Existence and Asymptotic Behavior of Solutions for a Class of Quasilinear Elliptic Problems with Condition Neumann

Ilma Marques *
IMECC - UNICAMP,
Caixa Postal 6065.
13081-970 Campinas-SP, Brazil
E-mail: imarques@ime.unicamp.br

Abstract

In this paper we study the existence of solutions and its asymptotic behavior for the following class of quasilinear elliptic problems in radial form

$$\begin{array}{l} -\epsilon^2(r^\alpha|u'|^\beta u')' = r^\gamma f(u), \quad \text{in } (0,R) \\ u'(0) = u'(R) = 0, \end{array}$$

where α, β, γ are given real numbers, $\epsilon > 0$ is a small parameter and $0 < R < \infty$.

1 Introdução

In this paper we consider the following class of quasilinear elliptic problems in radial form

$$(1) \qquad \left\{ \begin{array}{l} -\epsilon^2(r^\alpha|u'|^\beta u')' = r^\gamma f(u), \quad \text{in } (0,R) \\ u'(0) = u'(R) = 0 \end{array} \right.$$

where α, β, γ are given real numbers such that $\gamma \geq \alpha$ and $\beta \geq 0$, $\epsilon > 0$ is a small parameter and $0 < R < \infty$. For that matter we make the following assumptions on the nonlinearity f:

(f_1) $f : [a,b] \to \mathbb{R}$ is a function of the class C^1, with $f(a) = f(b) = 0$;

(f_2) f has exactly $2l+1$ zeros, $a = a_1 < 0 = a_2 < a_3 < ... < a_{2l+1} = b$ with $l = 1, 2, 3, ...$ such that $f(a_i) = 0$, $\forall\ i$ and $f'(a_i) < 0$, if i is odd;

(f_3) $\lim_{t \to a_{2l}} \frac{f(t-a_{2l})}{|t-a_{2l}|^\beta (t-a_{2l})} > 0$.

*These results appear in the author's UNICAMP doctoral dissertation.

It has been emphasized in [4] that most of the phenomena occuring the study of nonlinear elliptic equations can be more easily explained and understand when these equations are written in radial form.

Let us consider the operator

$$Lu := -(r^\alpha |u'|^\beta u')'$$

acting (weak sense) in absolutely continuous functions $u : (0, R) \to \mathbb{R}$.

The motivation for our study is the fact that this operator includes the following operators, when considered acting in radial symmetric functions defined, say in a ball of $\mathbb{R}^N$:
(i) Laplacian: $\alpha = \gamma = N - 1$, $\beta = 0$,
(ii) p-Laplacian ($1 < p < N$): $\alpha = \gamma = N - 1$, $\beta = p - 2$,
(iii) k-Hessian ($1 \geq k < k/2$): $\alpha = N - k$, $\gamma = N - 1$, $\beta = k - 1$.

We consider the Banach space $\widehat{X}_R$ of absolutely continuous functions $u : (0, R) \to \mathbb{R}$ such that

$$\|u\|^{\beta+2} := \epsilon^2 \int_0^R r^\alpha |u'(r)|^{\beta+2} dr + \int_0^R r^\alpha |u(r)|^{\beta+2} dr < \infty.$$

Let us denote by $L^q_\theta(0, R)$, $q \geq 1$ and $\theta > -1$, the Banach space of Lebesgue measurable functions $u : (0, R) \to \mathbb{R}$ such that

$$|u|_{L^q_\theta} := \left(\int_0^R r^\theta |u(r)|^q dr \right)^{1/q} < \infty.$$

Associated with each space X_R and weight θ we define the critical exponent:

$$q^* := \frac{(\theta + 1)(\beta + 2)}{\alpha - \beta - 1}, \tag{2}$$

under the assumption that $\alpha - \beta - 1 > 0$.

Using standard variational methods (local minimization and the Mountain Pass Theorem) we can prove that (1) has at least l nonconstant solutions and using monotone iteration methods we can prove that the nonconstant solutions approach either a_l or a_{2l+1} in $\overline{\Omega}$ as $\epsilon \to 0$ almost everywhere.

More difficult is to prove that the solutions given for Mountain pass Theorem are different of a_i. If i is even we use the existence of solutions and its asymptotic behavior for the problem (1) with Dirichlet boundary conditions, which we prove the follow. If i is odd we prove that a_i is a local minimum in $\widehat{X}_R$. This fact isn't trivial, since in this space we don't have any result saying that if a_i is a local minimum in C^1 is also a local minimum in $\widehat{X}_R$. This result was obtained in [10] for space $W_0^{1,p}(\Omega)$, where Ω a smooth bounded domain of $\mathbb{R}^N$.

Our main results are the followings

Theorem 1.1 *Suppose that f satisfies the assumptions (f_1), (f_2) and (f_3). Then, there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$, problem* (1) *has at least l nonconstant solutions satisfing*

$$a_1 < u_1(r) < a_3 < u_2(r) < a_5 < \ldots < a_{2l-1} < u_l(r) < a_{2l+1},$$

where $l = 1, 2, 3, ...$

Remark 1.1 *The theorem 1.1 still is true if we consider the* a_i'*s negatives for all* $i \geq 3$*, i.e.,* $a = a_{2l+1} < ... < a_3 < a_2 = 0 < a_1 = b$*. The condition* $a_2 = 0$ *is not essential only make easy the presentation.*

Theorem 1.2 *Suppose that* f *satisfies the assumptions* (f_1)*,* (f_2) *and* (f_3) *for* $l = 1$ *and let* u_ϵ *a nonconstant solution of problem* (1)*. Given any* $\delta > 0$*, let*

$$\Omega^+(\epsilon, \mu) := \{r \in (0, R) : 0 < u_\epsilon(r) < \mu < a_3\}$$

contain a open ball $B(r, w^*(\epsilon, \delta))$ *centered at some* $r = r(\epsilon, \mu) \in \Omega^+(\epsilon, \mu)$ *whose radius* $w^*(\epsilon, \mu)$ *is the maximum of radii of open balls in* $\Omega^+(\epsilon, \mu)$*. Then*

$$\lim_{\epsilon \to 0} w^*(\epsilon, \mu) = 0.$$

2 Auxiliary Results

A fundamental step in the proof of Theorem 1.1 consists in to show the existence of solutions and its asymptotic behavior for the following Dirichlet problems

$$(3) \qquad \begin{cases} -\epsilon^2(r^\alpha |u'|^\beta u')' = r^\gamma f(u), & \text{in } (0, R) \\ u'(0) = u(R) = 0, \end{cases}$$

where $0 < R < \infty$, α, γ, β are given real numbers such that $\gamma \geq \alpha$ and $\beta \geq 0$. $\epsilon > 0$ is a small parameter. The function f satisfy the assumptions (f_1), (f_2) and (f_3) for $l = 1$.

We consider the Banach space X_R of absolutely continuous functions $u : (0, R) \to \mathbb{R}$ such that $u(R) = 0$ and

$$\epsilon^2 \int_0^R r^\alpha |u'(r)|^{\beta+2} < \infty.$$

We recall that λ_1 denotes the first eigenvalue of the eigenvalue problem (see p.145 [4])

$$(4) \qquad \begin{cases} Lu = \lambda r^\delta |u|^\beta u, & \text{in } (0, R) \\ u'(0) = u(R) = 0, \end{cases}$$

where λ, δ are given real numbers, $0 < R < \infty$ and is characterized by

$$\lambda_1 = \inf_{u \in X_R \setminus \{0\}} \frac{\int_0^R r^\alpha |u'(r)|^{\beta+2} dr}{\int_0^R r^\delta |u(r)^{\beta+2} dr}.$$

We show the following theorem.

Theorem 2.1 *Let f satisfy the assumptions (f_1), (f_2) and $(f_3)$for $l = 1$. Then given any ball $B(0, r_1) \subset B(0, R)$, $0 < r_1 < R$, There exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon \leq \epsilon_0$, the Dirichlet problem*

$$
(5) \qquad \begin{cases} -\epsilon^2 (r^\alpha |u'|^\beta u^{'})^{'} = r^\gamma f(u), & in\ (0, r_1) \\ u'(0) = u(r_1) = 0, \end{cases}
$$

has a positive solution $0 < u_\epsilon(r) < a_3$ in $[0, r_1]$ such that $u_\epsilon \to a_3$ as $\epsilon \to 0$ on every compact subset of $(0, r_1)$. Moreover, the Dirichlet problem

$$
(6) \qquad \begin{cases} -\epsilon^2 (r^\alpha |u'|^\beta u')' = r^\gamma f(u), & in\ (r_1, R) \\ u(r_1) = u(R) = 0, \end{cases}
$$

has a negative solution $a_1 < v_\epsilon(r) < 0$ in $[r_1, R]$ such that $v_\epsilon \to a_1$ as $\epsilon \to 0$ on every compact subset of (r_1, R).

The prove of Theorem 2.1 is done by combining argument of truncation and method of lower and upper-solution.

A function $\underline{u} \in X_R \cap L^\infty_\gamma$ is said to be a lower solution of (3) if

$$
\begin{cases} \epsilon^2 \displaystyle\int_0^R r^\alpha |\underline{u}'|^\beta \underline{u}' \phi' dr \leq \int_0^R r^\gamma f(\underline{u}) \phi dr, \ \forall\ \phi \in X_R,\ \phi \geq 0 \\ \\ \underline{u}'(0) = 0,\ \underline{u}(R) \leq 0. \end{cases}
$$

In the same way, a function $\overline{u} \in X_R \cap L^\infty_\gamma$ is said to be a upper solution of (3) if

$$
\begin{cases} \epsilon^2 \displaystyle\int_0^R r^\alpha |(\overline{u})'|^\beta (\overline{u})' \phi' dr \geq \int_0^R r^\gamma f(\overline{u}) \phi dr, \ \forall\ \phi \in X_R,\ \phi \geq 0 \\ \\ (\overline{u})'(0) = 0,\ \overline{u}(R) \geq 0. \end{cases}
$$

Lemma 2.1 *Consider $g : \mathbb{R} \to \mathbb{R}$, a function continuous and increasing, such that $g(0) = 0$ and functions $u_1, u_2 \in X_R \cap L^\infty_\gamma$ such that, for all $\phi \in X_R$, $\phi \geq 0$*

$$
\begin{cases} \epsilon^2 \displaystyle\int_0^R r^\alpha |u_2'|^\beta u_2' \phi' dr + \int_0^R g(u_2(r)) \phi dr \leq \epsilon^2 \int_0^R r^\alpha |u_1'|^\beta u_1' \phi' dr + \int_0^R g(u_1(r)) \phi dr \\ \\ u_2(R) \leq u_1(R). \end{cases}
$$

Then $u_2 \leq u_1$ a.e. $(0, R)$.

The proof of this result is similarly the proof of the Lemma 2.2 in [3].

Lemma 2.2 *Let $g : \mathbb{R} \to \mathbb{R}$, a function continuous and increasing, such that $g(0) = 0$. Then, for every function $h \in L^{q'}_\gamma(0, R)$, where $1/q + 1/q' = 1$, the problem*

$$
(7) \qquad \begin{cases} -\epsilon^2 (r^\alpha |u'|^\beta u')' + g(u(r)) = h(r),\ r \in (0, R) \\ \\ u'(0) = u(R) = 0 \end{cases}
$$

admits a unique weak solution $u \in X_R$. Moreover, the associated operator $T : L^{q'}_{\gamma} \to X_R$, $h \mapsto u$ is continuous and nondecreasing.

Proof: We split the proof in two parts.
Part 1: Consider the function $\phi \in C^1(X_R, \mathbb{R})$ defined by

$$\phi(u) := \frac{\epsilon^2}{\beta+2}\int_0^R r^{\alpha}|u'|^{\beta+2}dr + \int_0^R G(u)dr - \int_0^R h(r)udr,$$

where

$$G(u) = \int_0^u g(t)dt.$$

Claim 1: ϕ is coercive. Indeed,

$$\begin{aligned}\phi(u) &= \tfrac{1}{\beta+2}\|u\|^{\beta+2} + \int_0^R G(u)dr - \int_0^R h(r)udr \\ &\geq \tfrac{1}{\beta+2}\|u\|^{\beta+2} - \int_0^R h(r)udr.\end{aligned}$$

Hence, $\phi(u) \to +\infty$ as $\|u\| \to +\infty$, i.e., ϕ is coercive.
Claim 2: ϕ is weakly lower semicontinuous. Indeed, let $u_n \rightharpoonup u$ in X_R. By the Proposition 1.1 [4] we have

$$\psi(u_n) := \int_0^R G(u_n) \to \psi(u) := \int_0^R G(u).$$

Hence, ψ is weakly lower semicontinuous. So, there exists $u_0 \in X_R$ such that $\phi(u_0) = \min_{u \in X_R} \phi(u)$.
Part 2: Let $u_1, u_2 \in X_R$ weak solutions of (7), i.e.,

$$\epsilon^2 \int_0^R (r^{\alpha}|u_1'|^{\beta}u_1')\varphi' dr + \int_0^R g(u_1)\varphi dr - \int_0^R h(r)\varphi dr = 0, \forall\ \varphi \in X_R \tag{8}$$

and

$$\epsilon^2 \int_0^R (r^{\alpha}|u_2'|^{\beta}u_2')\varphi' dr + \int_0^R g(u_2)\varphi dr - \int_0^R h(r)\varphi dr = 0, \forall\ \varphi \in X_R. \tag{9}$$

By (8) and (9) we have

$$\epsilon^2 \int_0^R [(r^{\alpha}|u_1'|^{\beta}u_1') - (r^{\alpha}|u_2'|^{\beta}u_2')]\varphi' dr + \int_0^R [g(u_1) - g(u_2)]\varphi dr = 0.$$

Choosing $\varphi = (u_1 - u_2)^+$ and using the fact that g is increasing, we obtain

$$
\begin{aligned}
0 &= \epsilon^2\int_0^R [(r^\alpha|u_1'|^\beta u_1') - (r^\alpha|u_2'|^\beta u_2')]((u_1-u_2)^+)'dr \\
&\quad + \int_0^R [g(u_1)-g(u_2)](u_1-u_2)^+dr \\
&= \epsilon^2\int_{u_1\geq u_2} r^\alpha \frac{|u_1'|^\beta+|u_2'|^\beta}{2}|((u_1-u_2)^+)'|^2dr \\
&\quad + \int_{u_1\geq u_2} r^\alpha \frac{|u_1'|^\beta-|u_2'|^\beta}{2}[|u_1'|^2-|u_2'|^2]dr \\
&\quad + \int_{u_1\geq u_2} [g(u_1)-g(u_2)](u_1-u_2)^+dr \geq 0.
\end{aligned}
$$

Hence, $(u_1-u_2)^+ = 0$ a.e. in $(0,R)$ or, equivalently, $u_1 \leq u_2$ a.e. in $(0,R)$.

Choosing $\varphi = (u_2-u_1)^+$, using similar argument we obtain that $(u_2-u_1)^+ = 0$ a.e. in $(0,R)$ or, equivalently, $u_2 \leq u_1$ a.e. in $(0,R)$.

Hence, the problem (7) has a unique weak solution $u \in X_R$.

The fact that T is nondecreasing follows from Lemma 2.1. □

3 Poof of Auxiliary Results

Proposition 3.1 *Suppose that f satisfies the assumptions (f_1), (f_2) and (f_3) for $l=1$. Then given any ball $B(0,r_1) \subset B(0,R)$, $0<r_1<R$, there exists $\epsilon_0>0$ such that for all $0<\epsilon\leq\epsilon_0$, the Dirichlet problem (5), has a positive solution $0<u_\epsilon(r)<a_3$ in $[0,r_1]$. Moreover, the Dirichlet problem (6) has a negative solution $a_1<v_\epsilon(r)<0$ in $[r_1,R]$.*

Proof: We start proving the existence of the positive solution for problem (5), for this we use an argument of truncation. Let $f_1:\mathbb{R}\to\mathbb{R}$ be defined by

$$
f_1(u) := \begin{cases} f(u), & \text{if} \quad 0\leq u\leq a_3 \\ 0, & \text{if} \quad u\leq 0 \\ 0, & \text{if} \quad u\geq a_3. \end{cases}
$$

Using (f_1), we have that there exists $M>0$ such that $r^\gamma f_1(t)+Mt$ is nondecreasing in t for $t\in[0,a_3]$.

Now, we consider the following auxiliary problem

$$
(10) \qquad \begin{cases} -\epsilon^2(r^\alpha|u'|^\beta u')' + Mu = r^\gamma f_1(u) + Mu, \text{ in } (0,r_1) \\ u'(0)=u(r_1)=0. \end{cases}
$$

Thus, we will to prove the existence of the positive solution for problem (10) by using the method of lower and upper-solution. We prove this in tree steps.

Step 1: Observe that the function $\overline{u}(r)\equiv a_3$, for $r\in[0,r_1]$ is a upper solution of (10).

Step 2: Construction of the lower solution of (10). Let

$$
a = \lim_{t\to 0}\frac{f_1(t)}{|t|^\beta t}
$$

and λ_1 is the first eigenvalue of $Lu := -(r^\alpha |u'|^\beta u')'$ in $(0, R)$ subject to the Dirichlet boundary condition. It follows by (f_3) that given $\overline{\delta} \in (0, a)$, there exist $t_0 > 0$ such that for all $| t | \leq t_0$ we have

$$a - \overline{\delta} \leq \frac{f_1(t)}{| t |^\beta t}. \tag{11}$$

Let $\varphi_1 > 0$ an eigenvalue corresponding to the first eigenvalue λ_1. Take $\beta_1 > 0$ such that $| \beta_1 \varphi_1(r) | \leq t_0$ and $\beta_1 \left(\max_{(0,r_1)} \varphi_1\right) < a_3$. From (11)

$$a - \overline{\delta} \leq \frac{f_1(\beta_1 \varphi_1)}{| \beta_1 \varphi_1 |^\beta \beta_1 \varphi_1}.$$

Choosing $\epsilon_0 > 0$ such that $\epsilon_0^2 \lambda_1 \frac{r^\delta}{r^\gamma} < a - \overline{\delta}$, we have that $\beta_1 \varphi_1$ is a lower solution of problem (10) for all $0 < \epsilon \leq \epsilon_0$.

Step 3: We will show that there exists a minimal (and, respectively, a maximal) weak solution u_* (resp. u^*) for problem (10) such that $\beta_1 \varphi_1 = \underline{u} \leq u_* \leq \overline{u} = a_3$. Consider the set

$$[\underline{u}, \overline{u}] := \{u \in L^\infty_\gamma(0, r_1) : \underline{u}(r) \leq u(r) \leq \overline{u}(r) \text{ a.e. in } (0, r_1)\}$$

with the topology a.e. of convergence, and define the operator $S : [\underline{u}, \overline{u}] \to L^q_\gamma$ by

$$Sv = r^\gamma f_1(v) + Mv \in L^\infty_\gamma(0, r_1) \subset L^{q'}_\gamma(0, r_1), \forall\, v \in [\underline{u}, \overline{u}],$$

where q' is such that $1/q + 1/q' = 1$. We get that S is nondecreasing and bounded . Moreover, if $v_n, v \in [\underline{u}, \overline{u}]$, then

$$\| Sv_n - Sv \|^q_{L^q_\gamma} = \int_0^{r_1} | r^\gamma f_1(v_n) + Mv_n - r^\gamma f_1(v) - Mv |^q \, dr.$$

Let $v_n \to v$ a.e. in Ω. Applying the Lebesgue dominated convergence theorem, we obtain that $\| Sv_n - Sv \|_{L^q_\gamma} \to 0$, and then, S is continuous.

Consider the continuous nondecreasing operator $F : [\underline{u}, \overline{u}] \to X_R$ defined by $F := ToS$, (where T is the continuous and nondecreasing defined in Lemma 2.2, i.e., for a function $v \in [\underline{u}, \overline{u}]$, $F(v)$ is the unique solution of problem (10) with $\epsilon = 1$.

Writing $u_1 = F(\underline{u})$ and $u^1 = F(\overline{u})$ we obtain that $\varphi \in X_R$, $\varphi > 0$,

$$\begin{aligned} \int_0^{r_1} r^\alpha | u_1' |^\beta u_1' \varphi' + \int_0^{r_1} M u_1 \varphi &= \int_0^{r_1} (r^\gamma f_1(\underline{u}) + M\underline{u})\varphi \\ &\geq \int_0^{r_1} r^\alpha | \underline{u}' |^\beta \underline{u}' \varphi' + \int_0^{r_1} M\underline{u}\varphi \end{aligned}$$

and

$$\begin{aligned} \int_0^{r_1} r^\alpha | (u^1)' |^\beta (u^1)' \varphi' + \int_0^{r_1} M u^1 \varphi &= \int_0^{r_1} (r^\gamma f_1(\overline{u}) + M\overline{u})\varphi \\ &\leq \int_0^{r_1} r^\alpha | (\overline{u})' |^\beta (\overline{u})' \varphi' + \int_0^{r_1} M\overline{u}\varphi. \end{aligned}$$

Applying Lemma 2.1 and taking into account that F is nondecreasing, we obtain

$$\underline{u} \leq F(\underline{u}) \leq F(u) \leq F(\overline{u}) \leq \overline{u}, \text{ a.e. in } (0, r_1), \ \forall\, u \in [\underline{u}, \overline{u}].$$

Repeating the same reasoning, we can prove the existence of sequences (u^n) and (u_n) satisfying $u^0 = \overline{u}$, $u^{n+1} = F(u^n)$, $u_0 = \underline{u}$, $u_{n+1} = F(u_n)$ and, for every weak solution $u \in [\underline{u}, \overline{u}]$ of problem (10) with $\epsilon = 1$, we obtain

$$\underline{u} = u_0 \leq u_1 \leq ... \leq u_n \leq u \leq u^n \leq ... \leq u^1 \leq u^0 = \overline{u}, \text{ a.e. in } (0, r_1).$$

Then, $u_n \to u_*$, $u^n \to u^*$, a.e. in $(0, r_1)$, with $u_*, u^* \in [\underline{u}, \overline{u}]$, and $u_* \leq u^*$ a.e. in $(0, r_1)$. Since $u_{n+1} = F(u_n) \to F(u_*)$ and $u^{n+1} = F(u^n) \to F(u^*)$ in X_R, by continuity of F, then $u_*, u^* \in X_R$ with $u_* = F(u_*)$ and $u^* = F(u^*)$. This completes the proof. Then, u_* is minimal weak solution (respectively, u^* maximal weak solution) of (10) with $\epsilon = 1$ such that $u_*, u^* \in [\underline{u}, \overline{u}]$, $\forall\ 0 < \epsilon \leq \epsilon_0$. In particular, every solution $u \in [\underline{u}, \overline{u}]$ of (10) with $\epsilon = 1$ satisfies also $u_* \leq u \leq u^*$, a.e. in $(0, r_1)$. Since the solutions u_* and u^* are between 0 and a_3 then u_* and u^* are solutions of (5). Therefore, there exists a solution $u_\epsilon : u_* \in [\beta_1\varphi_1, a_3]$ of (5), for all $0 < \epsilon \leq \epsilon_0$.

To prove the existence of the negative solution $v_\epsilon(r)$,where $a_1 \leq v_\epsilon \leq 0$, is enough to consider the truncation function $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_2(u) := \begin{cases} f(u), & \text{if} \quad a_1 \leq u \leq 0 \\ 0, & \text{if} \quad u \leq a_1, \\ 0, & \text{if} \quad u \geq 0, \end{cases}$$

and the problem

$$\begin{cases} -\epsilon^2(r^\alpha |u'|^\beta u')' = r^\gamma f_2(u), \text{ em } (r_1, R) \\ u(r_1) = u(R) = 0. \end{cases}$$

We consider here, λ_1 the first eigenvalue of problem

$$\begin{cases} -(r^\alpha |u'|^\beta u')' = \lambda r^\delta |u|^\beta u, \text{ in } (r_1, R) \\ u(r_1) = u(R) = 0 \end{cases}$$

and $\varphi_1 > 0$ the eigenfunction corresponding to the first eigenvalue λ_1. □

3.1 Asymptotic Behavior of a Class of Solutions

We have the following proposition which shows the asymptotic behavior of a class of solutions of (5) as $\epsilon \to 0$.

Proposition 3.2 *Let be* $0 < u_\epsilon < a_3$ *a positive solution of* (5) *and let be* $a_1 < v_\epsilon < 0$ *a negative solution of* (6)*. Then*
i) $u_\epsilon \to a_3$ *as* $\epsilon \to 0$ *uniformly on every compact subset of* $(0, r_1)$*;*
ii) $v_\epsilon \to a_1$ *as* $\epsilon \to 0$ *uniformly on every compact subset of* (r_1, R)*.*

Proof: **i)** The proof follows by adapting some arguments made in Theorem 4 of [7].

First, observe that exist $\mu \in (0,1)$ such that $u_\epsilon \in C^{1,\mu}(0, r_1)$, by Proposition 2.2 [4].

Consider the function $f_1 : \mathbb{R} \to \mathbb{R}$ defined at the Proposition 3.1 and $\varphi_1 > 0$ an eigenfunction corresponding to the first eigenvalue λ_1 of $Lu := -(r^\alpha |u'|^\beta u')'$ in $(0, r_1)$ subject to Dirichlet boundary conditions. Since $f_1 \geq 0$, $f_1 \not\equiv 0$ and $\varphi_1 > 0$ from Lemma 3.2 in [4], we have $u_\epsilon > 0$ in $(0, r_1)$, $u'_\epsilon \leq 0$ in $(0, r_1)$, $\varphi_1 > 0$ in $(0, r_1)$ and $\varphi'_1 \leq 0$ in $(0, r_1)$

Consequently, there exists $\beta_1 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$, we have $u_\epsilon(r) \geq \beta_1 \varphi_1$, and for a given $\eta > 0$ there is C_η such that

$$u_\epsilon(r) \geq C_\eta > 0, \tag{12}$$

for all $r \in \Omega_\eta := \{r \in (0, r_1) : dist(r, r_1) > \eta\}$. Take φ_1 such that $\parallel \varphi_1 \parallel = 1$. Since u_ϵ is solution of (10) it follows that

$$\epsilon^2 \int_0^{r_1} r^\alpha \mid u'_\epsilon \mid^\beta u'_\epsilon \varphi' dr = \int_0^{r_1} r^\gamma f_1(u_\epsilon) \varphi dr, \ \forall\ \varphi \geq 0,\ \varphi \in X_R. \tag{13}$$

In particular, for $\varphi = \varphi_1$, we obtain

$$\epsilon^2 \int_0^{r_1} r^\alpha \mid u'_\epsilon \mid^\beta u'_\epsilon \varphi'_1 dr = \int_0^{r_1} r^\gamma f_1(u_\epsilon) \varphi_1 dr. \tag{14}$$

Claim: The expression in the left-hand of (14) goes to zero as $\epsilon \to 0$. Indeed, observe that $0 < u_\epsilon \leq a_3$ and $f_1(u_\epsilon) \leq \widetilde{C}$. Thus, using the Hölder inequality and (13) with $\varphi = u_\epsilon$, we obtain

$$\begin{aligned} \epsilon^2 \int_0^{r_1} r^\alpha \mid u'_\epsilon \mid^\beta u'_\epsilon \varphi'_1 dr &\leq C\epsilon^2 \left(\tfrac{1}{\epsilon^2} \int_0^{r_1} r^\gamma f_1(u_\epsilon) u_\epsilon dr \right)^{\frac{(\beta+1)}{\beta+2}} \\ &\leq \widehat{C} \epsilon^{\frac{2}{\beta+2}}, \text{ for some constant } \widehat{C}. \end{aligned}$$

Define $d_\eta := inf\{\varphi_1(r) : r \in \Omega_\eta\} > 0$. Then,

$$d_\eta \int_{\Omega_\eta} r^\gamma f_1(u_\epsilon) dr \leq \int_{\Omega_\eta} r^\gamma f_1(u_\epsilon) \varphi_1 dr < \int_0^{r_1} r^\gamma f_1(u_\epsilon) \varphi_1 \to 0,\ \epsilon \to 0. \tag{15}$$

Now, suppose by contradiction that there are a number $C_1 > 0$ and a sequence $\epsilon_j \to 0$ such that the Lebesgue's measure of the sets

$$\Omega_{\eta,j} := \{r \in \Omega_\eta : u_{\epsilon_j}(r) < a_3 - \eta\} \tag{16}$$

are bounded from below by C_1. It follows from (15) that

$$I_j := \int_{\Omega_{\eta,j}} r^\gamma f_1(u_{\epsilon_j}) dr \to 0, \text{ as } \epsilon_j \to 0. \tag{17}$$

Observe that in $\Omega_{\eta,j}$, from (12) and (16), we have $C_\eta \leq u_{\epsilon_j} \leq a_3 - \eta$.

Since f_1 is bounded from below in the interval $[C_\eta, a_3 - \eta]$ by a number $d > 0$, from (16) it follows

$$I_j = \int_{\Omega_{\eta,j}} r^\gamma f_1(u_{\epsilon_j})dr \geq d \int_{\Omega_{\eta,j}} r^\gamma dr = dC \geq d'|\Omega_{\eta,j}| \geq d'C_1,$$

for any $0 < d' \leq \frac{dC}{|\Omega_{\eta,j}|}$, which contradicts (17). Therefore, $|\ \Omega_{\eta,j}\ |$ does not bounded from below, i.e., $u_\epsilon(r) \to a_3$, on every compact subset of $(0, r_1)$ as $\epsilon \to 0$.
ii) It follows similarly such as in the positive case i). □

Demonstração do Teorema 2.1: The proof follows directly from Proposition 3.1 and from the Proposition 3.2. □

4 Proof of Theorem 1.1

The proof is done by using a version of the Mountain Pass Theorem, due a Hofer [11] in order to show the existence of critical points of the mountain pass type.

4.1 Particular Case

First, we consider the particular case of the Theorem 1.1.

Theorem 4.1 *Let be f satisfying the assumptions (f_1), (f_2) and (f_3) for $l = 1$. Then there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$, the problem (1) has at least one nonconstant solution u_ϵ verifying $a_1 < u_\epsilon(r) < a_3$.*

In order to prove the Theorem 4.1, we use the results that we will prove below.

Lemma 4.1 *Let be f satisfying the assumptions (f_1), (f_2) and (f_3) for $l = 1$. Then there exists functions of class C^1, $f_1 : (-\infty, a_1] \to \mathbb{R}^+$, $f_2 : [a_3, +\infty) \to \mathbb{R}^-$ and real numbers η_1 and $\overline{\eta}_1$ such that:*
(i) *$f_1(a_1) = f(a_1)$, $f_1'(a_1) = f'(a_1)$ and $f_1(t) > 0$ for all $t \in (\eta_1, a_1)$;*
(ii) *$f_2(a_3) = f(a_3)$, $f_2'(a_3) = f'(a_3)$ and $f_2(t) < 0$ for all $t \in (a_3, \overline{\eta}_1)$;*
(iii) *η_1 and $\overline{\eta}_1$ are so that $\eta_1 < a_1 < a_3 < \overline{\eta}_1$,*

$$\int_{\eta_1}^{a_1} f_1(t)dt = \left|\int_{a_1}^{0} f(t)dt\right|, \qquad \left|\int_{a_3}^{\overline{\eta}_1} f_2(t)dt\right| = \int_{0}^{a_3} f(t)dt$$

and $\forall\ t \in [0, 1]$, we have

$$\eta_1 < t(1-t)a_3 + (a_3 - a_1)t + a_1 < \overline{\eta}_1$$

and

$$\eta_1 < t(1-t)a_1 + (a_3 - a_1)t + a_1 < \overline{\eta}_1.$$

Proof: We start proving the existence of η_1.

We take

$$\alpha(t) := t(1-t)a_1 + (a_3 - a_1)t + a_1.$$

Thus, $0 = \frac{d}{dt}(\alpha(t)) = -2ta_1 + a_3$ if, and only if, $t = \frac{a_3}{2a_1}$. Moreover, since $a_1 < 0$ and $\frac{d^2}{dt}\alpha(t) = -2a_1$ it follows that $\alpha(\frac{a_3}{2a_1}) = \frac{a_3^2}{4a_1} + a_1$ is the minimal value of $\alpha(t)$. Finally we define

$$\eta_1 := \frac{a_3^2}{4a_1} + 2a_1.$$

For the proof of the existence of $\overline{\eta}_1$ we take

$$\beta(t) := t(1-t)a_3 + (a_3 - a_1)t + a_1.$$

We note that $0 = \frac{d}{dt}(\beta(t)) = (1-2t)a_3 + a_3 - a_1$ if, and only if, $t = 1 - \frac{a_1}{2a_3}$. Since $\frac{d^2}{dt}\beta(t) = -2a_3$ and $a_3 > 0$ we have $\beta(1 - \frac{a_1}{2a_3}) = \frac{a_1^2}{4a_3} + a_3$ is the maximal value of β. Now, we take

$$\overline{\eta}_1 := \frac{a_1^2}{4a_3} + 2a_3.$$

Now, we are going to prove the existence of function f_1.

We take $g(t) = f'(a_1)(t - a_1)$ and ξ_1, ξ_2 functions of class C^1 so that $\xi_1 \equiv 1$ at a neighbourhood of a_1, $\xi_2 \equiv 1$ at a neighbourhood of η_1, $\xi_1(t) + \xi_2(t) = 1$ for all $t \in [\eta_1, a_1]$ and

$$\int_{\eta_1}^{a_1} g(t)\xi_1(t)dt < \left|\int_{a_1}^{0} f(t)dt\right|.$$

Now, we choose $r > 0$ such that

$$\int_{\eta_1}^{a_1} [r\xi_2(t)g(t) + \xi_1(t)g(t)]dt = \left|\int_{a_1}^{0} f(t)dt\right|.$$

We define $f_1 : (-\infty, a_1] \to \mathbb{R}^+$ by

$$f_1(t) := \begin{cases} r\xi_2(t)g(t) + \xi_1(t)g(t), & \eta_1 \le t \le a_1 \\ rf'(a_1)t - ra_1 f'(a_1), & t \le \eta_1. \end{cases}$$

We note that for $t \le \eta_1$ the graph of f_1 is the tangent line to f_1 at the point $(\eta_1, f_1(\eta_1))$.

Finally, we prove the existence of f_2.

We take $g(t) = f'(a_3)(t - a_3)$ and ξ_1, ξ_2 functions of class C^1 so that $\xi_1 \equiv 1$ at a neighbourhood of a_3, $\xi_2 \equiv 1$ at a neighbourhood of $\overline{\eta}_1$, $\xi_1(t) + \xi_2(t) = 1$ for all $t \in [a_3, \overline{\eta}_1]$ and

Now, we choose $r > 0$ such that

$$\left|\int_{a_3}^{\overline{\eta}_1} [r\xi_2(t)g(t) + \xi_1(t)g(t)]dt\right| = \int_0^{a_3} f(t)dt.$$

We define $f_2 : [a_3, +\infty) \to \mathbb{R}^-$ by

$$f_2(t) := \begin{cases} r\xi_2(t)g(t) + \xi_1(t)g(t), & a_3 \le t \le \overline{\eta}_1 \\ rf'(a_1)t - ra_1 f'(a_1), & t \ge \overline{\eta}_1. \end{cases}$$

We observe that for $t \ge \beta_1$ the graph of f_2 is the tangent line to f_2 at the point $(\overline{\eta}_1, f_2(\overline{\eta}_1))$.

Thus the proof of Lemma is concluded. □

As consequence of the Maximum Principle we have the following lemma.

Lemma 4.2 *If u_ϵ is a nonconstant solution of the Neumann problem*

$$\begin{cases} -\epsilon^2(r^\alpha |u'|^\beta u')' = r^\gamma \widehat{f}(u), & \text{in } (0,R) \\ u'(0) = u'(R) = 0, \end{cases} \tag{18}$$

where $\widehat{f} : \mathbb{R} \to \mathbb{R}$ is a truncation function defined by

$$\widehat{f}(t) := \begin{cases} f(t), & a_1 \le t \le a_3 \\ f_1(t), & t \le a_1 \\ f_2(t), & t \ge a_3, \end{cases}$$

and f_1 and f_2 are defined in Lemma 4.1. Then $a_1 \le u_\epsilon(r) \le a_3$, i.e., u_ϵ is nonconstant solution of (1).

Proof: We start proving that $u_\epsilon(r) \le a_3$. Indeed, let be

$$v(r) := \begin{cases} u_\epsilon(r) - a_3, & \text{if } u_\epsilon(r) \ge a_3 \\ 0, & \text{if } u_\epsilon(r) < a_3 \end{cases}$$

and $\Omega_+ := \{r \in (0,R) : u_\epsilon(r) \ge a_3\}$.

Notice that

$$\epsilon^2 \int_0^R r^\alpha |u'|^\beta u' v' dr = \int_0^R r^\gamma \widehat{f}(u_\epsilon) v dr = \int_{\Omega_+} r^\gamma f_2(u_\epsilon) v dr \le 0.$$

So,

$$\int_0^R r^\alpha |v'|^{\beta+2} \le 0.$$

Thus, it follows that $|v'| = 0$. Since u_ϵ is nonconstant, there exists $r \in (0,R)$ with $u_\epsilon(r) < a_3$. So that, $v \equiv 0$. Therefore, $u_\epsilon(r) \le a_3$, for all $r \in (0,R)$.

Similarly, we have $a_1 \le u_\epsilon$, taking

$$w(r) := \begin{cases} u_\epsilon(r) - a_1, & u_\epsilon(r) \le a_1 \\ 0, & u_\epsilon(r) > a_1 \end{cases}$$

and $\Omega_- := \{r \in (0,R) : u_\epsilon(r) \le a_1\}$.

This finishes the proof of Lemma. □

Thus, by Lemma 4.1, it follows that a solution for problem (18) is a solution for problem (1), since $f \equiv \widehat{f}$ in $[a_1, a_3]$. Therefore, to prove the Theorem 4.1 is equivalent to prove the following theorem.

Theorem 4.2 *Let be f satisfying the conditions (f_1), (f_2) and (f_3). Then there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$, the problem (18) has at least one nonconstant solution u_ϵ.*

Define a functional on $\widehat{X}_R$ by

$$J_\epsilon(u) := \frac{\epsilon^2}{\beta+2}\int_0^R r^\alpha |u'|^{\beta+2} dr - \int_0^R r^\gamma \widehat{F}(u) dr,$$

where

$$\widehat{F}(u) := \int_0^u \widehat{f}(t) dt,$$

which is C^1 with derivative given by

$$J'_\epsilon(u)\varphi = \epsilon^2 \int_0^R r^\alpha |u'|^\beta \varphi' dr - \int_0^R r^\gamma \widehat{f}(u)\varphi dr, \quad \forall\ \varphi \in \widehat{X}_R.$$

Critical points of J_ϵ are weak solutions (18).

We will prove that J_ϵ verifies the Mountain Pass geometry, namely,

The functional J_ϵ satisfies the Palais-Smale condition if every sequence $(u_n) \subset \widehat{X}_R$ satisfying $J_\epsilon(u_n) \to c$ and $J_\epsilon \to 0$ has a subsequence convergent.

Lemma 4.3 *The functional J_ϵ satisfies the condition Palais-Smale.*

Proof: Let $(u_n) \subset \widehat{X}_R$ be a sequence satisfing $J_\epsilon(u_n) \to c$ and $J'_\epsilon(u_n) \to 0$, as $n \to \infty$.

Hence,

$$(19)\quad |\ J_\epsilon(u_n)\ | = \left| \frac{\epsilon^2}{\beta+2}\int_0^R r^\alpha |u'|^{\beta+2} dr - \int_0^R r^\gamma \widehat{F}(u_n) dr \right| \leqslant d \ \text{ for some } d > 0,$$

and

$$(20)\qquad |\ J'_\epsilon(u_n)v\ | = \left| \epsilon^2 \int_0^R r^\alpha \mid u'_n \mid^\beta u'_n v' dr - \int_0^R r^\gamma \widehat{f}(u_n) v dr \right| \leq \delta_n \|v\|,$$

for all $v \in \widehat{X}_R$ where $\delta_n \to 0,\ \ n \to \infty$.

Claim: (u_n) is bounded. Indeed, define

$$v_n(r) = \begin{cases} u_n(r) - \overline{\eta}_1, & \text{if } \ u_n(r) \geq \overline{\eta}_1, \\ 0, & \text{if } \ u_n(r) < \overline{\eta}_1, \end{cases}$$

and $\Omega_1 := \{r \in (0,R) : u_n(r) \geq \overline{\eta}_1\}$.

From (20), we have

$$\left| \epsilon^2 \int_0^R r^\alpha |v'_n|^{\beta+2} dr - \int_0^R r^\gamma \widehat{f}(u_n) v_n dr \right| \leq \delta_n \|v_n\|.$$

Since $\widehat{f}(u_n) \leq -C$ in Ω_1, it follows that

$$\epsilon^2 \int_o^R r^\alpha |v_n'|^{\beta+2} dr + C \int_0^R r^\gamma v_n dr \leq \delta_n \|v_n\| \tag{21}$$

$$C \int_0^R r^\gamma v_n dr \leq \delta_n \|v_n\|. \tag{22}$$

Define $w_n := \frac{v_n}{\|v_n\|}$. Hence, $\|w_n\| = 1$. Taking subsequence if necessary we can assume that $w_n \rightharpoonup w$ weakly in $\widehat{X}_R$, $w_n \to w$ in L^p_θ and $w_n(r) \to w(r)$ a.e. in $(0, R)$. Dividing (22) by $\|v_n\|$, we have

$$C \int_0^R r^\gamma w_n dr \leq \delta_n.$$

Applying the limits on both sides and taking account that $w_n \geq 0$, we have that $w \equiv 0$ a.e. in $(0, R)$. Dividing (21) by $\|v_n\|$, and taking the limit as $n \to \infty$, we obtain

$$\epsilon^2 \int_0^R r^\alpha \frac{|v_n'|^{\beta+2}}{\|v_n\|} \to 0. \tag{23}$$

On the other hand,

$$1 = \|w_n\| = \epsilon^2 \int_0^R r^\alpha |w_n'|^{\beta+2} dr + \int_0^R r^\alpha |w_n|^{\beta+2} dr. \tag{24}$$

Thus,

$$\epsilon^2 \int_0^R r^\alpha |w_n'|^{\beta+2} dr \to 1.$$

Multiplying and dividing (23) by $\|v_n\|^{\beta+1}$, we obtin

$$\|v_n\|^{\beta+1} \epsilon^2 \int_0^R r^\alpha \left| \left(\frac{v_n}{\|v_n\|} \right)' \right|^{\beta+2} dr = \|v_n\|^{\beta+1} \epsilon^2 \int_0^R r^\alpha |w_n'|^{\beta+2} dr \to 0.$$

Also, from (24) it follows that $\|v_n\| \to 0$ as $n \to \infty$. Consequently, $\|v_n\| \leq C$.

Now, define

$$z_n(r) := \begin{cases} u_n(r), & \text{if } \eta_1 \leq u_n(r) \leq \overline{\eta}_1 \\ 0, & \text{if } \eta_1 \geq u_n(r) \\ 0, & \text{if } u_n(r) \geq \overline{\eta}_1 \end{cases}$$

and $\Omega_2 := \{r \in (0, R) : \eta_1 \leq u_n(r) \leq \overline{\eta}_1\}$.

From (20), we have

$$\left| \epsilon^2 \int_0^R r^\alpha |z_n'|^{\beta+2} dr - \int_0^R r^\gamma \widehat{f}(u_n) z_n dr \right| \leq \delta_n \|z_n\|_{\widehat{X}_R}.$$

Since $|\widehat{f}(u_n)| \leq \widehat{C}$ in Ω_2 and $|z_n| \leq c_1$ in Ω_2, where $c_1 = max\{|\eta_1|, |\overline{\eta}_1|\}$, we obtain

$$\epsilon^2 \int_0^R r^\alpha |z_n'|^{\beta+2} dr \leq \delta_n \|z_n\| + \int_0^R r^\gamma |\widehat{f}(u_n)||z_n| dr \leq \delta_n \|z_n\| + C$$

Hence,

$$\|z_n\|^{\beta+2} = \epsilon^2 \int_0^R r^\alpha |z_n'|^{\beta+2} dr + \int_0^R r^\alpha |z_n|^{\beta+2} dr \leq C + \delta_n \|z_n\|.$$

Since $\beta \geq 0$, then $\|z_n\| \leq C$.

Finally, we consider the sequence

$$t_n(r) := \begin{cases} u_n(r) - \eta_1, & \text{if } u_n(r) \leq \eta_1 \\ 0, & \text{if } u_n(r) > \eta_1 \end{cases}$$

and the set $\Omega_3 := \{r \in (0, R) : u_n(r) \leq \eta_1 < 0\}$.

Similarly, as in the case of the sequence (v_n), we have $\|t_n\| \leq C$.

Let $u_n := v_n + z_n + t_n$. Then, (u_n) in bounded in Ω. Thus, J_ϵ satisfies the (PS) condition. □

Lemma 4.4 *The functional J_ϵ is lower bounded.*

Proof: Let $\Omega_1 := \Omega_3 \bigcup \Omega_4$ and $\Omega_2 := \Omega_5 \bigcup \Omega_6$, with

$$\Omega_3 := \{r \in (0, R) : 0 \leq u(r) \leq a_3\}, \quad \Omega_4 := \{r \in (0, R) : a_3 < u(r) < \infty\},$$
$$\Omega_5 := \{r \in (0, R) : a_1 \leq u(r) \leq 0\}, \quad \Omega_6 := \{r \in (0, R) : -\infty < u(r) < a_1\}.$$

How Ω_3 and Ω_5 are bounded, we have

$$\begin{aligned} \int_{\Omega_1} r^\gamma \widehat{F}(u) dr &= \int_{\Omega_3} r^\gamma \widehat{F}(u) dr + \int_{\Omega_4} r^\gamma \widehat{F}(u) dr \\ &< \widetilde{C} + \int_{\Omega_4} r^\gamma \widehat{F}(a_3) dr \leq C \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega_2} r^\gamma \widehat{F}(u) dr &= \int_{\Omega_5} r^\gamma \widehat{F}(u) dr + \int_{\Omega_6} r^\gamma \widehat{F}(u) dr \\ &< \overline{C} + \int_{\Omega_6} r^\gamma \widehat{F}(a_1) dr < C. \end{aligned}$$

Also,

$$\int_0^R r^\gamma \widehat{F}(u) dr = \int_{\Omega_1} r^\gamma \widehat{F}(u) dr + \int_{\Omega_2} r^\gamma \widehat{F}(u) dr < C.$$

Hence,

$$\begin{aligned} J_\epsilon(u) &= \tfrac{\epsilon^2}{\beta+2} \int_0^R r^\alpha |u'|^{\beta+2} dr - \int_0^R r^\gamma \widehat{F}(u) dr \\ &\geq \tfrac{\epsilon^2}{\beta+2} \int_0^R r^\alpha |u'|^{\beta+2} dx] - C \geq -C. \end{aligned}$$

Therefore, J_ϵ is lower bounded. □

Lemma 4.5 *a_i, with i odd, is strict local minimum of J_ϵ in $\widehat{X}_R$.*

Proof: First of all, we will prove that J_ϵ has a local minimum in C^1. Also, let $\delta > 0$ such that $\widehat{F}(a_i) \geq \widehat{F}(t)$, to $|t - a_i| < \delta$ and $u \in C^1$ such that $\|u(r) - a_i\|_{C^1} = \max\{|u(r) - a_i|, |u'(r) - a_i|\} < \delta$. We claim that there exists $\eta > 0$ such that $\Omega_\eta := \{r \in (0, R) : |u(r) - a_i| > \eta\}$ has positive measure. Indeed, let $u \in C^1(0, R)$, $u \not\equiv a_i$. Then, there exists $r_0 \in (0, R)$ such that $u(r_0) \neq a_i$. Hence, there exists $\eta > 0$ such that either $u(r_0) > a_i + \eta$ or $u(r_0) < a_i - \eta$.

Since u is continuous there exists a ball $B_\delta(r_0)$ such that $|u(r) - a_i| > \eta$, for all $r \in B_\delta(r_0)$. Therefore, Ω_η has positive measure.

Now, we define $c_1 := max\{\widehat{F}(a_i - \eta), \widehat{F}(a_i + \eta)\}$. Since a_i a strict local maximum of $\widehat{F}$, we have

$$\begin{aligned}\int_0^R r^\gamma \widehat{F}(u)dr &\leq \int_{\Omega_\eta} r^\gamma c_1 dr + \int_{(0,R)\setminus\Omega_\eta} r^\gamma \widehat{F}(a_i)dr \\ &< \widehat{F}(a_i)\int_{\Omega_\eta} r^\gamma dr + \int_{(0,R)\setminus\Omega_\eta} r^\gamma \widehat{F}(a_i)dr = \int_0^R r^\gamma \widehat{F}(a_i)dr.\end{aligned}$$

Therefore,

$$J_\epsilon(u) \geq -\int_0^R r^\gamma \widehat{F}(u)dr > -\int_0^R r^\gamma \widehat{F}(a_i)dr = J_\epsilon(a_i),$$

i.e., a_i is a strict local minimum of J_ϵ in C^1.

Take $a_i = a_3$. Assume that a_3 is a local minimum of J_ϵ in C^1 and a_3 is not a local minimum in $\widehat{X}_R$. Then for all $\eta > 0$ small enough, there exists $v_\eta \in B_\eta \subset \widehat{X}_R$ such that

$$J_\epsilon(v_\eta) = \min_{v \in B_\eta} J_\epsilon(v) < J_\epsilon(a_3), \tag{25}$$

where B_η is a ball of radius η centered at a_3,

$$B_\eta := \{v \in \widehat{X}_R : \|v - a_3\| \leq \eta\},$$

since, B_η is weakly close and J_ϵ is lower bounded in B_η.

Therefore, v_η satisfies the Euler equation

$$J'_\epsilon(v_\eta)\varphi = G'(v_\eta)\varphi, \ \forall\ \varphi \in \widehat{X}_R, \text{ where } G(v_\eta) := \|v_\eta - a_3\|, \tag{26}$$

i.e.,

$$\epsilon^2 \int_0^R r^\alpha |v'_\eta|^\beta v'_\eta \varphi' - \int_0^R r^\gamma \widehat{f}(v_\eta)\varphi = \mu_\eta \left[\epsilon^2 \int_0^R r^\alpha |v'_\eta|^\beta v'_\eta \varphi' + \int_0^R r^\alpha |v_\eta - a_3|^\beta (v_\eta - a_3)\varphi\right],$$

for all $\varphi \in \widehat{X}_R$ and for some Lagrange multiplier μ_η. Since v_η is minimum of J_ϵ in B_η then $\mu_\eta \leq 0$.

Rewriting the equation above, we have

$$(27)\quad (1-\mu_\eta)\epsilon^2\int_0^R r^\alpha|v_\eta'|^\beta v_\eta'\varphi' = \int_0^R [r^\gamma \widehat{f}(v_\eta)+\mu_\eta r^\alpha|v_\eta-a_3|^\beta(v_\eta-a_3)]\varphi,$$

for all $\varphi\in\widehat{X}_R$. Thus, v_η is weak solution of problem

$$(28)\qquad \begin{cases} -(1-\mu_\eta)\epsilon^2(r^\alpha|v_\eta'|^\beta v_\eta')' &= \mu_\eta r^\alpha|v_\eta-a_3|^\beta(v_\eta-a_3) \\ & \quad +r^\gamma\widehat{f}(v_\eta), \quad \text{in} \quad (0,R) \\ v_\eta'(0)=v_\eta'(R)=0. & \end{cases}$$

We will show that $a_1\le v_\eta(r)\le a_3$, for all $r\in(0,R)$.
Claim 1: v_η is nonconstant. Indeed, if $v_\eta\equiv$ constant then $v_\eta\equiv a_3$, since $v_\eta\to a_3$ in $\widehat{X}_R$, that contradict (25).
Claim 2: $v_\eta\le a_3$. Indeed, let

$$v(r):=\begin{cases} v_\eta(r)-a_3, & \text{if} \quad v_\eta(r)\ge a_3 \\ 0, & \text{if} \quad v_\eta(r)<a_3 \end{cases}$$

and $\Omega_+:=\{r\in(0,R): v_\eta(r)\ge a_3\}$.

From (27), we have

$$\begin{aligned}(1-\mu_\eta)\epsilon^2\int_0^R r^\alpha|v_\eta'|^\beta v_\eta' v' dr &= \int_0^R [r^\gamma\widehat{f}(v_\eta)+\mu_\eta r^\alpha|v_\eta-a_3|^\beta(v_\eta-a_3)]v dr \\ &= \int_{\Omega_+}[r^\gamma f_2(v_\eta)v+\mu_\eta r^\alpha|v_\eta-a_3|^{\beta+2}]dr\le 0,\end{aligned}$$

a.e. since, $f_2(t)\le 0$, $v\ge 0$, $\mu_\eta\le 0$. The function f_2 is a function defined in Lemma 4.1.

Since, $(1-\mu_\eta)\ge 0$, $\epsilon^2>0$ e $v_\eta'=v'$, we have

$$\int_0^R r^\alpha|v'|^{\beta+2}\le 0.$$

It follows $v\equiv$ constant. Also, if $v_\eta(r)\ge a_3$, for all $r\in(0,R)$ it follows that $v\equiv v_\eta-a_3$. Hence $v_\eta\equiv$ constant. It is a contradiction. Therefore, there exists $r\in(0,R)$ such that $v_\eta(r)<a_3$, i.e., $v(r)=0$. Then $v(r)\equiv 0$, for all $r\in(0,R)$, i.e., $v_\eta\le a_3$ for all $r\in(0,R)$.
Claim 3: $a_1\le v_\eta$. Indeed, it follows similarly as in the claim 2 if we take

$$w(r):=\begin{cases} v_\eta(r)-a_1, & v_\eta(r)\le a_1 \\ 0, & v_\eta(r)>a_1 \end{cases}$$

and $\Omega_-:=\{r\in(0,R): v_\eta(r)\le a_1\}$.

Therefore, of the claim 2 and 3, we have that $a_1\le v_\eta(r)\le a_3$, i.e., $\sup_{(0,R)}|v_\eta|\le C^*$.

Then from Theorem 2 in [14], we have that there exists a positive constant $\theta \in (0,1)$ such that $v_\eta \in C^{1,\theta}$ and moreover, there exists a positive constant $C = C((0,R), C^*, \beta)$ such that $\|v_\eta\|_{C^{1,\theta}} \leq C$.

Thus, from Ascoli-Arzelá Theorem we get there exists a subsequence of $(v_{\eta_j}) = (v_\eta)$ such that $v_\eta \to a_3$ in C^1. It contradict the fact that a_3 is local minimum of J_ϵ in C^1. Indeed, a_3 local minimum of J_ϵ in C^1 follows $J_\epsilon(a_3) \leq J_\epsilon(v)$ for all $v \in C^1$ such that $0 < \|v - a_3\|_{C^1} < \delta_0$ for some $\delta_0 > 0$. Since $v_\eta \to a_3$ in C^1 it follows $J_\epsilon(a_3) \leq J_\epsilon(v_\eta)$.

On the other hand, we have from (25) that $J_\epsilon(v_\eta) < J_\epsilon(a_3)$, for all $\eta > 0$ small enough. Hence the contradict.

Therefore, a_3 is local minimum of J_ϵ in $\widehat{X}_R$.

Similarly, it follows that a_1 is local minimum of J_ϵ in $\widehat{X}_R$.

Without loss of generality, we suppose that a_i is strict local minimum of J_ϵ in $\widehat{X}_R$. Contrary case, for all $\delta > 0$, there exists $v_\delta \in \widehat{X}_R$ such that $J_\epsilon(v_\delta) = J_\epsilon(a_i)$. Hence, v_δ is a critical point of J_ϵ in $\widehat{X}_R$. □

We will use arguments of [5] to obtain the next lemma.

Lemma 4.6 *If a is a strict local minimum of J_ϵ, i.e.,*

$$J_\epsilon(a) < J_\epsilon(u) \tag{29}$$

for all $u \in \widehat{X}_R$ such that $0 < \|u - a\| < \delta_0$ for some $\delta_0 > 0$. Then, for any $0 < \rho < \delta_0$,

$$\inf\{J_\epsilon(u) : u \in \widehat{X}_R \text{ and } \|u - a\| = \rho\} > J_\epsilon(a). \tag{30}$$

Proof: Assume by contradiction that the infimum in (30) is equal to $J_\epsilon(a)$ for some ρ with $0 < \rho < \delta_0$. So there exists a sequence $u_n \in \widehat{X}_R$ with $\|u_n - a\| = \rho$ and, say, $J_\epsilon(u_n) \leq J_\epsilon(a) + \frac{1}{2n^2}$. Call

$$A := \{u \in \widehat{X}_R : \rho - \delta \leq \|u - a\| \leq \rho + \delta\},$$

where $\delta > 0$ is chosen so that $0 < \rho - \delta$ and $\rho + \delta < \delta_0$. In view of our contradiction hypothesis an (29), it follows that $inf\{J_\epsilon(u) : u \in A\} = J_\epsilon(a)$.

We now apply for each n Ekeland's principle to the functional J_ϵ on A to get the existence of $v_n \in A$ such that

$$J_\epsilon(v_n) \leq J_\epsilon(u_n), \tag{31}$$

$$\|v_n - u_n\| \leq \frac{1}{n}, \tag{32}$$

$$J_\epsilon(v_n) \leq J_\epsilon(u) + \frac{1}{n}\|u - v_n\|, \quad \forall \ u \in A. \tag{33}$$

Our purpose is to show that v_n is a Palais-Smale sequence for J_ϵ in $\widehat{X}_R$, i.e., $J_\epsilon(v_n) \leq C$ (it follows by (31)) and $J'_\epsilon(v_n) \to 0$, as $n \to \infty$.

Once this is proved, we get, that v_n has a convergent subsequence. Denote this subsequence by v_n we have that $v_n \to v$ in $\widehat{X}_R$. Notice that $v \in A$, since A is complete. Hence, $v \in \widehat{X}_R$ and therefore it satisfies $\|v - a\| = \rho$ and $J_\epsilon(v) = J_\epsilon(a)$, which contradicts (29).

For we conclude the proof we will prove that $J'_\epsilon(v_n) \to 0$, as $n \to \infty$, We first fix $n > \frac{1}{\delta}$, take $w \in \widehat{X}_R$ and $u_t := v_n + tw$. We observe that for $|t|$ sufficiently small, $u_t = v_n + tw \in A$. Indeed,

$$\lim_{t\to 0} \|u_t - a\| = \|v_n - a\| \le \|v_n - u_n\| + \|u_n - a\| \le \frac{1}{n} + \rho < \delta + \rho.$$

On the other hand,

$$\|v_n - a\| \ge \|a - u_n\| - \|u_n - v_n\| \ge \rho - \frac{1}{n} > \rho - \delta.$$

Also, we can take $u = u_t$ in (33),and then, for $t > 0$,

$$\frac{J_\epsilon(v_n) - J_\epsilon(v_n + tw)}{t} \le \frac{1}{n}\frac{1}{t}\|v_n - tw - v_n\| \le \frac{1}{nt}\|tw\|. \tag{34}$$

Taking the limit in (34) as $t \to 0$, we obtain $\langle J'_\epsilon(v_n), w\rangle \le \frac{1}{n}\|w\|$. Consequently,

$$|\langle J'_\epsilon(v_n), w\rangle| \le \frac{1}{n}\|w\|, \quad \forall\, w \in \widehat{X}_R.$$

Also, $J'_\epsilon(v_n) \to 0$, $n \to \infty$ and v_n is a sequence (PS) for J_ϵ in $W^{1,p}(\Omega)$. Therefore, the proof of the lemma is concluded. □

Demonstração do Teorema 4.2: We define

$$\Gamma := \{h \in C([0,1], \widehat{X}_R) : h(0) = a_1 \text{ e } h(1) = a_3\}$$

and

$$\gamma_\epsilon := \inf_{h\in\Gamma} \max_{t\in[0,1]} J_\epsilon(h(t)).$$

By Lemma 4.6 $\gamma_\epsilon := \inf_{h\in\Gamma} \max_{t\in[0,1]} J_\epsilon(h(t)) > c = max\{J_\epsilon(a_1), J_\epsilon(a_3)\}$.

Since J_ϵ satisfies the condition (PS), it follows from Theorem of [11] that there exists $\overline{u}$ critical point of J_ϵ such that $J_\epsilon(\overline{u}) = \gamma_\epsilon$. Moreover, $\overline{u}$ is the mountain-pass type, since if the critical points are not isolated in $\widehat{X}_R$ then there exists an infinite of critical points of J_ϵ. Since a_1 and a_3 are strict local minimum then $\overline{u} \ne a_1$ and $\overline{u} \ne a_3$. Therefore, in order to show the existence of a nonconstant critical point of J_ϵ, we only need to prove that $\gamma_\epsilon < 0$, since $J_\epsilon(0) = 0$.

We claim that $\gamma_\epsilon < 0$. In fact, we consider $B \subset (0, R)$ the open ball and we define

$$u_0(r) := \begin{cases} v_\epsilon(r), & r \in \overline{B} \\ w_\epsilon(r), & r \in [0,R]\backslash\overline{B}, \end{cases}$$

where v_ϵ is the positive solution for the Dirichlet problem (5) in $\overline{B}$ and w_ϵ is the negative solution for the Dirichlet problem (6) in $[0, R]\backslash\overline{B}$.

Since $(0,R)\in C^1$ and $v_\epsilon, w_\epsilon \in L^p(0,R)$ it follows that $u_0 \in \widehat{X}_R$.

Now, given $\epsilon>0$ we consider the special path

$$h_\epsilon(t) := t(1-t)u_0(r) + (a_3-a_1)t + a_1, \quad \text{in} \ \ \Gamma.$$

Then we claim that there exists a small number $\epsilon_0>0$, such that, for all $0<\epsilon<\epsilon_0$ and for all $t\in[0,1]$ $\max J_\epsilon(h_\epsilon(t))<0$. In fact, suppose by contradiction that there is not exist this $\epsilon_0>0$. Then for any $\epsilon_0>0$, there is an $\epsilon<\epsilon_0$ such that

$$J_\epsilon(h_\epsilon(t_\epsilon)) \geq 0, \tag{35}$$

for some $t_\epsilon\in[0,1]$. Then choose a sequence (ϵ_k) such that

$$\lim_{k\to\infty}\epsilon_k = 0, \quad J_\epsilon(h_{\epsilon_k}(t_{\epsilon_k}))\geq 0, \quad 0\leq \lim_{k\to 0} t_{\epsilon_k} = \theta \leq 1.$$

Without loss of generality, let be $\epsilon_k=\epsilon$ and $t_{\epsilon_k}=t$. Now, given any open ball $B\subset[0,R]$, we have

$$\begin{aligned} J_\epsilon(h_\epsilon(t)) &= \tfrac{\epsilon^2}{\beta+2}\int_B r^\alpha |h'_\epsilon(t)|^{\beta+2}dr + \frac{\epsilon^2}{\beta+2}\int_{(0,R)\backslash B} r^\alpha |h'_\epsilon(t)|^{\beta+2}dr \\ &\quad -\int_0^R r^\gamma \widehat{F}(h_\epsilon(t))dr \\ &= \tfrac{\epsilon^2}{\beta+2}t^{\beta+2}(1-t)^{\beta+2}\left(\int_B r^\alpha|v'_\epsilon|^{\beta+2}dr + \int_{(0,R)\backslash B} r^\alpha |w'_\epsilon|^{\beta+2}dr\right) \\ &\quad -\int_0^R r^\gamma \widehat{F}(t(1-t)u_0+(a_3-a_1)t+a_1)dr. \end{aligned}$$

Since

$$\epsilon^2\int_B r^\alpha |v'_\epsilon|^{\beta+2}dr = \int_B r^\gamma \widehat{f}(v_\epsilon)v_\epsilon dr$$

and

$$\epsilon^2\int_{(0,R)\backslash B} r^\alpha |w'_\epsilon|^{\beta+2}dr = \int_{(0,R)\backslash B} r^\gamma \widehat{f}(w_\epsilon)w_\epsilon dr$$

we get

$$\begin{aligned} J_\epsilon(h_\epsilon(t)) &= \tfrac{t^{\beta+2}(1-t)^{\beta+2}}{\beta+2}\left[\int_B r^\gamma \widehat{f}(v_\epsilon)v_\epsilon dr + \int_{(0,R)\backslash B} r^\gamma\widehat{f}(w_\epsilon)w_\epsilon dr\right] \\ &\quad -\int_0^R r^\gamma \widehat{F}(t(1-t)u_0+(a_3-a_1)t+a_1)dr. \end{aligned} \tag{36}$$

Since $v_\epsilon\to a_3$ as $\epsilon\to 0$ uniformly on every compact subset of B and $w_\epsilon\to a_1$ as $\epsilon\to 0$ uniformly on every compact subset of $\Omega\backslash B$, so

$$\lim_{\epsilon\to 0}\int_B r^\gamma \widehat{f}(v_\epsilon)v_\epsilon dr = 0 \ \text{ and } \ \lim_{\epsilon\to 0}\int_{(0,R)\backslash B} r^\gamma \widehat{f}(w_\epsilon)w_\epsilon dr = 0.$$

If we take the limit on both sides of (36), it follows from the dominated convergence theorem and the above facts that

$$\begin{aligned}\lim_{\epsilon\to 0} J_\epsilon(h_\epsilon(t)) &= \lim_{\epsilon\to 0}\left[-\int_0^R r^\gamma \widehat{F}(t(1-t)u_0(r,\epsilon)+(a_3-a_1)t+a_1)dr\right]\\ &= -r^\gamma \widehat{F}(\theta(1-\theta)a_3+(a_3-a_1)\theta+a_1)|B|\\ &\quad -r^\gamma \widehat{F}(\theta(1-\theta)a_1+(a_3-a_1)\theta+a_1)|(0,R)\backslash B|,\end{aligned}$$

where $|A|$ is the Lebesgue measure of $A\subset\mathbb{R}^N$. By an assumptions for $\widehat{f}$, Lemma 4.1, we have

$$-\widehat{F}(\theta(1-\theta)a_3+(a_3-a_1)\theta+a_1)\le 0 \tag{37}$$

and

$$-\widehat{F}(\theta(1-\theta)a_1+(a_3-a_1)\theta+a_1)\le 0. \tag{38}$$

Since,

$$\eta_1<\theta(1-\theta)a_3+(a_3-a_1)\theta+a_1<\overline{\eta}_1,$$

$$\eta_1<\theta(1-\theta)a_1+(a_3-a_1)\theta+a_1<\overline{\eta}_1$$

and $\widehat{F}$ is zero only in η_1, 0 and $\overline{\eta}_1$ then if (37) and (38) are zeros, we have that

$$\theta(1-\theta)a_3+(a_3-a_1)\theta+a_1=0=\theta(1-\theta)a_1+(a_3-a_1)\theta+a_1,$$

i.e., $\theta(1-\theta)a_3=\theta(1-\theta)a_1$. Since $a_3\ne a_1$, then $\theta(1-\theta)=0$, i.e., either $\theta=0$ or $\theta=1$. Hence either $\widehat{F}(a_1)=0$ or $\widehat{F}(a_3)=0$. Then either $a_1=0$ or $a_3=0$, which is impossible. We conclude that either

$$-\widehat{F}(\theta(1-\theta)a_3+(a_3-a_1)\theta+a_1)<0$$

or

$$-\widehat{F}(\theta(1-\theta)a_1+(a_3-a_1)\theta+a_1)<0.$$

Hence, $\lim_{\epsilon\to 0} J_\epsilon(h_\epsilon(t))<0$, which is a contradiction to (35). Also, $\gamma_\epsilon<0$. □

Corollary 4.1 *Let f be satisfying (f_1), (f_2) and (f_3) for $l=1$ and let u_ϵ the nonconstant solution of* (1) *such that $J_\epsilon(u_\epsilon)=\gamma_\epsilon$. Then*

$$\lim_{\epsilon\to 0} sup J_\epsilon(u_\epsilon)<0.$$

Proof: From the proof of Theorem 4.2 there exists a number ϵ_1 positive such that $J_\epsilon(h_{\epsilon_1}(t))<0$ for all $t\in[0,1]$, where

$$h_{\epsilon_1}(t)=t(1-t)u_0(r,\epsilon_1)+(a_3-a_1)t+a_1,$$

u_0 is defined in proof of Theorem 4.2.

Hence, for all $0 < \epsilon < \epsilon_1$, $J_\epsilon(h_{\epsilon_1}(t)) \leq J_\epsilon(h_{\epsilon_1}(t)) < 0$ for all $t \in [0,1]$. It follows from the intermediate value theorem that there exists $t_\epsilon \in [0,1]$ such that $h_{\epsilon_1}(t_\epsilon) = u_\epsilon$. Consequently, $J_\epsilon(h_{\epsilon_1}(t_\epsilon)) = J_\epsilon(u_\epsilon)$, for some $t_\epsilon \in [0,1]$, and so the assertion is true. □

Demonstração do Teorema 4.1: The proof follows directly from the Theorem 4.2 and Lemma 4.2 □

Demonstração do Teorema 1.1: For each $l = 1, 2, ...$ we consider the function $\widetilde{f} : [a_{2l-1} - a_{2l}, a_{2l+1} - a_{2l}] \to \mathbb{R}$ of class C^1 defined by $\widetilde{f}(t) := f(t + a_{2l})$. Then from Theorem 4.1, the problem

$$\left\{ \begin{array}{l} -\epsilon^2(r^\alpha|v'|^{\beta+2} + v')' = r^\gamma \widetilde{f}(v),\ r \in (0,R) \\ v'(0) = v'(R) = 0, \end{array} \right. \tag{39}$$

has a nonconstant solution $v_l(r)$ such that $a_{2l-1} - a_{2l} < v_l(r) < a_{2l+1} - a_{2l}$, where $v = u - a_{2l}$. So, $u_l = v_l + a_{2l}$ is a nonconstant solution for the problem (1) with $a_{2l-1} < u_l < a_{2l+1}$.

Hence, there exists at least l nonconstants solutions for the problem (1) satisfy

$$a_1 < u_1(r) < a_3 < u_2(r) < a_5 < ... < a_{2l-1} < u_l(r) < a_{2l+1}.$$

□

5 Asymptotic Behavior for a Class of Solutions

Demonstração do Teorema 1.2: Let $0 < \mu < a_3$. We shall prove by contradiction. Suppose that $\lim_{\epsilon \to 0} w^*(\epsilon, \mu) \neq 0$. Then there is a convergent sequence $\{w^*(\epsilon_k, \mu)\}$ such that $\lim_{k\to\infty} w^*(\epsilon_k, \mu) = \alpha_\mu > 0$. This means that, for each $\epsilon_k > 0$, there is $r_k = r(\epsilon_k, \mu) \in \Omega^+(\epsilon_k, \mu)$ so that the ball $B(r_k, \alpha_\mu)$, centered at the point r_k with the radius α_μ, is contained in $\Omega^+(\epsilon_k, \mu)$.

Notice that u_ϵ is a upper solution of the Dirichlet problem

$$\left\{ \begin{array}{l} -\epsilon_k^2(r^\alpha|u'|^{\beta+2}u')' = r^\gamma f(u), r \in B(r_k, \alpha_\mu) \\ u'(0) = u(r) = 0,\ r \in \partial B(r_k, \alpha_\mu). \end{array} \right. \tag{40}$$

Claim: There are ϵ_{k_0} and $\beta_1 > 0$ such that $\beta_1\varphi_1$ is a lower solution for the problem (40) for all $0 < \epsilon_k \leq \epsilon_{k_0}$, where φ_1 is an eigenfunction corresponding to the first eigenvalue λ_1 of operator $Lu := -(r^\alpha|u'|^{\beta+2}u')'$ subject to Dirichlet boundary condition. In fact, this follow similarly as in the step 2 in proof of the Proposition 3.1.

Also, from Theorem 2.1 we know that the problem (40) has a minimal solution u^* with $\beta_1\varphi_1 \leq u^* \leq u_\epsilon$ and so that $u^* \to a_3$ on every compact subset of $B(r_k, \alpha_\mu)$ as $\epsilon_k \to 0$. This leads to a contradiction for $\mu < a_3$. □

Remark 5.1 *If we choose an open ball* $B = B(x, w^*(\epsilon, \mu))$ *centered at some point* $r = r(\varepsilon, \mu)$ *whose radius* $w^*(\epsilon, \mu)$ *is the maximum of the radii of balls in*

$$\Omega^-(\epsilon, \mu) = \{r \in (0, R) : a_1 < -\mu < u(r, \epsilon) < 0\}$$

we can prove, by a similar method, that $\lim_{\epsilon \to 0} w^*(\epsilon, \mu) = 0$.

Remark 5.2 *By a translation we can remark that Theorem 1.2 and remark 5.1 give us the asymptotic behavior of any solution* $u_l(x)$ *obtained in the Theorem 1.1.*

5.1 Examples

Example 5.1 *Let* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(u) := \frac{|u|^{p-2}u(p - p|u|^p)}{(1 + |u|^p)}.$$

Hence, f satisfies the conditions (f_1), (f_2), (f_3), *for* $l = 1$, *where* $a_1 = -1$, $a_2 = 0$ *and* $a_3 = 1$.

Also, by Theorem 1.1 there is $\epsilon_0 > 0$ so that the Neumann problem

$$\begin{cases} -\epsilon^2(r^\alpha |u'|^\beta u')' = r^\gamma \frac{|u|^{p-2}u(p-p|u|^p)}{(1+|u|^p)}, & \text{in } (0, R) \\ u'(0) = u'(R) = 0, \end{cases}$$

has a nonconstant solution $a_1 \leq u_\epsilon \leq a_3$ for all $0 < \epsilon < \epsilon_0$ and $p \geq 2$.

Example 5.2 *Let* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(u) = u(a - u^2)$, *where* $a \in \mathbb{R}^+$. *Hence, f satisfies the conditions* (f_1), (f_2), (f_3), *for* $l = 1$, *where* $a_1 - \sqrt{a}$, $a_2 = 0$ *and* $a_3 = \sqrt{a}$.

Since, by Theorem 1.1 there is $\epsilon_0 > 0$ so that the Neumann problem

$$\begin{cases} -\epsilon^2(r^\alpha |u'|^\beta u')' = r^\gamma u(a - u^2), & \text{in } (0, R) \\ u'(0) = u'(R) = 0, \end{cases}$$

has at least l nonconstant solutions satisfy

$$a_1 \leq u_1(x) \leq a_3 \leq u_2(x) \leq a_5 < ... < u_l(x) < a_{2l+1}$$

for all $0 < \epsilon < \epsilon_0$.

Example 5.3 *Let* $f : [-\pi, (2l-1)\pi] \to \mathbb{R}$ *defined by* $f(u) = sen(u)$. *Hence, f satisfies the conditions* (f_1), (f_2), (f_3), *for* $p = 2$, $l = 1, 2, 3, ...$ *where*

$$a_1 = -\pi < a_2 = 0 < a_3 = \pi < ... < a_{2l} = (2l - 2)\pi < a_{2l+1} = (2l - 1)\pi.$$

Hence, from Theorem 1.1 there is $\epsilon_0 > 0$ so that the Neumann problem

$$\begin{cases} -\epsilon^2(r^\alpha |u'|^\beta u')' = r^\gamma sen(u), & \text{in } (0, R) \\ u'(0) = u'(R) = 0, \end{cases}$$

has at least l nonconstant solutions satisfy

$$a_1 \leq u_1(x) \leq a_3 \leq u_2(x) \leq a_5 < ... < u_l(x) < a_{2l+1}$$

for all $0 < \epsilon < \epsilon_0$.

References

[1] Brézis, H., *Analyse fonctionelle théorie et applications*, Masson, Paris (1987).

[2] Boccardo, L. e Murat, F., *Almost Everywhere Convergence of the Gradients of Solutions to Elliptic and Parabolic Equations*, Nonl. Anal. TMA, 6 (1992), 581-597.

[3] Cañada, A., Drábek, P. e Gamez, J., *Existence of positive solutions for some problems with nonlinear difusion*, Trans. AMS, 349 (1997), 4231-4249.

[4] Clément, P., De Figueiredo, D. G. e Mitidieri, E., *Quasilinear elliptic equations with critical exponents*, Nonl. Anal. TMA, 7 (1996), 133-170.

[5] Cuesta, M., De Figueiredo, D. G. e Gossez, J. P., *The beginning of the Fučik spectrum for the p-Laplacian*, J. Diff. Eqns., 159 (1999), 212-238.

[6] De Figueiredo, D. G.,*Lectures on the Ekeland Variational Principle with Applications and Détours*, TATA Institute, Springer- Verlag, New York/Berlin (1989).

[7] De Figueiredo, D. G., *On the existence of multiple ordered solutions of nonlinear eigenvalue problems*, Nonl. Anal. TMA, 11 (1987), 481-492.

[8] De Figueiredo, D. G., Gonçalves, J. V. e Miyagaki, O. H., *On a class of quasilinear elliptic problems involving critical exponents*, Comm. Contemp. Math., 2 (2000), 47-59.

[9] Díaz, J. I., *Nonlinear partial differential equations and free boundaries*, Vol. 1, Elliptic Equations, Pitman, London (1985).

[10] García Azorero, J. P., Peral Alonso, I. e Manfredi, J. J., *Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations*, Comm. Contemp. Math., 3 (2000), 385-404.

[11] Hofer, H., *A geometric description of the neighborhood of a critical point given by the moutain-pass theorem*, J. London Math. Soc, 31 (1985), 566-570.

[12] Ko, B. S., *The existence of nonconstant solutions of a class of semilinear elliptic Neumann problems with a small parameter*, Nonl. Anal. TMA, 28 (1997), 1249-1263.

[13] Kufner, A. e Opic, B., *Hardy-type inequalities*,Pitman Res. Notes in Math., vol 219, Longman Scientific and Technical, Harlow (1990).

[14] Lieberman, G. M., *Boundary regularity for solutions of degenerate elliptic equations*, Nonl. Anal. TMA, 12 (1988), 1203-1219.

[15] Manríquez, C. C., *Existência e multiplicidade de soluções para certos problemas elípticos semilineares*, Tese de Doutorado, UNICAMP, (1992).

[16] Tolksdorf, P., *Regularity for a more general class of quasilinear elliptic equations*, J. Diff. Eqns., 51 (1984), 126-150.

[17] Willem, M., *Minimax Theorems*, PNDLE, Vol 24, Birkhäuser, Berlin (1996).