Accurate Evaluation of Elliptic Integrals

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Abstract

We make use of polynomial approximations and get accurate estimates of elliptic integrals. The estimates differ from exact values by very small errors, some of order 0.05% (or even less). The methods explained here can also be used to prove transcendental inequalities.

0. Introduction

Elliptic integrals come out very frequently in the study of complex-based theories like Minimal Surfaces, Algebraic Curves, Riemann Surfaces, Quantum Physics and many others. In order to reckon these integrals, one either tries to use handbooks or special functions, but in both cases without the guarantee of finding the answer. Moreover, before trying these two alternatives, one generally has to make simplifications like changes of variable, integration by parts, and so on. Once again, this procedure can be fruitless, for either the integrals cannot be simplified or even so they lack in handbooks.

Depending on the research area, elliptic integrals are not allowed to be evaluated by computer softwares without the appropriate theoretical error estimates. We refer the reader to [**3**] for a brief explanation about different kinds of errors. Among them, the *integration method error* is the hardest one to evaluate, and this task frequently turns out to be unpractical. There are several reasons for not trusting numerically computed integrals, and some arguments were already discussed in [**3**], [**4**] and [**5**].

This Technical Report deals with an alternative for the evaluation of elliptic integrals by means of very accurate polynomial approximations. In other words, given $\int f$ which cannot be usually computed, we take polynomials p and q to approximate f by two rational functions r and R, in such a way that $r \leq f \leq R$ throughout the whole integration interval. We shall have $r \geq 0$ for all cases studied here, which is a mild restriction, since one can simply add integrals over disjoint intervals. Of course, the inequality $r \leq f \leq R$ must be formally proved, $\max\{\deg(p),\deg(q)\}$ should not be too high, and the relative error $\int(R-r)/\int(R+r)$ must be of very small order (0.05% or even less). In this present work we use polynomial estimates already proved in [**3**] and deal with single evaluations, namely *either* $\int f > \int r$ *or* $\int f < \int R$.

Moreover, most of the estimating rational integrals in this report can be reckoned by Equation (7). The final numeric expression is given by

$$(term_1)\ln(term_2) + (term_3)[\arctan(term_4) + \arctan(term_5)],$$

where $term_n$ is real for every n. Since all computations presented here can be verified by hand, the estimates take $term_n$ with finite decimal part. At this point it is important to remark that $(\tilde{x},\tilde{y}) \in (x,X) \times (y,Y) \subset \mathbb{R}_+ \times (0,1)$ implies $X \ln y < \tilde{x} \ln \tilde{y} < x \ln Y$. But how many decimals should $term_n$ have? In fact, $term_2$ is allowed to have 4 decimals or more (up to 7 in this work), because of the logarithm algebraic properties. For instance, right after Equation (22) we have an expression with $\ln 0.1732568 = -7\ln 10 + 3\ln 2 + \ln 216571$. Now $\ln 216571 = \varepsilon + \ln 216571.2 = \varepsilon - \ln 10 + 4\ln 2 + \ln 3 + \ln 45119$. In [**1**] we have tables of logarithm with 15 exact decimals, while ε is of order 10^{-7}. So $\ln 0.1732568 < -1.75297$.

The *arctan* function is much harder to deal with. Because of that, terms 4 and 5 are never taken here with more than 6 decimals. Besides [**1**], we can use the following formulae to compute *arctan* with more exact decimals:

$$\arctan(x+h) = \arctan x + \arctan \frac{h}{1+hx+x^2},$$

and

$$\arctan x = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{(1+x^2)^{n+1}}.$$

This rapidly converging series is due to Castellanos (see details in [**2**]). For instance, we can use $x = 0.4159$ and $h = 0.000088$ to compute $\arctan 0.415988 > \arctan 0.4159 + \arctan(7.5 \cdot 10^{-5}) > 0.39421$.

The readers who prefer a dynamical verification of our assertions are invited to access "http://www.ime.unicamp.br/~valerio/softwares.html" and download "tecrep61_03.m" for Matlab. The programme computes $term_n$ for each integral studied here. One can follow each Lemma and Equation number in this report at the corresponding step by running the software. Before running the programme, we suggest the reader to fit the command line window into the whole left-hand side of the screen.

1. First Approach

In this section we make some upper and lower estimates of elliptic integrals like $\int t^{n/2}/P^{\frac{1}{2}}$, where $n \in \mathbb{N}$ and $P = QR^2$ for trinomials Q and R. We make strong use of Lemma 2.1 in [**3**] and shall refer to the equations there as ((5)), ((6)), etc.

Proposition 1.1. *The following inequalities hold*:

$$\mathcal{J}_1 := \int_0^1 \frac{t^{\frac{1}{2}}dt}{(t^2+0.1t+1)^{\frac{1}{2}}(t^2+0.15t+1.06)} > 0.3774; \tag{1}$$

$$\mathcal{J}_2 := \int_0^1 \frac{t^{\frac{1}{2}}dt}{(t^2+0.1t+1)^{\frac{1}{2}}(1+0.15t+1.06t^2)} > 0.3895; \tag{2}$$

$$\mathcal{J}_3 := \int_0^1 \frac{2dt}{(t^4+0.1t^2+1)^{\frac{1}{2}}(t^4+0.15t^2+1.06)} < 1.49845; \tag{3}$$

$$\mathcal{J}_4 := \int_0^1 \frac{t^{3/2}dt}{(t^2+0.1t+1)^{\frac{1}{2}}(1+0.15t+1.06t^2)} < 0.19961. \tag{4}$$

Proof

¿From ((5)) it follows that

$$\frac{\mathcal{J}_1}{2} > \int_0^1 \frac{t^2[-0.31t^2+1+0.2555t^2(t^2-1)^2]dt}{t^4+0.15t^2+1.06}. \tag{5}$$

The numerator p_1 of the integrand at (5) is bigger than

$$p_2 := (0.2555t^4-0.549325t^2-0.24293125)(t^4+0.15t^2+1.06)+1.6187t^2+0.2575.$$

Therefore,

$$\mathcal{J}_1 > -0.7498799\dot{3} + \int_0^1 \frac{3.2374t^2+0.515}{t^4+0.15t^2+1.06}dt. \tag{6}$$

If α, β, A and B are real numbers with $A^2 < 4B$, then

$$\int \frac{(\alpha t^2+\beta)dt}{t^4+At^2+B} = \frac{\alpha b-\beta}{4ab}\ln\left(\frac{t^2-at+b}{t^2+at+b}\right)+\frac{\alpha b+\beta}{2bc}\left(\arctan\frac{2t-a}{c}+\arctan\frac{2t+a}{c}\right), \tag{7}$$

where $a=(-A+2B^{\frac{1}{2}})^{\frac{1}{2}}$, $b=B^{\frac{1}{2}}$ and $c=(A+2B^{\frac{1}{2}})^{\frac{1}{2}}$. We use this and (6) to conclude that

$$\mathcal{J}_1 > -0.74988+0.495254\ln 0.18991+1.25734(\arctan 0.415988+\arctan 2.2752). \tag{8}$$

We finally conclude that $\mathcal{J}_1 > 0.3774$. In the case of $\mathcal{J}_2$ we have

$$\frac{\mathcal{J}_2}{2} > 0.943396\int_0^1 \frac{t^2[-0.31t^2+1+0.2555t^2(t^2-1)^2]dt}{t^4+0.14151t^2+0.9434}. \tag{9}$$

The numerator of the integrand at (9) is again p_1, which is bigger than

$$p_3 := (0.2555t^4 - 0.54716t^2 - 0.218111)(t^4 + 0.14151t^2 + 0.9434) + 1.54705t^2 + 0.20575.$$

Therefore,

$$\mathcal{J}_2 > -0.943396 \cdot 0.6988 + 0.943396 \int_0^1 \frac{3.0941t^2 + 0.4115}{t^4 + 0.14151t^2 + 0.9434} dt. \tag{10}$$

By means of (7) and (10) we conclude that

$$\frac{\mathcal{J}_2 + 0.65925}{0.943396} > 0.4974591 \ln 0.1899149 + 1.218369(\arctan 0.45576 + \arctan 2.31501),$$

and finally one has $\mathcal{J}_2 > 0.3895$. Now we use ((6)) in order to get

$$\frac{\mathcal{J}_3}{2} < \int_0^1 \frac{[-0.39t^2 + 1.07385 + 0.24(t^2 - 0.65)^3]dt}{t^4 + 0.15t^2 + 1.06}. \tag{11}$$

Since the numerator p_4 of (11) is equal to

$$p_5 := (0.24t^2 - 0.504)(t^4 + 0.15t^2 + 1.06) - 0.2646t^2 + 1.54218,$$

we now use (7) to conclude that

$$\mathcal{J}_3 < -0.848 - 0.637795 \ln 0.1899 + 0.82977(\arctan 0.416 + \arctan 2.27524),$$

which finally gives us $\mathcal{J}_3 < 1.49845$. In the case of $\mathcal{J}_4$ we have

$$\frac{\mathcal{J}_4}{2} < 0.9434 \int_0^1 \frac{t^4[-0.39t^2 + 1.07385 + 0.24(t^2 - 0.65)^3]dt}{t^4 + 0.1415t^2 + 0.9433}. \tag{12}$$

The numerator of (12) is $t^4 p_4 =: p_6$, which is smaller than

$$p_7 := (0.24t^6 - 0.50196t^4 - 0.24116t^2 + 1.5156)(t^4 + 0.1415t^2 + 0.9433) + 0.01304t^2 - 1.4296.$$

We now use (7) to conclude that

$$\mathcal{J}_4 < 0.9434[2.7382141 + 0.553269 \ln 0.18992 - 1.010599(\arctan 0.4558 + \arctan 2.315)],$$

hence $\mathcal{J}_4 < 0.19961$.

q.e.d.

2. Second Approach

In this section we make further evaluations by upper and lower bounds, this time for simpler elliptic integrals like $\int t^{n/2}/P^{\frac{1}{2}}$, where $n \in \mathbb{N}$ and $P = QR^2$ for binomials Q and R. Lemma 2.1 in [**3**] will be again strongly referred to, and its equations indicated by ((7)), ((8)), etc.

Proposition 2.1. *The following inequalities hold*:

$$\mathcal{J}_5 := \int_0^1 \frac{2t^2 dt}{(t^4+1)^{\frac{1}{2}}(t^4+0.7164)} > 0.5442; \tag{13}$$

$$\mathcal{J}_6 := \int_0^1 \frac{t^{\frac{1}{2}} dt}{(t^2+1)^{\frac{1}{2}}(1+0.7164t^2)} > 0.4521; \tag{14}$$

$$\mathcal{J}_7 := \int_0^1 \frac{2dt}{(t^4+1)^{\frac{1}{2}}(t^4+0.7164)} < 2.20891; \tag{15}$$

$$\mathcal{J}_8 := \int_0^1 \frac{t^{3/2} dt}{(t^2+1)^{\frac{1}{2}}(1+0.7164t^2)} < 0.24097. \tag{16}$$

Proof

¿From ((7)) it follows that

$$\frac{\mathcal{J}_5}{2} > \int_0^1 \frac{t^2[-0.293t^2+1+0.22t^2(t^2-1)(t^2-1.22)]dt}{t^4+0.7164}, \tag{17}$$

and the numerator p_8 of (17) is bigger than

$$p_9 := (0.22t^4 - 0.4884t^2 - 0.18221)(t^4+0.7164) + 1.34988t^2 + 0.13053.$$

Therefore,

$$\mathcal{J}_5 > -0.60202 + \int_0^1 \frac{2.69976t^2+0.26106}{t^4+0.7164} dt. \tag{18}$$

Now we apply (7) to (18) and get

$$\mathcal{J}_5 > -0.60202 + 0.45949 \ln 0.173257 + 1.156(\arctan 0.53718 + \arctan 2.53718),$$

which finally implies $\mathcal{J}_5 > 0.54424$. In the same way one has

$$\frac{\mathcal{J}_6}{2} > 1.395868 \int_0^1 \frac{t^2[-0.293t^2+1+0.22t^2(t^2-1)(t^2-1.22)]dt}{t^4+1.396}, \tag{19}$$

and the numerator of (19) is again p_8, which is bigger than

$$p_{10} := (0.22t^4 - 0.4884t^2 - 0.33172)(t^4 + 1.396) + 1.6818t^2 + 0.46308.$$

It follows that

$$\mathcal{J}_6 > -1.395868 \cdot 0.90104 + 1.395868 \int_0^1 \frac{3.3636t^2 + 0.92616}{t^4 + 1.396} dt,$$

and from (7) we have

$$\mathcal{J}_6 > 1.395868[-0.90104 + 0.4195446 \ln 0.173258 + 1.34901(\arctan 0.301 + \arctan 2.301)].$$

Therefore, $\mathcal{J}_6 > 0.45211$. Regarding $\mathcal{J}_7$, from ((8)) we have

$$\frac{\mathcal{J}_7}{2} < \int_0^1 \frac{[-0.3766t^2 + 1.08471 + 0.22(t^2 - 1)(t^2 - 0.6)^2]dt}{t^4 + 0.7164}, \tag{20}$$

and the numerator p_{11} of (20) is smaller than

$$p_{12} := (0.22t^2 - 0.484)(t^4 + 0.7164) - 0.191t + 1.35225.$$

It follows that

$$\mathcal{J}_7 < -0.821\dot{3} + \int_0^1 \frac{-0.382t^2 + 2.7045}{t^4 + 0.7164},$$

and from (7) we have

$$\mathcal{J}_7 < -0.821\dot{3} - 0.687368 \ln 0.173257 + 1.08113365(\arctan 0.5372 + \arctan 2.5372),$$

which finally implies $\mathcal{J}_7 < 2.20891$. Now we use ((8)) once more to get

$$\frac{\mathcal{J}_8}{2} < 1.39587 \int_0^1 \frac{t^4[-0.3766t^2 + 1.08471 + 0.22(t^2 - 1)(t^2 - 0.6)^2]dt}{t^4 + 1.3958}, \tag{21}$$

of which the numerator is $t^4 p_{11} =: p_{13}$, and this latter is smaller than

$$p_{14} := (0.22t^6 - 0.484t^4 - 0.34047t^2 + 1.68108)(t^4 + 1.3958) + 0.47524t^2 - 2.346445.$$

Because of that we have

$$\mathcal{J}_8 < 1.39587 \cdot 3.00444 + 1.39587 \int_0^1 \frac{0.9505t^2 - 4.6928}{t^4 + 1.3958} dt, \tag{22}$$

which implies

$$\mathcal{J}_8 < 1.39587[3.00444 + 0.800597 \ln 0.1732568 - 0.982848(\arctan 0.3011 + \arctan 2.3011)].$$

This finally gives us $\mathcal{J}_8 < 0.24097$.

q.e.d.

3. Third Approach

We now deal with much heavier kinds of integrals like $\int RQ^{\pm 1/2}$, in which R is rational and Q is a trinomial. All lower and upper approximation formulae from [**3**] will be frequently used.

Proposition 3.1. *The following inequalities hold*:

$$\mathcal{J}_9 := \int_0^1 \frac{(t^4-0.1t^2+1)^{-\frac{1}{2}}}{(t^2-0.1)^2+0.7164}(1-t^2)dt < 0.8645; \tag{23}$$

$$\mathcal{J}_{10} := \int_0^1 \frac{t^2(t^4+1)^{-\frac{1}{2}}}{1+1.06t^4}(1-t^2)dt > 0.1005; \tag{24}$$

$$\mathcal{J}_{11} := \int_0^1 \frac{[-0.15-t^2+0.7222(1+t^2+t^4)]}{(1+1.06t^4)(t^4+1.06)}(t^4-0.1t^2+1)^{\frac{1}{2}}dt > 0.443; \tag{25}$$

$$\mathcal{J}_{12} := \int_0^1 \left[\frac{1}{t^4+1.06}+\frac{t^4}{1+1.06t^4}\right]\frac{dt}{(t^4+1)^{\frac{1}{2}}} > 0.8812; \tag{26}$$

$$\mathcal{J}_{13} := \int_0^1 \frac{t^2(t^4-0.1t^2+1)^{-1/2}}{(t^2-0.0764)^2+0.7164}dt < 0.3018; \tag{27}$$

$$\mathcal{J}_{14} := \int_0^1 \frac{t^2(t^4-0.1t^2+1)^{-1/2}}{(1-0.0764t^2)^2+0.7164t^4}dt < 0.24994. \tag{28}$$

Proof

¿From ((17)) it follows that

$$\mathcal{J}_9 < 0.473^{-1}\int_0^1 \frac{(t^4-0.2t^2+2.11416)^{-1}(1-t^2)}{t^4-0.2t^2+0.7264}dt <$$

$$1.52344\int_0^1\left(\frac{1-t^2}{t^2-0.2+0.7264}+\frac{t^2-1}{t^2-0.2+2.11416}\right)dt.$$

We now use (7) in order to get

$$\frac{\mathcal{J}_9}{1.52344} < -0.3937\ln 0.14609+0.070645(\arctan 0.505403+\arctan 2.75561)+$$

$$0.2393\ln 0.16388+0.09488(\arctan 0.14405+\arctan 2.28667).$$

hence $\mathcal{J}_9 < 0.86444$. Now we use ((7)) to obtain

$$1.06\mathcal{J}_{10} > \int_0^1 \frac{p_8(1-t^2)}{t^4+0.9434}dt,$$

and $p_8(1-t^2) =: p_{15}$ is bigger than

$$p_{16} := (-0.22t^6+0.7084t^4-0.2563t^2-1.693)(t^4+0.9434)+1.2417t^2+1.597.$$

Therefore, from (7) it follows that

$$1.06\mathcal{J}_{10} > -1.6682-0.072198\ln 0.17163+1.03529(\arctan 0.43496+\arctan 2.43496),$$

so $\mathcal{J}_{10} > 0.1005$. Regarding (25), we first use the fact that

$$\frac{1}{(1+1.06t^4)(t^4+1.06)} > \frac{1.06^{-1}}{0.1166}\left(\frac{1}{t^4+0.9434}-\frac{1}{t^4+1.06}\right),$$

and apply it together with ((17)) in order to obtain

$$1.06\cdot 0.1166\mathcal{J}_{11} > 0.03983-\int_0^1 \frac{0.146t^2+0.0853}{t^4+0.9434}dt+\int_0^1 \frac{0.1202t^2+0.1242}{t^4+1.06}dt.$$

¿From (7) it follows that

$$0.123596\mathcal{J}_{11} > 0.03983-[0.0104\ln 0.171625+0.0838814(\arctan 0.435+\arctan 2.435)]+$$
$$0.083916(\arctan 0.39376+\arctan 2.39376).$$

Because of that, $\mathcal{J}_{11} > 0.44301$. Now, from ((7)) we have

$$\mathcal{J}_{12} > \int_0^1 \left(\frac{1}{t^4+1.06}+\frac{t^4}{1+1.06t^4}\right)[-0.293t^2+1+0.22t^2(t^2-1)(t^2-1.22)]dt.$$

Therefore,

$$\mathcal{J}_{12} > 0.873159+\int_0^1 \left(\frac{1}{t^4+1.06}-\frac{0.89}{t^4+0.943}\right)(0.22t^6-0.4884t^4-0.0246t^2+1)dt$$
$$> 0.8275+0.89\int_0^1 \frac{0.232t^2-1.4606}{t^4+0.943}dt+\int_0^1 \frac{-0.2578t^2+1.5177}{t^4+1.06}dt.$$

¿From (7) it follows that

$$\mathcal{J}_{12} > 0.8275+0.89[0.31144\ln 0.171625-0.4564013(\arctan 0.43512+\arctan 2.43512)]+$$
$$-0.3017\ln 0.1716244+0.423815(\arctan 0.39376+\arctan 2.39376).$$

Hence $\mathcal{J}_{12} > 0.8812$. Now we analyse $\mathcal{J}_{13}$ and use ((17)) to assert that

$$\mathcal{J}_{13} < \int_0^1 \frac{t^2(0.473t^4 - 0.0946t^2+1)^{-1}}{(t^2-0.0764)^2+0.7164}dt <$$

$$\frac{2.114165}{1.928678}\int_0^1 \left(\frac{1.3918t^2 - 0.0472\cdot 0.7222}{t^4-0.1528t^2+0.7222} - \frac{1.3918t^2-0.0472\cdot 2.114}{t^4-0.2t^2+2.114}\right) <$$

$$1.0962\int_0^1 \left(\frac{1.3918t^2-0.034}{t^4-0.1528t^2+0.7222} - \frac{1.3918t^2-0.1}{t^4-0.2t^2+2.114}\right).$$

¿From (7) it follows that

$$\frac{\mathcal{J}_{13}}{1.0962} < 0.2629977\ln 0.15223 + 0.543446(\arctan 0.51375 + \arctan 2.70241) +$$

$$-0.2071237\ln 0.16387 - 0.40199(\arctan 0.144 + \arctan 2.28669),$$

thus $\mathcal{J}_{13} < 0.3018$. Finally, by using again ((17)) we have

$$\mathcal{J}_{14} < \int_0^1 \frac{t^2(0.473t^4 - 0.0946t^2+1)^{-1}}{0.7222t^4-0.1528t^2+1}dt <$$

$$1.384658\frac{2.114165}{0.534}\int_0^1 \left(\frac{0.7294t^2+0.0116\cdot 1.3846}{t^4-0.2116t^2+1.3846} - \frac{0.7294t^2+0.0116\cdot 2.114}{t^4-0.2t^2+2.114}\right) <$$

$$5.48201401\int_0^1 \left(\frac{0.7294t^2+0.0161}{t^4-0.2116t^2+1.3846} - \frac{0.7294t^2+0.0245}{t^4-0.2t^2+2.114}\right).$$

¿From (7) it follows that

$$\mathcal{J}_{14} < 5.48201401[0.111722\ln 0.1522 + 0.253875(\arctan 0.27226 + \arctan 2.46095) +$$

$$-0.10105\ln 0.16387 - 0.226744(\arctan 0.144065 + \arctan 2.28669)],$$

thus $\mathcal{J}_{14} < 0.249935$.

References

[**1**] Abramowitz, M. and Stegun, I. A. *Handbook of mathematical functions with formulas, graphs, and the mathematical tables*. Washington D.C., Nat. Bur. of Standards (1964).

[**2**] Castellanos, D. *The ubiquitous Pi. Part I*. Math. Mag. **61** (1988) 67–98.

[**3**] Ramos Batista, V. *Polynomial estimates in the unitary interval*. UNICAMP Technical Report 60/03, University of Campinas, 2003.
http://www.ime.unicamp.br/rel_pesq/2003/rp60-03.html

[**4**] Ramos Batista, V. *The use of unitary functions in the behaviour analysis of elliptic integrals*. UNICAMP Technical Report 03/03, University of Campinas, 2003. http://www.ime.unicamp.br/rel_pesq/2003/rp03-03.html

[**5**] Ramos Batista, V. *Theoretical evaluation of elliptic integrals based on computer graphics*, UNICAMP Technical Report 71/02, University of Campinas, 2002. http://www.ime.unicamp.br/rel_pesq/2002/rp71-02.html