \circ dM_t] \\ X(0) = X_0 \end{cases} \tag{1}$$

is called a stochastic Lax equation.

Proposition 3.3 *The solution of (1) is given by:*

$$X = Ad(\epsilon(-\int Ad(\epsilon(M)) \circ dM)) \ X_0$$

Proof: Once the adjoint is an isomorphism, there exists a unique process u_t in G such that the solution $X_t = Ad(u_t^{-1})X_0$. A direct calculation shows that

$$dX_t = [X_t, \ \circ d \log u_t]$$

Then, by uniqueness of the solution, X_t is the solution of the Lax equation (1) if and only if

$$M = \log u,$$

that is, the solution is given by:

$$X = Ad(\epsilon(M)^{-1})X_0 = Ad(\epsilon(-\int Ad(\epsilon(M)) \circ dM))X_0$$

$\square$

Corollary 3.4 *Let $G = K \cdot S$, where K and S are Lie subgroups of G with the corresponding Lie algebras $\mathcal{K}$ and $\mathcal{S}$. Assume that $K \cap S = \{e\}$ and that $Ad(K)\mathcal{S} \subseteq \mathcal{S}$. The solution of (1) is given by*

$$X = Ad(\epsilon(\int Ad(\epsilon(M_{\mathcal{K}})) \circ dM_{\mathcal{S}})\epsilon(M_{\mathcal{K}}))X_0$$

where $M_{\mathcal{K}}$ and $M_{\mathcal{S}}$ are the corresponding projections of M on $\mathcal{K}$ and $\mathcal{S}$.

Proof:
Let $X_t = Ad(u_t^{-1})X_0$ be the solution of (1), we have that

$$\log u = M_{\mathcal{K}} + M_{\mathcal{S}},$$

hence, by the stochastic Campbell-Hausdorff formula (Thm. 2.4):

$$u = \epsilon(M_{\mathcal{K}} + M_{\mathcal{S}}) = \epsilon\left(\int Ad(\epsilon(M_{\mathcal{K}})) \circ dM_{\mathcal{S}}\right)\epsilon(M_{\mathcal{K}}).$$

$\square$

Example: Lax equation in the Heisenberg Lie algebra.

Let M_t be a martingale in the Lie algebra of the Heisenberg group given by:

$$M_t = \begin{pmatrix} 0 & M_t^1 & M_t^2 \\ 0 & 0 & M_t^3 \\ 0 & 0 & 0 \end{pmatrix},$$

where M_t^1, M_t^2 and M_t^3 are real martingales with respect to a certain filtration $(\mathcal{F}_t)_{t\geq 0}$. Consider the equation:

$$dX_t = [X_t, \ \circ dM_t],$$

with

$$X_0 = \begin{pmatrix} 0 & x_0 & y_0 \\ 0 & 0 & z_0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A direct calculation from the definition shows that

$$\epsilon(M)^{-1} = \begin{pmatrix} 0 & -M_t^1 & \left(M_t^1 M_t^3 - M_t^2 - \int_0^t M_s^1 \ \circ dM_s^3\right) \\ 0 & 0 & -M_t^3 \\ 0 & 0 & 0 \end{pmatrix}$$

We remark that in this example, the left exponential equals the right exponential (hence $\epsilon(M)^{-1} = \epsilon(-M)$) if and only if $\int M^1 \ dM^3 = \int M^3 \ dM^1$. Finally, Proposition 3.3 states that the solution $X = Ad(\epsilon(M)^{-1})X_0$ is given by:

$$X_t = \begin{pmatrix} 0 & x_0 & \left(x_0 \ M_t^3 + y_0 - z_0 \ M_t^1\right) \\ 0 & 0 & z_0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3.2 Remark on Rotation Matrix

Given a stochastic dynamical system on a Riemannian manifold M, the Lyapunov exponents and the rotation numbers (or more generally, the rotation matrix) provide, respectively, the asymptotic radial and angular behaviour of the system (see e.g. L. Arnold [1] and the references therein). We recall that given an initial orthonormal basis u_0 in the orthonormal frame bundle OM, the difference between the induced flow in OM (by Gram-Schmidt orthonormalization of the linearized flow) and the parallel transport in OM is given by a process g_t in the structural group $O(n, \mathbb{R})$ of the principal bundle $\pi : OM \to M$.

With the formulae given in the previous section, the matrix of rotation, as defined in Ruffino [13] can be written as

$$\mathcal{R}(u_0) = \lim_{t \to \infty} \frac{1}{t} \log g_t,$$

when the limit exists (for details and existence results for stochastic systems see [13]). The definition, as stated by the formula above, shows that we can consider the matrix of rotation as a Lyapunov exponent of the system induced in the structural group $O(n, \mathbb{R})$ in the same way that the original Lyapunov exponents of the system in M (as stated in Arnold [1]) are the asymptotic exponents of the system induced in structural group $Gl(n, \mathbb{R})$ of the frame bundle $BM \to M$.

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