Polynomial Estimates in the Unitary Interval

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Abstract

We show how to get accurate polynomial estimates in a fixed real interval. We explain the methods used to get theoretical exactness of up to 0.05% for evaluation of transcendental integrals. A discussion of further applications of these estimates is also presented.

0. Introduction

Throughout the centuries, all practical problems involving measurement of sizes, mapping and handling of shapes, tracking of movements and many others, have been accomplished by no other means but mathematical modelling. Mathematics is, in fact, our utmost approach at predicting, explaining and reproducing natural phenomena, even if total exactness cannot be reached. Although innumerable improvements have been added to mathematical models, one steadily needs more exactness, and so the modelling enhances with increasing complexity. This is the typical case for Nuclear Physics, Micro-engineerings, Digital Systems and many others.

Irrational numbers are a simple instance that total exactness cannot be attained in practical life. However, modern computers are equipped to handle them with overshooting precisions. But some numeric integrations fail to succeed if one does not previously apply suitable changes of variables: $\int_0^\pi \sin\frac{1}{t}dt$, $\int_0^e [(1-\ln t)/(e-t)]^{1/2}dt$, etc. Moreover, in order to assure the exactness of numeric integration, one needs to control some different errors: *integration method* *error*, *truncation error* and *rounding error*. For example, the first one is at most $\max_{[0,1]}|f^{\mathrm{IV}}|/(180n^4)$ for $\int_0^1 f\,dt$ calculated with Simpson's Method, with $[0,1]$ equally partitioned by $n \in 2\mathbb{N}\setminus\{0\}$, where f^{IV} is the 4^{th} derivative of f. Numeric software integration normally works with iteration algorithms which disregard the method error analysis, a theoretically rather unsafe procedure (see further comments in [**2**]). The second kind of error is typical for series that one truncates in order to compute the approximate sum.

The latter refers to machine precision, because numbers like $1/3$ and $\sqrt{2}$ will be taken with finite decimal part. Some softwares can handle them symbolically, but sometimes we get residual answers. For instance, the software Matlab 6.0 for Linux gives $(\sqrt{5}-1)/4-\sin(\pi/10)\equiv 5.55\cdot 10^{-17}$ in long precision mode. Similar problems also occur for Mathematica, Maple and others. The rounding error also refers to the smallest and biggest positive numbers the machine works with. For example, Matlab 6.0 takes $2^{-1075}=0$ but $2^{-1074}>0$. This means that the computer graph of any $f:[0,1]\to\mathbb{R}$ will never show us what happens between two "consecutive numbers". Although in practical problems so much precision is indeed irrelevant, for Pure Mathematics every assertion must be irrefutably proved. But what made Mathematics so inexorable in this sense? The reason is that *intuition alone is insufficient to assure the truth about facts.* This can be exemplified by classical problems like *the brachistochone, the Peano curve, the horned sphere, the long line, the differentiable f with nowhere continuous f'*[1], *Gödel's incompleteness theorem*, and many others.

For these reasons, when a mathematical model uses transcendental integrals, we must either evaluate them by controlling all numeric errors or make use of subtable approximate integrands with explicit primitives. This second choice is preferable when the theoretical estimate of $\max_{[0,1]}|f^{\mathrm{IV}}|$ becomes impractical. In this work we show how to deal with this second alternative through accurate polynomial approximations. We use them to substitute terms in integrands in order to get explicit primitives. These primitives give us lower and upper bounds of the desired transcendental function. For example, from Lemma 1.1 in this report we have that

$$P(s)\sin s - Q(s)\cos s < \int_0^s \frac{\sin t\,dt}{\sqrt{t^2+0.1t+1}} < R(s)\sin s - S(s)\cos s,\ \forall\ s\in[0,1],$$

where $P=0.7665s^2-1.022s-1.5875$, $Q=0.2555s^3-0.511s^2-1.5875s+2.022$, $R=0.72s^2-0.936s-1.5258$ and $S=0.24s^3-0.468s^2-1.5258s+1.9439$. In

[1] See more about the Köpcke function in [**1**], p. 228.

this example our theoretical error is smaller than 0.04%.

The readers who prefer a dynamical verification of our assertions are invited to access "http://www.ime.unicamp.br/∼valerio/softwares.html" and download "tecrep60_03.m" for Matlab. The programme does not work with other operations except addition, subtraction and multiplication of numbers with finite decimal part. All numbers are shown with 14 decimals in the programme, and at any expression products altogether never exceed 13 decimals with a *sole exception* at Equation 15 (see below). One can follow each Lemma and Equation number in this report at the corresponding step by running the software. In this paper, and in the programme, each polynomial inequality like $p_i > p_j$ is lastly proved through a term-by-term comparison. That is, we shall have $p_i = \sum_{k=0}^{n} a_{ki}t^k > p_j = \sum_{k=0}^{n} b_{kj}t^k$ because $a_{ki} \geq b_{kj}$ $\forall k$ with at least *one* sharp inequality. Before running the programme, we suggest the reader to fit the command line window into the whole left-hand side of the screen.

1. First Approach

In this section we obtain some basic inequalities for functions of the form $A^{\frac{1}{2}}/B$, where both A and B are 2nd-order polynomials with real coefficients. One works with the special case when $A^{1/2}/B$ is bounded by and quite close to an affine function in the unitary interval.

Lemma 1.1. *The following inequalities hold for* $t \in (0, 1)$

$$\frac{(t^2+1)^{\frac{1}{2}}}{t^2+0.15t+1.06} > -0.3001t + 0.94; \tag{1}$$

$$\frac{(t^2+1)^{\frac{1}{2}}}{1+0.15t+1.06t^2} > -0.37t + 1; \tag{2}$$

$$\frac{(t^2+0.1t+1)^{\frac{1}{2}}}{t^2+0.7164} < -0.69t + 1.54; \tag{3}$$

$$\frac{(t^2+0.1t+1)^{\frac{1}{2}}}{1+0.7164t^2} < 1.003; \tag{4}$$

Proof

We notice that

$$p_1 := (t^2+1)-[(t^2+0.15t+1.06)(-0.3001t+0.94)]^2 > p_2 := -0.09007t^6+0.53716t^5+$$
$$-0.9072972t^4+0.9150537t^3-0.814893t^3+0.3529368t+0.0071097,$$

and $p_2(1)=0$. Hence (1) will hold providing $p_3 := [p_2/(1-t)-0.0071097]/t > 0$. But

$$p_3 > p_4 := 0.09t^4-0.45t^3+0.46t^2-0.46t+0.36,$$

with $p_4(1)=0$. Thus, (1) will be valid if $p_5 := p_4/(1-t) > 0$, where $p_5 = -0.09t^3+0.36t^2-0.1t+0.36$. Since $0.36 > 0.1 > 0.09$, then p_5 is positive in $[0,1]$ and this implies (1).

Now rewrite (2) as

$$p_6 := -0.15382084t^6+0.7879298t^5-1.18158825t^4+1.22638t^3-1.0574t^2+0.44t > 0$$

and consider the assertion

$$p_6/t > p_7 := -0.187t^5+0.787t^4-1.2t^3+1.22t^2-1.06t+0.44 > 0.$$

The polynomial p_7 has a root at $t=1$, so we need to show that $p_8 := p_7/(1-t)-0.007 > 0$. Once again $p_8(1)=0$, so consider $p_9 := p_8/(1-t) = -0.187t^3+0.413t^2-0.187t+0.433$. Since $p_9 > (-0.187t+0.413)(t^2+1)$, we finally conclude (2). Regarding (3), first notice that

$$(t^2+0.1t+1)-(0.69t-1.54)^2(t^2+0.7164)^2 < p_{10}, \quad \text{where}$$

$$p_{10} := -0.47t^6+2.13t^5-3.05t^4+3.045t^3-2.64t^2+1.2t-0.215.$$

One sees that $p_{10}(1)=0$, hence consider $p_{11} := p_{10}/(1-t)+0.685t^5$. Once again $p_{11}(1)=0$ and so we take $p_{12} := p_{11}/(5-5t)$. It is not difficult to verify that

$$p_{12} < p_{13} := (-0.231t^2-0.085186t-0.20814)(t-0.403)^2+0.00009(t-1),$$

which is always negative in $[0,1]$. Hence (3) is valid, indeed. It remains to verify (4), of which the left-hand side takes its maximum at $t=T$, where $0.07164T^2-2T+1.3328=0$. Since $0.07164t^2 \geq 0$, then $T > 0.6664$. So the replacement

$1+0.7164T^2 = 20T - 12.328$ makes sense, namely T is real. By substituting T^2 in (4) we get

$$\frac{2.07164T - 1.325636}{20T - 12.328} < 1.003^2,$$

which is true, indeed. If not, we would get $-18.04854T \geq -11.076442952$, which would imply $T \leq 0.614$ (contradiction). This means that (4) is also valid.

2. Second Approach

This section is devoted to much finer estimates, which can be applied to get very small theoretical errors. They can be of order 0.05% or even less, depedending on the use of these approximations. A little difference between positive functions can give a much smaller difference for their integrals.

Lemma 2.1 *For $t \in (0,1)$ the following equations hold*:

$$(t^2+0.1t+1)[-0.31t+1+0.2555t(t-1)^2]^2 =: p_{14} < 1; \tag{5}$$

$$(t^2+0.1t+1)[-0.39t+1.07385+0.24(t-0.65)^3]^2 =: p_{20} > 1; \tag{6}$$

$$(t^2+1)[-0.293t+1+0.22t(t-1)(t-1.22)]^2 =: p_{27} < 1; \tag{7}$$

$$(t^2+1)[-0.3766t+1.08471+0.22(t-1)(t-0.6)^2]^2 =: p_{30} > 1; \tag{8}$$

Proof

A thorough computation shows that

$$p_{15} := \frac{1-p_{14}}{t} > p_{16} := -0.0653t^7 + 0.25459t^6 - 0.27244t^5 - 0.329t^4$$
$$+0.72908t^3 - 0.3558t^2 + 0.0299t + 0.00897, \tag{9}$$

which is divisible by $(1-t)$. We have

$$\frac{p_{16}}{1-t} = 0.0653t^6 - 0.18929t^5 + 0.08315t^4 + 0.41215t^3 - 0.31693t^2 + 0.03887t + 0.00897.$$

Now observe that

$$p_{17} := \frac{p_{16}}{1-t} > (t-0.4433)^2 p_{18} - 0.0004272t + 0.0004274 =: p_{19}, \tag{10}$$

where $p_{18} := 0.0653t^4 - 0.1317t^3 - 0.0473t^2 + 0.396t + 0.04345$. Since $0.396 > 0.1317 + 0.0473$, then $p_{18} > 0$ in $[0,1]$. This proves (5). Now we have that

$$p_{21} := p_{20} - 1 > (t-0.7645)^2 p_{22} - 0.00057t + 0.00063 =: p_{23} \tag{11}$$

where $p_{22} := 0.0576t^6 - 0.1309t^5 - 0.0209t^4 + 0.4018t^3 - 0.0754t^2 - 0.0526t + 0.0262$. This polynomial has the following property:

$$p_{24} := p_{22} - 0.0576t^6 > (t - 0.2935)^2 p_{25} - 0.01t + 0.013 =: p_{26}, \tag{12}$$

where $p_{25} := -0.1309t^3 - 0.0978t^2 + 0.3556t + 0.1417$. Since $0.3556 > 0.1309 + 0.0978$, then $p_{25} > 0$ in $[0, 1]$. This proves (6). Regarding (7), first observe that $p_{28} := (1 - p_{27})/t$ is such that

$$p_{28} > p_{29} := 0.95t^7 + 0.97t^6 + 1.19t^5 + 0.59t^4 + 0.83t^3 + 1.737t^2 - 0.412t + 2.12, \tag{13}$$

which is always positive in $[0, 1]$. This proves (7). Finally observe that

$$p_{30} - 1 > (t - 0.309)^2 p_{31} + 0.76 \cdot 10^{-4}(1 - t) =: p_{32}, \tag{14}$$

where $p_{31} := 0.0484t^6 - 0.18305t^5 + 0.15t^4 + 0.3715t^3 - 0.5374t^2 + 0.04t + 0.1148$. In order to simplify our further analyses, we shall take $p_{31} > p_{33} := 0.048t^6 - 0.184t^5 + 0.15t^4 + 0.37t^3 - 0.5374t^2 + 0.04t + 0.1148$. We have

$$p_{33} > (t - 0.9372)^2 p_{34}, \tag{15}$$

where

$$p_{34} := 0.048t^4 - 0.09403t^3 - 0.068412t^2 + 0.32435t + 0.1306508, \tag{16}$$

which is always positive in $[0, 1]$. This concludes the proof of Lemma 2.1.

3. Third Approach

This last section exemplifies how we can increase exactness by further refinements in our approximations. The following lemma shows a two-step way of getting increasingly close to $\sqrt{t^2 - 0.1t + 1}$ by lower bounding polynomials.

Lemma 3.1 *For $t \in (0, 1)$ the following equations hold*:

$$p_{35} := [(t^2 - 0.1t + 1) - (0.473t^2 - 0.0946t + 1)^2]/t > 0; \tag{17}$$

$$p_{37} := p_{35} - 0.132(t + 0.55)(1 - t^2) > 0. \tag{18}$$

Proof

Consider $p_{36} := (p_{35} - 1.344 \cdot 10^{-5})/(1 - t)$. Since all coefficients of p_{36} are positive, then (17) holds. Now take $p_{38} := (p_{37} - 1.344 \cdot 10^{-5})/(1 - t)$. We have $10^3 \cdot p_{38} > 90t^2 - 71t + 16$, with no real roots, which implies (18).

References

[**1**] A. Denjoy, *Sur les fonctions dérivées sommables*, Bull. Soc. Math. France 43 (1915), 161–248.

[**2**] V. Ramos Batista, *Theoretical evaluation of elliptic integrals based on computer graphics*, UNICAMP Technical Report 71/02, University of Campinas, 2002. Home page http://www.ime.unicamp.br/rel_pesq/2002/rp71-02.html