# Existence and Asymptotic Behavior of Solutions for a Class of Neumann Problems

Ilma Marques *

IMECC - UNICAMP,

Caixa Postal 6065.

13081-970 Campinas-SP, Brazil

*E-mail*: imarques@ime.unicamp.br

**Abstract**

In this paper we study the existence of solutions $u \in W^{1,p}(\Omega)$ and their asymptotic behavior for the problem

$$\begin{cases} -\epsilon^2 \triangle_p u = f(u), & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\triangle_p$ is the p-Laplace operator.

*These results appear in the author's UNICAMP doctoral dissertation.

## 1 Introduction

In this paper we study the existence of nonconstant solutions and their asymptotic behavior for the following class of Neumann problem

$$(1)\qquad \begin{cases} -\epsilon^2 \triangle_p u &= f(u), \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0, \qquad \text{on } \partial\Omega, \end{cases}$$

where $\triangle_p$ denotes the p-Laplace operator, $p \geq 2$, $\epsilon > 0$ is a small parameter, $\Omega$ is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, with smooth boundary $\partial\Omega$. $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of $u$ on $\partial\Omega$. For that matter we make the following assumptions on the nonlinearity $f$:

$(f_1)$ $f : [a, b] \to \mathbb{R}$ is a function of class $C^1$, where $a$ and $b$ are zeros of the $f$;

$(f_2)$ $f$ has exactly $2l + 1$ zeros, $a = a_1 < 0 = a_2 < a_3 < ... < a_{2l+1} = b$ with $l = 1, 2, 3, ...$ so that $f(a_i) = 0,\ \forall\ i$ and $f'(a_i) < 0$, if $i$ odd;

$(f_3)$ $\lim_{t\to a_{2l}} \frac{f(t-a_{2l})}{|t-a_{2l}|^{p-2}(t-a_{2l})} > 0$.

A function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of (1) if

$$\int_\Omega \epsilon^2 |\nabla u|^{p-2} \nabla u \nabla \varphi dx - \int_\Omega f(u)\varphi dx = 0, \quad \forall\ \varphi \in W^{1,p}(\Omega).$$

It has been proved by Ko [8], that under hypotheses $(f_1)$, $(f_2)$, $(f_3)$ with $l = 1$, $p = 2$ and

$(f_4)$ $|\int_0^{a_1} f(t)dt| = \int_0^{a_3} f(t)dt$

there exists $\epsilon_0 > 0$ such that problem (1) possesses one solution $a_1 \leq u_\epsilon \leq a_3$ for all $0 < \epsilon < \epsilon_0$. His proof uses variational arguments. [8] also shows that the nonconstant solution $u_\epsilon$ converge either to $a_1$ or $a_3$ a.e. on $\overline{\Omega}$ as $\epsilon \to 0$ using monotone iteration methods. The same problem had studied by Manríquez [10], who proved under hypotheses $(f_1)$, $(f_2)$, $(f_3)$, $p = 2$, $\epsilon = 1$,

$(f_5)$ $|f(t)| \leq a|t|^\sigma + b$ with $1 \leq \sigma < 2^*$ where $2^* = 2N/(N-2)$ if $N > 2$ and $2^* = \infty$ if $N = 2$ or $N = 1$

and

$(f_6)$ $|f'(t)| \leq a|t|^\sigma + d$ with $1 \leq \sigma < 2^* - 2$

that problem (1) possesses at least $2l$ nonconstant solutions. His proof uses a combination of the variational and the topological degree arguments.

Using standard variational methods (local minimization and the Mountain Pass Theorem) we prove the existence of $\epsilon_0 > 0$ such that problem (1) has at least $l$ nonconstant solutions for all $0 < \epsilon < \epsilon_0$. Using monotone iteration methods we show that the nonconstant solutions converge either $a_l$ or $a_{2l+1}$ a.e. on $\overline{\Omega}$ as $\epsilon \to 0$.

The hardest step pf our proof is to show that the solutions $u_\epsilon$ given for Mountain Pass Theorem is different of $a_i$, if $i$ even. In order to show this we use the existence of solutions and their asymptotic behavior for the problem (1) with Dirichlet boundary conditions, which we establish in Section 2.

Our main results are the following.

**Theorem 1.1** *Suppose that $f$ satisfies the assumptions $(f_1)$, $(f_2)$ and $(f_3)$. Then, there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$, problem* (1) *has at least*

*$l$ nonconstant solutions satisfing*

$$a_1 < u_1(x) < a_3 < u_2(x) < a_5 < ... < a_{2l-1} < u_l(x) < a_{2l+1},$$

*where $l = 1, 2, 3, ....$*

**Remark 1.1** *From the Theorem of regularity of Tolksdorf ([[11], Thm 1]) there exists $\alpha \in (0,1)$ such that $u_l \in C^{1,\alpha}(\Omega)$.*

**Remark 1.2** *The theorem 1.1 still is true if we consider the $a_i's$ negative for all $i \geq 3$, i.e., $a = a_{2l+1} < ... < a_3 < a_2 = 0 < a_1 = b$. The condition $a_2 = 0$ is not essential; it only makes the presentation easies.*

**Theorem 1.2** *Let $f$ be satisfing the conditions $(f_1)$, $(f_2)$ and $(f_3)$ for $l = 1$ and let $u_\epsilon(x)$ a nonconstant solution of problem (1). Given any $\delta > 0$, let*

$$\Omega^+(\epsilon, \delta) := \{x \in \Omega : 0 < u_\epsilon(x) < \delta < a_3\}$$

*contain a open ball $B(x, w^*(\epsilon, \delta))$ centered at some $x = x(\epsilon, \delta) \in \Omega^+(\epsilon, \delta)$ whose radius $w^*(\epsilon, \delta)$ is the maximum of radii of open balls in $\Omega^+(\epsilon, \delta)$. Then*

$$\lim_{\epsilon \to 0} w^*(\epsilon, \delta) = 0.$$

## 2 Dirichlet Problem

To prove Theorem 1.1, we need to show the existence of solutions and their asymptotic behavior for the following Dirichlet problem

$$\begin{cases} -\epsilon^2 \triangle_p u &= f(u), \quad \text{in } \Omega \\ u &= 0, \quad\quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

We prove the following Theorem.

**Theorem 2.1** *Let $f$ be a function satisfing the assumptions $(f_1)$, $(f_2)$ and $(f_3)$ with $l = 1$. Then there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon \leq \epsilon_0$, the Dirichlet problem (2) has a positive solution $0 < u_\epsilon < a_3$ in $\overline{\Omega}$ such that $u_\epsilon \to a_3$ as $\epsilon \to 0$, uniformly in every compact subset of $\Omega$, and a negative solution $a_1 < v_\epsilon < 0$ in $\overline{\Omega}$ such that $v_\epsilon \to a_1$ as $\epsilon \to 0$ uniformly in every compact subset of $\Omega$.*

## 2.1 Existence of Solutions

The prove is done in two steps.

**Proposition 2.1** *Suppose that $f$ satisfies the assumptions $(f_1)$, $(f_2)$ and $(f_3)$ with $l = 1$. Then there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon \leq \epsilon_0$, the Dirichlet problem (2) has a positive solution $0 < u_\epsilon < a_3$ in $\overline{\Omega}$ and a negative solution $a_1 < v_\epsilon < 0$ in $\overline{\Omega}$.*

**Proof:** We start by proving the existence of a positive solution for problem (2). Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_1(u) := \begin{cases} f(u), & \text{if} \quad 0 \leq u \leq a_3 \\ 0, & \text{if} \quad u \leq 0 \\ 0, & \text{if} \quad u \geq a_3. \end{cases}$$

By $(A_1)$, we have that there exists $M > 0$ such that $f_1(t) + Mt$ is nondecreasing in $t$ for $t \in [0, a_3]$.

Now, we consider the following auxiliary problem

$$(3) \qquad \begin{cases} -\epsilon^2 \Delta_p u + Mu = f_1(u) + Mu, & \text{in} \quad \Omega \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

Thus, we will show the existence of a positive solution for problem (3) by using the method of lower and upper-solution. We prove this in three steps:

**Step 1:** Observe that the function $\overline{u}(x) \equiv a_3$, for $x \in \overline{\Omega}$ is a upper solution of (3).

**Step 2:** Construction of the lower solution of (3).

Let

$$\gamma = \lim_{t \to 0} \frac{f_1(t)}{|t|^{p-2}t}$$

and $\lambda_1$ is the first eigenvalue of $-\triangle_p$ subject to the Dirichlet boundary condition. It follows from $(f_3)$ that given $\delta > 0$, (take $\delta < \gamma$), there exists $t_0 > 0$ such that for all $| t | \leq t_0$ we have

$$\gamma - \delta \leq \frac{f_1(t)}{| t |^{p-2} t}. \tag{4}$$

Let $\varphi_1 > 0$ an eigenfunction corresponding to the first eigenvalue $\lambda_1$. Take $\beta > 0$ such that $| \beta\varphi_1(x) | \leq t_0$ and $\beta (\max_{\Omega} \varphi_1) < a_3$. By (4), we obtain

$$\gamma - \delta \leq \frac{f_1(\beta\varphi_1)}{| \beta\varphi_1 |^{p-2} \beta\varphi_1}.$$

Choosing $\epsilon_0 > 0$ such that $\epsilon_0^2\lambda_1 < \gamma - \delta$, we have that $\beta\varphi_1$ is a lower solution of problem (3) for all $0 < \epsilon \leq \epsilon_0$.

**Step 3:** We will show that there exists a minimal (and, respectively, a maximal) weak solution $u_*$ (resp. $u^*$) for problem (3), such that $\beta\varphi_1 = \underline{u} \leq u_* \leq \overline{u} = a_3$. Consider the set

$$[\underline{u}, \overline{u}] := \{u \in L^\infty(\Omega) : \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ a.e. in } \Omega\}$$

with the topology of a.e. convergence, and define the operator $S : [\underline{u}, \overline{u}] \to L^q(\Omega)$ by

$$Sv = f_1(v) + Mv \in L^\infty(\Omega) \subset L^q(\Omega), \forall\ v \in [\underline{u}, \overline{u}],$$

where $q$ denotes the conjugate exponent to $p$, i.e., $1/p + 1/q = 1$. We get that $S$ is nondecreasing and bounded. Moreover, if $v_n, v \in [\underline{u}, \overline{u}]$, then

$$\| Sv_n - Sv \|_{L^q}^q = \int_\Omega | f_1(v_n) + Mv_n - f_1(v) - Mv |^q \, dx.$$

Let $v_n \to v$ a.e. in $\Omega$. Applying the Lebesgue dominated convergence theorem, we obtain that $\| Sv_n - Sv \|_{L^q} \to 0$, and then $S$ is continuous.

Define the operator $T : L^q(\Omega) \to W^{1,p}(\Omega),\ f \mapsto u$. $T$ is continuous and nondecreasing.

Consider the continuous nondecreasing operator $F : [\underline{u}, \overline{u}] \to W_0^{1,p}(\Omega)$ defined by $F := ToS$, i.e., for a function $v \in [\underline{u}, \overline{u}]$, $F(v)$ is the unique weak solution of problem (3).

Writing $u_1 = F(\underline{u}),\ \ u^1 = F(\overline{u})$, we obtain that for all $\varphi \in W_0^{1,p}$, $\varphi \geq 0$,

$$\begin{aligned} \int_\Omega | \nabla u_1 |^{p-2} \nabla u_1 \nabla \varphi + \int_\Omega M u_1 \varphi &= \int_\Omega (f_1(\underline{u}) + M\underline{u})\varphi \\ &\geq \int_\Omega | \nabla \underline{u} |^{p-2} \nabla \underline{u} \nabla \varphi + \int_\Omega M \underline{u} \varphi \end{aligned}$$

and

$$\begin{aligned} \int_\Omega | \nabla u^1 |^{p-2} \nabla u^1 \nabla \varphi + \int_\Omega M u^1 \varphi &= \int_\Omega (f_1(\overline{u}) + M\overline{u})\varphi \\ &\leq \int_\Omega | \nabla \overline{u} |^{p-2} \nabla \overline{u} \nabla \varphi + \int_\Omega M \overline{u} \varphi. \end{aligned}$$

Applying Lemma 2.2 of [3] and taking into account that $F$ is nondecreasing, we obtain

$$\underline{u} \leq F(\underline{u}) \leq F(u) \leq F(\overline{u}) \leq \overline{u},\ \text{a.e. in } \Omega,\ \forall\ u \in [\underline{u}, \overline{u}].$$

Repeating the same reasoning, we can prove the existence of sequences $(u^n)$ and $(u_n)$ satisfying $u^0 = \overline{u}$, $u^{n+1} = F(u^n)$, $u_0 = \underline{u}$, $u_{n+1} = F(u_n)$ and, for every weak solution $u \in [\underline{u}, \overline{u}]$ of problem (3) with $\epsilon = 1$, we have

$$\underline{u} = u_0 \leq u_1 \leq ... \leq u_n \leq u \leq u^n \leq ... \leq u^1 \leq u^0 = \overline{u}, \text{ a.e. in } \Omega.$$

Then, $u_n \to u_*$, $u^n \to u^*$, a.e. in $\Omega$, with $u_*, u^* \in [\underline{u}, \overline{u}]$, $u_* \leq u^*$ a.e. in $\Omega$. Since $u_{n+1} = F(u_n) \to F(u_*)$, and $u^{n+1} = F(u^n) \to F(u^*)$ in $W_0^{1,p}(\Omega)$ by continuity of $F$, then $u_*, u^* \in W_0^{1,p}(\Omega)$ with $u_* = F(u_*)$ and $u^* = F(u^*)$. This completes the proof. Then, $u_*$ is minimal weak solution (respectively, $u^*$ maximal weak solution) for (3) with $\epsilon = 1$ such that $u_*, u^* \in [\underline{u}, \overline{u}]$, for all $0 < \epsilon \leq \epsilon_0$. In particular, every weak solution $u \in [\underline{u}, \overline{u}]$ of (3) with $\epsilon = 1$ satisfies also $u_* \leq u \leq u^*$, a.e. in $\Omega$. Since the solutions $u_*$ and $u^*$ are between 0 and $a_3$ then $u_*$ and $u^*$ are solutions of (2). Therefore there exists a solution for the problem (2) $u_\epsilon := u_*$, for all $0 < \epsilon \leq \epsilon_0$ such that $u_\epsilon \in [\beta\varphi_1, a_3]$.

To prove the existence of the negative solution $v_\epsilon(x)$, where $a_1 \leq v_\epsilon(x) \leq 0$ it is enough to consider the truncation function $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_2(u) := \begin{cases} f(u), & \text{if} \quad a_1 \leq u \leq 0 \\ 0, & \text{if} \quad u \leq a_1 \\ 0, & \text{if} \quad u \geq 0 \end{cases}$$

and then the proof follows similarly. $\square$

## 2.2 Asymptotic Behavior of the Solutions of the Dirichlet Problem

We have the following proposition which shows the asymptotic behavior of the solutions of (2) as $\epsilon \to 0$.

**Proposition 2.2** *Let* $0 < u_\epsilon < a_3$ *be a positive solution of* (2) *and let* $a_1 < v_\epsilon < 0$ *be a negative solution of* (2). *Then*

**i)** $u_\epsilon \to a_3$ *as* $\epsilon \to 0$ *uniformly on every compact subset of* $\Omega$*;*

**ii)** $v_\epsilon \to a_1$ *as* $\epsilon \to 0$ *uniformly on every compact subset of* $\Omega$*.*

**Proof:** **i)** The proof follows by adapting some arguments from Theorem 4 in [5].

First, observe that there exists $\alpha \in (0,1)$ such that $u_\epsilon \in C^{1,\alpha}(\Omega)$, by using either Theorem 1 in [11] or Theorem 2 in [9].

Consider the function $f_1 : \mathbb{R} \to \mathbb{R}$ defined at proof of the Proposition 2.1, and $\varphi_1 > 0$ an eigenfunction corresponding to the first eigenvalue $\lambda_1$ of $-\triangle_p$ in $\Omega$ subject to Dirichlet boundary conditions. From the Maximum Principle of Vásquez ([[12], Thm 5]), we have $u_\epsilon > 0$ in $\Omega$, $\frac{\partial u_\epsilon}{\partial \nu} < 0$ on $\partial\Omega$ and $\frac{\partial \varphi_1}{\partial \nu} < 0$ on $\partial\Omega$.

Consequently, there exist $\beta > 0$ such that for all $0 < \epsilon \leq \epsilon_0$, we have $u_\epsilon(x) \geq \beta\varphi_1$ and, for a given $\eta > 0$ there is $C_\eta$ such that

$$u_\epsilon(x) \geq C_\eta > 0, \tag{5}$$

for all $x \in \Omega_\eta := \{x \in \Omega : dist(x, \partial\Omega) > \eta\}$. Take $\varphi_1$ such that $\| \varphi_1 \| = 1$. Since $u_\epsilon$ is solution of (3) it follows that

$$\epsilon^2 \int_\Omega \mid \nabla u_\epsilon \mid^{p-2} \nabla u_\epsilon \nabla \varphi dx = \int_\Omega f_1(u_\epsilon)\varphi dx, \ \forall \ \varphi \geq 0, \ \varphi \in W_0^{1,p}(\Omega). \tag{6}$$

In particular, for $\varphi = \varphi_1$ we have

$$\epsilon^2 \int_\Omega \mid \nabla u_\epsilon \mid^{p-2} \nabla u_\epsilon \nabla \varphi_1 dx = \int_\Omega f_1(u_\epsilon)\varphi_1 dx. \tag{7}$$

**Claim**: The expression in the left-hand side of (7) goes to zero as $\epsilon \to 0$. In fact, observe that $0 < u_\epsilon \leq a_3$ and $f_1(u_\epsilon) \leq \widetilde{C}$. Thus, using Hölder inequality and (6) with $\varphi = u_\epsilon(x)$ we obtain

$$\begin{aligned} \epsilon^2 \int_\Omega \mid \nabla u_\epsilon \mid^{p-2} \nabla u_\epsilon \nabla \varphi_1 dx &\leq C\epsilon^2 \left( \frac{1}{\epsilon^2} \int_\Omega f_1(u_\epsilon) u_\epsilon dx \right)^{\frac{(p-1)}{p}} \\ &\leq \widehat{C}\epsilon^{\frac{2}{p}} \to 0 \ \text{ as } \ \epsilon \to 0, \ \text{for some constant } \widehat{C}. \end{aligned}$$

Define $d_\eta := inf\{\varphi_1(x) : x \in \Omega_\eta\} > 0$. Then,

$$d_\eta \int_{\Omega_\eta} f_1(u_\epsilon) dx \leq \int_{\Omega_\eta} f_1(u_\epsilon)\varphi_1 dx < \int_\Omega f_1(u_\epsilon)\varphi_1 \to 0, \ \epsilon \to 0. \tag{8}$$

Now suppose by contradiction that there are a number $C_1 > 0$ and a sequence $\epsilon_j \to 0$ such that the Lebesgue's measure of the sets

$$\Omega_{\eta,j} := \{x \in \Omega_\eta : u_{\epsilon_j}(x) < a_3 - \eta\} \tag{9}$$

are bounded from below by $C_1$. It follows from (8) that

$$I_j := \int_{\Omega_{\eta,j}} f_1(u_{\epsilon_j}) dx \to 0, \ \text{as } \epsilon_j \to 0. \tag{10}$$

Observe that in $\Omega_{\eta,j}$, from (5) and (9), we have $C_\eta \leq u_{\epsilon_j} \leq a_3 - \eta$.

Since $f_1$ is bounded from below in the interval $[C_\eta, a_3 - \eta]$ by a number $d > 0$, from (9) it follows

$$I_j = \int_{\Omega_{\eta,j}} f_1(u_{\epsilon_j})dx \geq d \mid \Omega_{\eta,j} \mid \geq dC_1,$$

which contradicts (10). Therefore, $\mid \Omega_{\eta,j} \mid$ is not bounded from below, i.e., $u_\epsilon(x) \to a_3$, on every compact subset of $\Omega$ as $\epsilon \to 0$.

**ii)** It follows similarly as in the previous case i). □

**Proof of Theorem 2.1:** The proof follows directly from Propositions 2.1 and 2.2. □

# 3 Proof of Theorem 1.1

The proof is done by using a version of the Mountain Pass Theorem, due a Hofer [7], and shows the existence of critical points of the mountain pass type.

## 3.1 A Particular Case

First, we prove a particular case of Theorem 1.1, that is case when $f$ has only three zeros.

**Theorem 3.1** *Let be $f$ satisfying the assumptions $(f_1)$, $(f_2)$ and $(f_3)$ for $l = 1$. Then there exists $\epsilon_0 > 0$ so that, for all $0 < \epsilon < \epsilon_0$, there is at least one nonconstant solution $u_\epsilon$ for the problem (1) verifying $a_1 < u_\epsilon(x) < a_3$.*

In order to prove Theorem 3.1, we use the two lemmas below.

**Lemma 3.1** *Let $f$ satisfy the assumptions $(f_1)$, $(f_2)$ and $(f_3)$ for $l = 1$. Then there exist functions of class $C^1$, $f_1 : (-\infty, a_1] \to \mathbb{R}^+$, $f_2 : [a_3, +\infty) \to \mathbb{R}^-$ and real numbers $\eta_1$ and $\beta_1$ such that:*

**(i)** *$f_1(a_1) = f(a_1)$, $f_1'(a_1) = f'(a_1)$ and $f_1(t) > 0$ for all $t \in (\eta_1, a_1)$;*

**(ii)** *$f_2(a_3) = f(a_3)$, $f_2'(a_3) = f'(a_3)$ and $f_2(t) < 0$ for all $t \in (a_3, \beta_1)$;*

**(iii)** *$\eta_1$ and $\beta_1$ are so that $\eta_1 < a_1 < a_3 < \beta_1$,*

$$\int_{\eta_1}^{a_1} f_1(t)dt = \left|\int_{a_1}^{0} f(t)dt\right|, \qquad \left|\int_{a_3}^{\beta_1} f_2(t)dt\right| = \int_0^{a_3} f(t)dt$$

*and for all $t \in [0, 1]$, we have*

$$\eta_1 < t(1-t)a_3 + (a_3 - a_1)t + a_1 < \beta_1$$

*and*

$$\eta_1 < t(1-t)a_1 + (a_3 - a_1)t + a_1 < \beta_1.$$

**Proof:** We start proving the existence of $\eta_1$.

We take

$$\alpha(t) := t(1-t)a_1 + (a_3 - a_1)t + a_1.$$

Thus, $0 = \frac{d}{dt}(\alpha(t)) = -2ta_1 + a_3$ if, and only if, $t = \frac{a_3}{2a_1}$. Moreover, since $a_1 < 0$ and $\frac{d^2}{dt}\alpha(t) = -2a_1$ it follows that $\alpha(\frac{a_3}{2a_1}) = \frac{a_3^2}{4a_1} + a_1$ is the minimal value of $\alpha(t)$. Now we define

$$\eta_1 := \frac{a_3^2}{4a_1} + 2a_1.$$

To prove the existence of $\beta_1$, we take

$$\beta(t) := t(1-t)a_3 + (a_3 - a_1)t + a_1.$$

We note that $0 = \frac{d}{dt}(\beta(t)) = (1-2t)a_3 + a_3 - a_1$ if, and only if, $t = 1 - \frac{a_1}{2a_3}$. Since $\frac{d^2}{dt}\beta(t) = -2a_3$ and $a_3 > 0$, we have $\beta(1 - \frac{a_1}{2a_3}) = \frac{a_1^2}{4a_3} + a_3$ is maximal value of $\beta$. Now, we take

$$\beta_1 := \frac{a_1^2}{4a_3} + 2a_3.$$

Now, we are going to prove the existence of the function $f_1$.

We take $g(t) = f'(a_1)(t - a_1)$ and $\xi_1, \xi_2$ functions of class $C^1$ so that $\xi_1 \equiv 1$ at a neighbourhood of $a_1$, $\xi_2 \equiv 1$ at a neighbourhood of $\eta_1$, $\xi_1(t) + \xi_2(t) = 1$ for all $t \in [\eta_1, a_1]$ and

$$\int_{\eta_1}^{a_1} g(t)\xi_1(t)dt < \left|\int_{a_1}^{0} f(t)dt\right|.$$

Now, we choose $r > 0$ such that

$$\int_{\eta_1}^{a_1} [r\xi_2(t)g(t) + \xi_1(t)g(t)]dt = \left|\int_{a_1}^{0} f(t)dt\right|.$$

We define $f_1 : (-\infty, a_1] \to \mathbb{R}^+$ by

$$f_1(t) := \begin{cases} r\xi_2(t)g(t) + \xi_1(t)g(t), & \eta_1 \le t \le a_1 \\ rf'(a_1)t - ra_1 f'(a_1), & t \le \eta_1. \end{cases}$$

We note that for $t \le \eta_1$ the graph of $f_1$ is the tangent line to $f_1$ at the point $(\eta_1, f_1(\eta_1))$.

Finally, we prove the existence of $f_2$.

We take $g(t) = f'(a_3)(t - a_3)$ and $\xi_1, \xi_2$ functions of class $C^1$ so that $\xi_1 \equiv 1$ at a neighbourhood of $a_3$, $\xi_2 \equiv 1$ at a neighbourhood of $\beta_1$, $\xi_1(t) + \xi_2(t) = 1$ for all $t \in [a_3, \beta_1]$ and

Now, we choose $r > 0$ such that

$$\left|\int_{a_3}^{\beta_1} [r\xi_2(t)g(t) + \xi_1(t)g(t)]dt\right| = \int_{0}^{a_3} f(t)dt.$$

We define $f_2 : [a_3, +\infty) \to \mathbb{R}^-$ by

$$f_2(t) := \begin{cases} r\xi_2(t)g(t) + \xi_1(t)g(t), & a_3 \le t \le \beta_1 \\ rf'(a_1)t - ra_1 f'(a_1), & t \ge \beta_1. \end{cases}$$

We observe that for $t \ge \beta_1$ the graph of $f_2$ is the tangent line to $f_2$ at the point $(\beta_1, f_2(\beta_1))$.

Thus the proof of Lemma is concluded. $\square$

As consequence of the Maximum Principle we have the following lemma.

**Lemma 3.2** *If $u_\epsilon$ is a nonconstant solution of the Neumann problem*

$$\begin{cases} -\epsilon^2 \triangle_p u &= \widehat{f}(u), \quad x \in \Omega \\ \frac{\partial u}{\partial \nu} &= 0, \qquad x \in \partial\Omega, \end{cases} \tag{11}$$

*where $\widehat{f} : \mathbb{R} \to \mathbb{R}$ is the function defined by*

$$\widehat{f}(t) := \begin{cases} f(t), & a_1 \le t \le a_3 \\ f_1(t), & t \le a_1 \\ f_2(t), & t \ge a_3, \end{cases}$$

*where $f_1$ and $f_2$ are defined in Lemma 3.1. Then $a_1 \le u_\epsilon \le a_3$.*

**Proof:** We start proving that $u_\epsilon \le a_3$. In fact, let

$$v(x) := \begin{cases} u_\epsilon(x) - a_3, & \text{if} \quad u_\epsilon(x) \ge a_3 \\ 0, & \text{if} \quad u_\epsilon(x) < a_3 \end{cases}$$

and $\Omega_+ := \{x \in \Omega : u_\epsilon(x) \ge a_3\}$.

Notice that

$$\epsilon^2 \int_\Omega |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon \nabla v dx = \int_\Omega \widehat{f}(u_\epsilon) v dx = \int_{\Omega_+} f_2(u_\epsilon) v dx \le 0.$$

So,

$$\int_\Omega |\nabla v|^p \leq 0.$$

Thus, it follows that $|\nabla v| = 0$. Since $u_\epsilon$ is nonconstant, there exists $x \in \Omega$ where $u_\epsilon(x) < a_3$. So that, $v \equiv 0$. Therefore, $u_\epsilon(x) \leq a_3$, for all $x \in \Omega$.

Similarly, we have $a_1 \leq u_\epsilon$, by taking

$$w(x) := \begin{cases} u_\epsilon(x) - a_1, & u_\epsilon(x) \leq a_1 \\ 0, & u_\epsilon(x) > a_1 \end{cases}$$

and $\Omega_- := \{x \in \Omega : u_\epsilon(x) \leq a_1\}$.

This finishes the proof of the Lemma. □

Thus, by Lemma 3.2, it follows that a solution for problem (11) is a solution for problem (1), since $f \equiv \widehat{f}$ in $[a_1, a_3]$. Therefore, to prove Theorem 3.1 it suffices to prove the following theorem.

**Theorem 3.2** *Let $f$ satisfy the conditions $(f_1)$, $(f_2)$ and $(f_3)$ for $l = 1$. Then there exists $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$, problem (11) has at least one nonconstant solution $u_\epsilon$.*

We consider the functional $J_\epsilon : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J_\epsilon(u) := \frac{\epsilon^2}{p} \int_\Omega |\nabla u|^p dx - \int_\Omega \widehat{F}(u) dx,$$

where $\widehat{F}(u) := \displaystyle\int_0^u \widehat{f}(t) dt$.

By Sobolev imbedding this functional is well defined for $u \in W^{1,p}(\Omega)$. Moreover, $J_\epsilon \in C^1(W^{1,p}(\Omega), \mathbb{R})$ with the derivative given by

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx - \int_\Omega \widehat{f}(u) \varphi dx, \quad \forall\, \varphi \in W^{1,p}(\Omega).$$

Thus, the critical points of $J_\epsilon$ are weak solutions of (11).

**Lemma 3.3** *The functional $J_\epsilon$ satisfies the Palais-Smale condition.*

**Proof:** Let $(u_n) \subset W^{1,p}(\Omega)$ be a sequence satisfing $J_\epsilon(u_n) \to c$ and $J'_\epsilon(u_n) \to 0$, as $n \to \infty$. Hence

$$(12) \qquad \mid J_\epsilon(u_n) \mid = \left| \frac{\epsilon^2}{p} \int_\Omega |\nabla u_n|^p \, dx - \int_\Omega \widehat{F}(u_n) dx \right| \leqslant d \text{ for some } d > 0,$$

and

$$(13) \qquad \mid J'_\epsilon(u_n)v \mid = \left| \epsilon^2 \int_\Omega \mid \nabla u_n \mid^{p-2} \nabla u_n \nabla v dx - \int_\Omega \widehat{f}(u_n) v dx \right| \leq \delta_n \|v\|,$$

for all $v \in W^{1,p}(\Omega)$ where $\delta_n \to 0$, $n \to \infty$.

**Claim:** $(u_n)$ is bounded. In fact, define

$$v_n(x) = \begin{cases} u_n(x) - \beta_1, & \text{if } u_n(x) \geq \beta_1, \\ 0, & \text{if } u_n(x) < \beta_1, \end{cases}$$

and $\Omega_1 := \{x \in \Omega : u_n(x) \geq \beta_1\}$.

By (13) we have

$$\left| \epsilon^2 \int_\Omega |\nabla v_n|^p dx - \int_\Omega \widehat{f}(u_n) v_n dx \right| \leq \delta_n \|v_n\|.$$

Since $\widehat{f}(u_n) \leq -C$ in $\Omega_1$, it follows

$$(14) \qquad \epsilon^2 \int_\Omega |\nabla v_n|^p dx + C \int_\Omega v_n dx \leq \delta_n \|v_n\|$$

$$(15) \qquad C \int_\Omega v_n dx \leq \delta_n \|v_n\|.$$

Define $w_n := \frac{v_n}{\|v_n\|}$. Hence, $\|w_n\| = 1$. Taking subsequence if necessary we can assume that $w_n \rightharpoonup w$ weakly in $W^{1,p}(\Omega)$, $w_n \to w$ in $L^p(\Omega)$ and $w_n(x) \to$

$w(x)$ a.e. in $\Omega$. Dividing (15) by $\|v_n\|$, we have

$$C\int_\Omega w_n dx \le \delta_n.$$

Taking limits of both sides and observing that $w \ge 0$, we have $w \equiv 0$. Dividing (14) by $\|v_n\|$ and taking the limit as $n \to \infty$, we conclude that

$$\epsilon^2 \int_\Omega \frac{|\nabla v_n|^p}{\|v_n\|} \to 0. \tag{16}$$

On the other hand,

$$1 = \|w_n\| = \epsilon^2 \int_\Omega |\nabla w_n|^p dx + \int_\Omega |w_n|^p.$$

Thus,

$$\epsilon^2 \int_\Omega |\nabla w_n|^p dx \to 1. \tag{17}$$

Multiplying and dividing (16) by $\|v_n\|^{p-1}$, we obtain

$$\|v_n\|^{p-1}\epsilon^2 \int_\Omega \left|\nabla\left(\frac{v_n}{\|v_n\|}\right)\right|^p dx = \|v_n\|^{p-1}\epsilon^2 \int_\Omega |\nabla w_n|^p dx \to 0.$$

Also, from (17) it follows that $\|v_n\| \to 0$ as $n \to \infty$. Consequently, $\|v_n\| \le C$.

Next, define

$$z_n(x) := \begin{cases} u_n(x), & \text{if } \eta_1 \le u_n(x) \le \beta_1 \\ 0, & \text{if } \eta_1 \ge u_n(x) \\ 0, & \text{if } u_n(x) \ge \beta_1 \end{cases}$$

and $\Omega_2 := \{x \in \Omega : \eta_1 \le u_n(x) \le \beta_1\}$.

By a similar argument, we prove that $\|z_n\| \le C$.

Finally, we consider the sequence

$$r_n(x) := \begin{cases} u_n(x) - \eta_1, & \text{if } u_n(x) \le \eta_1 \\ 0, & \text{if } u_n(x) > \eta_1 \end{cases}$$

and the set $\Omega_3 := \{x \in \Omega : u_n(x) \leq \eta_1 < 0\}$.

Similarly, as in the previous case, we prove that $\|r_n\| \leq C$.

Let $u_n := v_n + z_n + r_n$. Then, we conclude that $(u_n)$ is bounded in $\Omega$. Also, using the Boccardo-Murat convergence lemma in [1], we obtain that $J_\epsilon$ satisfies the (PS) condition. □

**Lemma 3.4** *The functional $J_\epsilon$ is lower bounded.*

**Proof:** Let $\Omega_1 := \Omega_3 \bigcup \Omega_4$ and $\Omega_2 := \Omega_5 \bigcup \Omega_6$, with

$$\Omega_3 := \{x \in \Omega : 0 \leq u(x) \leq a_3\}, \quad \Omega_4 := \{x \in \Omega : a_3 < u(x) < \infty\},$$

$$\Omega_5 := \{x \in \Omega : a_1 \leq u(x) \leq 0\}, \quad \Omega_6 := \{x \in \Omega : -\infty < u(x) < a_1\}.$$

Since, $u$ is bounded in $\Omega_3$ and in $\Omega_5$, we have

$$\int_{\Omega_1} \widehat{F}(u)dx = \int_{\Omega_3} \widehat{F}(u)dx + \int_{\Omega_4} \widehat{F}(u)dx < \widetilde{C} + \int_{\Omega_4} \widehat{F}(a_3)dx \leq C$$

and

$$\int_{\Omega_2} \widehat{F}(u)dx = \int_{\Omega_5} \widehat{F}(u)dx + \int_{\Omega_6} \widehat{F}(u)dx < \overline{C} + \int_{\Omega_6} \widehat{F}(a_1)dx < C.$$

Also,

$$\int_{\Omega} \widehat{F}(u)dx = \int_{\Omega_1} \widehat{F}(u)dx + \int_{\Omega_2} \widehat{F}(u)dx < C.$$

Hence,

$$J_\epsilon(u) \quad \geq \quad \tfrac{\epsilon^2}{p} \int_{\Omega} |\nabla u|^p dx - C \geq -C.$$

Therefore, $J_\epsilon$ is lower bounded. □

**Lemma 3.5** *$a_i$, with $i$ odd, is strict local minimum of $J_\epsilon$ in $W^{1,p}$.*

**Proof:** First of all, we will prove that $J_\epsilon$ has a local minimum in $C^1$ at $a_i$. Let $\delta > 0$ such that $\widehat{F}(a_i) \geq \widehat{F}(t)$, for $|t - a_i| < \delta$, and let $u \in C^1$ so that $\|u(x) - a_i\|_{C^1} = \max\{|u(x) - a_i|, |u'(x) - a_i|\} \leq \delta$. We claim that there exists $\eta > 0$ such that $\Omega_\eta := \{x \in \Omega : |u(x) - a_i| > \eta\}$ has positive measure. In fact, let $u \in C^1(\Omega)$, $u \not\equiv a_i$. Then, there exists $x_0 \in \Omega$ such that $u(x_0) \neq a_i$. Hence, there exists $\eta > 0$ such that either $u(x_0) > a_i + \eta$ or $u(x_0) < a_i - \eta$.

Since $u$ is continuous there exists a ball $B_{\tilde{\delta}}(x_0)$ such that $|u(x) - a_i| > \eta$, for all $x \in B_{\tilde{\delta}}(x_0)$. Therefore, $\Omega_\eta$ has positive measure.

Now, we define $c_1 := max\{\widehat{F}(a_i - \eta), \widehat{F}(a_i + \eta)\}$. Since, $a_i$ a strict local maximum of $\widehat{F}$, we have

$$\begin{aligned}\int_\Omega \widehat{F}(u)dx &\leq \int_{\Omega_\eta} c_1 dx + \int_{\Omega\setminus\Omega_\eta} \widehat{F}(a_i)dx \\ &< \widehat{F}(a_i)\int_{\Omega_\eta} dx + \int_{\Omega\setminus\Omega_\eta} \widehat{F}(a_i)dx = \int_\Omega \widehat{F}(a_i)dx.\end{aligned}$$

Therefore,

$$J_\epsilon(u) \geq -\int_\Omega \widehat{F}(u)dx > -\int_\Omega \widehat{F}(a_i)dx = J_\epsilon(a_i),$$

i.e., $a_i$ is a strict local minimum of $J_\epsilon$ in $C^1$.

Therefore, by Teorema 1.2 in [6] $a_i$ is local minimum of $J_\epsilon$ in $W^{1,p}(\Omega)$.

Without loss of generality, we suppose that $a_i$ is strict local minimum of $J_\epsilon$ in $W^{1,p}(\Omega)$. On the contrary case, for all $\delta > 0$, there exist $v_\delta \in W^{1,p}(\Omega)$ such that $J_\epsilon(v_\delta) = J_\epsilon(a_i)$. So, $v_\delta$ is a critical point of $J_\epsilon$ in $W^{1,p}(\Omega)$. □

We will use arguments of [4] in order to obtain the next lemma.

**Lemma 3.6** *If $a$ is a strict local minimum of $J_\epsilon$, i.e.,*

$$J_\epsilon(a) < J_\epsilon(u) \tag{18}$$

*for all* $u \in W^{1,p}(\Omega)$ *such that* $0 < \|u - a\| < \delta_0$ *for some* $\delta_0 > 0$. *Then, for any* $0 < \alpha < \delta_0$,

$$\inf\{J_\epsilon(u) : u \in W^{1,p}(\Omega) \text{ and } \|u - a\| = \alpha\} > J_\epsilon(a). \tag{19}$$

**Proof:** Assume by contradiction that the infimum in (19) is equal to $J_\epsilon(a)$ for some $\alpha$ with $0 < \alpha < \delta_0$. So there exists a sequence $u_n \in W^{1,p}(\Omega)$ with $\|u_n - a\| = \alpha$ and, say, $J_\epsilon(u_n) \leq J_\epsilon(a) + \frac{1}{2n^2}$. Call

$$A := \{u \in W^{1,p}(\Omega) : \alpha - \delta \leq \|u - a\| \leq \alpha + \delta\},$$

where $\delta > 0$ is chosen so that $0 < \alpha - \delta$ and $\alpha + \delta < \delta_0$. In view of our contradiction hypothesis an (18), it follows that $inf\{J_\epsilon(u) : u \in A\} = J_\epsilon(a)$.

We now apply the Ekeland Variational principle to the functional $J_\epsilon$ on $A$ in order to get the existence of $v_n \in A$ such that

$$J_\epsilon(v_n) \leq J_\epsilon(u_n), \tag{20}$$

$$\|v_n - u_n\| \leq \frac{1}{n}, \tag{21}$$

$$J_\epsilon(v_n) \leq J_\epsilon(u) + \frac{1}{n}\|u - v_n\|, \quad \forall \; u \in A. \tag{22}$$

Our purpose is to show that $v_n$ is a (PS) sequence for $J_\epsilon$ in $W^{1,p}(\Omega)$, i.e., $J_\epsilon(v_n) \leq C$ (which is obvious by (20)) and $J'_\epsilon(v_n) \to 0$, as $n \to \infty$.

Once this is proved, we get, that $v_n$ has a convergent subsequence. Denote this subsequence by $v_n$ we have that $v_n \to v$ in $W^{1,p}(\Omega)$. Notice that $v \in A$, since $A$ is complete. Hence, $v \in W^{1,p}(\Omega)$ and therefore it satisfies $\|v - a\| = \alpha$ and $J_\epsilon(v) = J_\epsilon(a)$, which contradicts (18).

So in order to conclude the proof of Lemma 3.6 we will prove that $J'_\epsilon(v_n) \to 0$, as $n \to \infty$. Take $w \in W^{1,p}(\Omega)$ and consider $u_t := v_n + tw$, for $n > 1/\delta$. We observe that for $|t|$ sufficiently small, $u_t = v_n + tw \in A$. Indeed

$$\lim_{t\to 0} \|u_t - a\| = \|v_n - a\| \leq \|v_n - u_n\| + \|u_n - a\| \leq \frac{1}{n} + \alpha < \delta + \alpha.$$

On the other hand,

$$\|v_n - a\| \geq \|a - u_n\| - \|u_n - v_n\| \geq \alpha - \frac{1}{n} > \alpha - \delta.$$

Also, we can take $u = u_t$ in (22), and then, for $t > 0$,

$$\frac{J_\epsilon(v_n) - J_\epsilon(v_n + tw)}{t} \leq \frac{1}{n}\frac{1}{t}\|v_n - tw - v_n\| \leq \frac{1}{nt}\|tw\|. \tag{23}$$

Taking the limit in (23) as $t \to 0$, we obtain $\langle J'_\epsilon(v_n), w\rangle \leq \frac{1}{n}\|w\|$. Consequently,

$$|\langle J'_\epsilon(v_n), w\rangle| \leq \frac{1}{n}\|w\|, \quad \forall\, w \in W^{1,p}(\Omega).$$

So, $J'_\epsilon(v_n) \to 0$, as $n \to \infty$ and the proof of the lemma is concluded. □

**Proof of Theorem 3.2:** We define

$$\Gamma := \{h \in C([0,1], W^{1,p}(\Omega)) : h(0) = a_1 \text{ e } h(1) = a_3\}$$

and

$$\gamma_\epsilon := \inf_{h\in\Gamma} \max_{t\in[0,1]} J_\epsilon(h(t)).$$

By Lemma 3.6 $\gamma_\epsilon := \inf_{h\in\Gamma} \max_{t\in[0,1]} J_\epsilon(h(t)) > c = max\{J_\epsilon(a_1), J_\epsilon(a_3)\}$. Since $J_\epsilon$ satisfies the condition (PS), using the Mountain Pass Theorem by [7] it follows that there exist $\overline{u}$ critical point of $J_\epsilon$ such that $J_\epsilon(\overline{u}) = \gamma_\epsilon$. If the critical points are not isolated in $W^{1,p}(\Omega)$ then there exist an infinite number of critical points of $J_\epsilon$. Otherwise, $\overline{u}$ is of the mountain-pass type, see [7]. Since $a_1$ and $a_3$ are strict local minimum then $\overline{u} \neq a_1$ and $\overline{u} \neq a_3$. Therefore, in order to show the existence of a nonconstant critical point of $J_\epsilon$, we only need to prove that $\gamma_\epsilon < 0$, since $J_\epsilon(0) = 0$.

We claim that $\gamma_\epsilon < 0$. In fact, we consider $B \subset \Omega$ an open ball in $\Omega$ and we define

$$u_0(x) := \begin{cases} v_\epsilon(x), & x \in \overline{B} \\ w_\epsilon(x), & x \in \overline{\Omega} \backslash \overline{B}, \end{cases}$$

where $v_\epsilon$ is the positive solution for the Dirichlet problem (2) in $\overline{B}$ and $w_\epsilon$ is the negative solution for the Dirichlet problem (2) in $(\Omega \backslash \overline{B}) \bigcup \partial(\Omega \backslash \overline{B})$.

Since $\Omega \in C^1$ and the functions $v_\epsilon, w_\epsilon \in L^p(\Omega)$, it follows by Proposition $IX.18$ of [2] that $u_0 \in W^{1,p}(\Omega)$.

Now, given $\epsilon > 0$ we consider the special path

$$h_\epsilon(t) := t(1-t)u_0(x) + (a_3 - a_1)t + a_1, \quad \text{in} \ \ \Gamma.$$

Then we claim that there exists a small number $\epsilon_0 > 0$, so that, for all $0 < \epsilon < \epsilon_0$ and for all $t \in [0,1]$ $\max_{t \geq 0} J_\epsilon(h_\epsilon(t)) < 0$. In fact, suppose by contradiction that there is no such a $\epsilon_0 > 0$. Then for any $\epsilon_0 > 0$ there is a $0 < \epsilon < \epsilon_0$ so that

$$J_\epsilon(h_\epsilon(t_\epsilon)) \geq 0, \tag{24}$$

for some $t_\epsilon \in [0,1]$. Then choose a sequence $(\epsilon_k)$ so that

$$\lim_{k \to \infty} \epsilon_k = 0, \quad J_\epsilon(h_{\epsilon_k}(t_{\epsilon_k})) \geq 0, \quad 0 \leq \lim_{k \to 0} t_{\epsilon_k} = \alpha \leq 1.$$

For simplicity we drop the indices and write $\epsilon_k = \epsilon$ and $t_{\epsilon_k} = t$. So

$$\begin{aligned} J_\epsilon(h_\epsilon(t)) &= \tfrac{\epsilon^2}{p}\int_B |\nabla h_\epsilon(t)|^p dx + \frac{\epsilon^2}{p}\int_{\Omega\setminus B} |\nabla h_\epsilon(t)|^p dx - \int_\Omega \widehat{F}(h_\epsilon(t))dx \\ &= \tfrac{\epsilon^2}{p} t^p(1-t)^p\Big(\int_B |\nabla v_\epsilon|^p dx + \int_{\Omega\setminus B} |\nabla w_\epsilon|^p dx\Big) \\ &\quad - \int_\Omega \widehat{F}(t(1-t)u_0 + (a_3 - a_1)t + a_1)dx. \end{aligned}$$

Since

$$\epsilon^2 \int_B |\nabla v_\epsilon|^p dx = \int_B \widehat{f}(v_\epsilon)v_\epsilon dx \quad \text{and} \quad \epsilon^2 \int_{\Omega\setminus B} |\nabla w_\epsilon|^p dx = \int_{\Omega\setminus B} \widehat{f}(w_\epsilon)w_\epsilon dx$$

we get

$$\begin{aligned} J_\epsilon(h_\epsilon(t)) &= \tfrac{t^p(1-t)^p}{p}\Big[\int_B \widehat{f}(v_\epsilon)v_\epsilon dx + \int_{\Omega\setminus B} \widehat{f}(w_\epsilon)w_\epsilon dx\Big] \\ &\quad - \int_\Omega \widehat{F}(t(1-t)u_0 + (a_3 - a_1)t + a_1)dx. \end{aligned} \tag{25}$$

Since $v_\epsilon \to a_3$ as $\epsilon \to 0$ uniformly on every compact subset of $B$ and $w_\epsilon \to a_1$ as $\epsilon \to 0$ uniformly on every compact subset of $\Omega\setminus B$, so

$$\lim_{\epsilon\to 0} \int_B \widehat{f}(v_\epsilon)v_\epsilon dx = 0 \quad \text{and} \quad \lim_{\epsilon\to 0} \int_{\Omega\setminus B} \widehat{f}(w_\epsilon)w_\epsilon dx = 0.$$

If we take the limit on both sides of (25), it follows from the dominated convergence theorem and the above facts that

$$\begin{aligned} \lim_{\epsilon\to 0} J_\epsilon(h_\epsilon(t)) &= \lim_{\epsilon\to 0}\Big[-\int_\Omega \widehat{F}(t(1-t)u_0(x,\epsilon) + (a_3 - a_1)t + a_1)dx\Big] \\ &= -\widehat{F}(\alpha(1-\alpha)a_3 + (a_3 - a_1)\alpha + a_1)|B| \\ &\quad -\widehat{F}(\alpha(1-\alpha)a_1 + (a_3 - a_1)\alpha + a_1)|\Omega\setminus B|, \end{aligned}$$

where $|A|$ is the Lebesgue measure of $A \subset \mathbb{R}^N$. It follows from Lemma 3.1that

$$-\widehat{F}(\alpha(1-\alpha)a_3 + (a_3 - a_1)\alpha + a_1) \le 0 \tag{26}$$

and

$$-\widehat{F}(\alpha(1-\alpha)a_1 + (a_3 - a_1)\alpha + a_1) \le 0. \tag{27}$$

In fact we have the strict inequalities below

$-\widehat{F}(\alpha(1-\alpha)a_3+(a_3-a_1)\alpha+a_1) < 0$ or $-\widehat{F}(\alpha(1-\alpha)a_1+(a_3-a_1)\alpha+a_1) < 0$.

Indeed, since

$$\eta_1 < \alpha(1-\alpha)a_3 + (a_3 - a_1)\alpha + a_1 < \beta_1,$$

$$\eta_1 < \alpha(1-\alpha)a_1 + (a_3 - a_1)\alpha + a_1 < \beta_1$$

and $\widehat{F}$ is zero only at $\eta_1$, 0 and $\beta_1$, we conclude that if (26) and (27) are zero, then

$$\alpha(1-\alpha)a_3 + (a_3 - a_1)\alpha + a_1 = 0 = \alpha(1-\alpha)a_1 + (a_3 - a_1)\alpha + a_1,$$

i.e., $\alpha(1-\alpha)a_3 = \alpha(1-\alpha)a_1$. Since $a_3 \neq a_1$, then $\alpha(1-\alpha) = 0$, i.e., either $\alpha = 0$ or $\alpha = 1$. Hence either $\widehat{F}(a_1) = 0$ or $\widehat{F}(a_3) = 0$. Then either $a_1 = 0$ or $a_3 = 0$, which is impossible. Hence, $\lim_{\epsilon\to 0} J_\epsilon(h_\epsilon(t)) < 0$, which is in contradiction with (24). So we get finally that $\gamma_\epsilon < 0$. □

**Corollary 3.1** *Let $f$ be satisfying $(f_1)$, $(f_2)$ and $(f_3)$ for $l = 1$ and let $u_\epsilon$ the nonconstant solution of* (1) *such that $J_\epsilon(u_\epsilon) = \gamma_\epsilon$. Then*

$$\lim_{\epsilon\to 0} sup J_\epsilon(u_\epsilon) < 0.$$

**Proof:** From the proof of Theorem 3.2 there exists a number $\epsilon_1$ positive such that $J_\epsilon(h_{\epsilon_1}(t)) < 0$ for all $t \in [0, 1]$, where

$$h_{\epsilon_1}(t) = t(1-t)u_0(\epsilon_1) + (a_3 - a_1)t + a_1.$$

Hence, for all $0 < \epsilon < \epsilon_1$, $J_\epsilon(h_{\epsilon_1}(t)) \leq J_{\epsilon_1}(h_{\epsilon_1}(t)) < 0$ for every $t \in [0,1]$. It follows from the intermediate value theorem that there exists $t_\epsilon \in [0,1]$ such that $h_{\epsilon_1}(t_\epsilon) = u_\epsilon(x)$, since, $h_{\epsilon_1}(0) = a_1$, $h_{\epsilon_1}(1) = a_3$ and $a_1 \leq u_\epsilon(x) \leq a_3$. So $J_\epsilon(h_{\epsilon_1}(t_\epsilon)) = J_\epsilon(u_\epsilon(x))$, for some $t_\epsilon \in [0,1]$, and so the assertion is true. □

**Proof of Theorem 3.1:** The proof follows directly from the Theorem 3.2 e Lemma 3.2. □

**Proof of Theorem 1.1:** For each $l = 1, 2, ....$ we consider the function $\widetilde{f} : [a_{2l-1} - a_{2l}, a_{2l+1} - a_{2l}] \to \mathbb{R}$ of class $C^1$ defined by $\widetilde{f}(t) := f(t + a_{2l})$. Then from Theorem 3.1, the problem

$$\left\{ \begin{array}{rcll} -\epsilon^2 \triangle_p v & = & \widetilde{f}(v), & x \in \Omega \\ \frac{\partial v}{\partial \nu} & = & 0, & x \in \partial\Omega \end{array} \right. \tag{28}$$

has a nonconstant solution $v_l(x)$ such that $a_{2l-1} - a_{2l} < v_l(x) < a_{2l+1} - a_{2l}$, where $v = u - a_{2l}$. So, $u_l = v_l + a_{2l}$ is a nonconstant solution for the problem (1) with $a_{2l-1} < u_l < a_{2l+1}$.

Hence, there exists at least $l$ nonconstants solutions for the problem (1) satisfy $a_1 < u_1(x) < a_3 < u_2(x) < a_5 < ... < a_{2l-1} < u_l(x) < a_{2l+1}$. □

# 4 Asymptotic Behavior of the Solutions of the Neumann Problem

We prove Theorem 1.2 which shows the limiting behavior of the nonconstant solutions of (1).

**Proof of Theorem 1.2:** Let $0 < \delta < a_3$. We shall prove by contradiction. Suppose that $\lim_{\epsilon \to 0} w^*(\epsilon, \delta) \neq 0$. Then, there is a convergent sequence $\{w^*(\epsilon_k, \delta)\}$ so that $\lim_{k\to\infty} w^*(\epsilon_k, \delta) = \alpha_\delta > 0$. This means that, for each $\epsilon_k > 0$, there is $x_k = x(\epsilon_k, \delta) \in \Omega^+(\epsilon_k, \delta)$ so that the ball $B(x_k, \alpha_\delta)$, centered at the point $x_k$ with the radius $\alpha_\delta$, is contained in $\Omega^+(\epsilon_k, \delta)$.

Notice that $u_\epsilon(x)$ is an upper solution of the Dirichlet problem

$$\begin{cases} -\epsilon_k^2 \triangle_p u = f(u), & \text{in } B(x_k, \alpha_\delta) \\ u = 0, & \text{on } \partial B(x_k, \alpha_\delta). \end{cases} \tag{29}$$

**Claim**: There are $\epsilon_{k_0}$ and $\beta > 0$ so that $\beta\varphi_1$ is a lower solution for the problem (29) for all $0 < \epsilon_k \leq \epsilon_{k_0}$, where $\varphi_1$ is an eigenfunction corresponding to the first eigenvalue $\lambda_1$ of $-\triangle_p$ in $\Omega$ subject to Dirichlet boundary condition. In fact, this follow similarly as in the step 2 in proof of the Proposition 2.1.

Also, from Theorem 2.1 we know that the problem (29) has a minimal solution $u^*$ with $\beta\varphi_1 \leq u^* \leq u_\epsilon$ and so that $u^* \to a_3$ on every compact subset of $B(x_k, \alpha_\delta)$ as $\epsilon_k \to 0$. This leads to a contradiction for $\delta < a_3$. □

**Remark 4.1** *If we choose an open ball $B = B(x, w^*(\epsilon, \delta))$ centered at some point $x = x(\varepsilon, \delta)$ whose radius $w^*(\epsilon, \delta)$ is the maximum of the radii of balls in*

$$\Omega^-(\epsilon, \delta) = \{x \in \Omega : a_1 < -\delta < u(x, \epsilon) < 0\}$$

*we can prove, by a similar method, that* $\lim_{\epsilon\to 0} w^*(\epsilon, \delta) = 0$.

**Remark 4.2** *By a translation we can remark that Theorem 1.2 and remark 4.1 give us the asymptotic behavior of any solution $u_l(x)$ obtained in the Theorem 1.1.*

## 4.1 Examples

**Example 4.1** *Let* $f:\mathbb{R}\to\mathbb{R}$ *defined by*

$$f(u) := \frac{|u|^{p-2}u(p-p|u|^p)}{(1+|u|^p)}.$$

*Hence,* $f$ *satisfies the conditions* $(f_1)$*,* $(f_2)$*,* $(f_3)$*, for* $l=1$*, where* $a_1=-1$*,* $a_2=0$ *and* $a_3=1$*.*

So by Theorem 3.1 there is an $\epsilon_0>0$ so that the Neumann problem

$$\begin{cases} -\epsilon^2\triangle_p u &= \frac{|u|^{p-2}u(p-p|u|^p)}{(1+|u|^p)}, \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial\Omega, \end{cases}$$

has a nonconstant solution $a_1\le u_\epsilon\le a_3$ for all $0<\epsilon<\epsilon_0$ and $p\ge 2$.

**Example 4.2** *Let* $f:\mathbb{R}\to\mathbb{R}$ *defined by* $f(u)=u(a-u^2)$*, where* $a\in\mathbb{R}^+$*. Hence,* $f$ *satisfies the conditions* $(f_1)$*,* $(f_2)$*,* $(f_3)$*, for* $l=1$*, where* $a_1=-\sqrt{a}$*,* $a_2=0$ *and* $a_3=\sqrt{a}$*.*

Since, by Theorem 1.1 there is $\epsilon_0>0$ so that the Neumann problem

$$\begin{cases} -\epsilon^2\triangle u &= u(a-u^2), \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial\Omega, \end{cases}$$

has at least $l$ nonconstant solutions satisfy

$$a_1\le u_1(x)\le a_3\le u_2(x)\le a_5<...<u_l(x)<a_{2l+1}$$

for all $0<\epsilon<\epsilon_0$.

**Example 4.3** *Let* $f:[-\pi,(2l-1)\pi]\to\mathbb{R}$ *defined by* $f(u)=sen(u)$*. Hence,* $f$ *satisfies the conditions* $(f_1)$*,* $(f_2)$*,* $(f_3)$*, for* $p=2$*,* $l=1,2,3,...$ *where*

$$a_1=-\pi<a_2=0<a_3=\pi<...<a_{2l}=(2l-2)\pi<a_{2l+1}=(2l-1)\pi.$$

Hence, from Theorem 1.1 there is $\epsilon_0 > 0$ so that the Neumann problem

$$\begin{cases} -\epsilon^2 \triangle u &= sen(u), \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial\Omega, \end{cases}$$

has at least $l$ nonconstant solutions satisfy

$$a_1 \leq u_1(x) \leq a_3 \leq u_2(x) \leq a_5 < ... < u_l(x) < a_{2l+1}$$

for all $0 < \epsilon < \epsilon_0$.

## References


[1] L. Boccardo, F. Murat, Almost Everywhere Convergence of the Gradients of Solutions to Elliptic and Parabolic Equations, Nonlinear Analysis TMA, 6 (1992) 581-597.

[2] H. Brézis, Analyse fonctionelle théorie et applications, Masson, Paris (1987).

[3] A. Cañada, P. Drábek, J. Gamez, Existence of positive solutions for some problems with nonlinear difusion, Trans. AMS, 349 (1997) 4231-4249.

[4] M. Cuesta, D. G. De Figueiredo, J. P. Gossez, The beginning of the Fučik spectrum for the p-Laplacian, J. Diff. Eqns., 159 (1999) 212-238.

[5] D. G. De Figueiredo, On the existence of multiple ordered solutions of nonlinear eigenvalue problems, Nonlinear Analysis TMA, 11 (1987) 481-492.

[6] J. P. García Azorero, I. Peral Alonso, J. J. Manfredi, Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations, Comm. Contemp. Math., 3 (2000) 385-404.

[7] H. Hofer, A geometric description of the neighborhood of a critical point given by the moutain-pass theorem, J. London Math. Soc, 31 (1985) 566-570.

[8] B. S. Ko, The existence of nonconstant solutions of a class of semilinear elliptic Neumann problems with a small parameter, Nonlinear Analysis TMA, 28 (1997) 1249-1263.

[9] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Analysis TMA, 12 (1988) 1203-1219.

[10] C. C. Manríquez, Existência e multiplicidade de soluções para certos problemas elípticos semilineares, Tese de Doutorado, UNICAMP, (1992).

[11] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Diff. Eqns., 51 (1984) 126-150.

[12] J. L. Vázquez,A strong maximum principle for quasilinear elliptic equations, Appl. Math. Optim., 12 (1984) 191-202.