On the resolvent technique for stability of plane Couette flow

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Abstract

We discuss the application of the resolvent technique to prove stability of plane Couette flow. Using this technique, we derive a threshold amplitude for perturbations that can lead to turbulence in terms of the Reynolds number. Our main objective is to show exactly how much control one should have over the perturbation to assure stability via this technique.

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1 Introduction

We discuss stability of plane Couette flow via the resolvent method. Applying this method, one can derive lower bounds for the norms of perturbations that can lead to turbulence. Our aim is to discuss and clarify a point that has been overseen so far, which is to determine how much control over the perturbations one should assume to derive the stability result via the resolvent technique. We discuss the two dimensional case, but all the considerations can be used for the three dimensional case with minor technical changes. The main difference between the two and three spatial dimensions is that different resolvent estimates hold for each case, leading to different thresholds. This point will be made clear later on. We begin describing the problem and discussing previous works using the resolvent method.

2 The problem and known results

We are interested in the following initial boundary value problem:

$$\begin{cases} u_t + (u \cdot \nabla)u + \nabla p = \dfrac{1}{R}\Delta u \\ \nabla \cdot u = 0 \\ u(x,0,t) = (0,0) \\ u(x,1,t) = (1,0) \\ u(x,y,t) = u(x+1,y,t) \\ u(x,y,0) = f(x,y) \end{cases} \tag{1}$$

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where $u : \mathbb{R} \times [0,1] \times [0,\infty) \longrightarrow \mathbb{R}^2$ is the unknown function $u(x,y,t) = (u_1(x,y,t), u_2(x,y,t))$. The positive parameter R is the Reynolds number. The initial condition $f(x,y)$ is assumed to be smooth, divergence free and compatible with the boundary conditions. The pressure $p(x,y,t)$ can be determined in terms of u by the elliptic problem

$$\begin{cases} \Delta p = -\nabla \cdot ((u \cdot \nabla) u) \\ p_y(x,0,t) = \dfrac{1}{R} u_{2yy}(x,0,t) \\ p_y(x,1,t) = \dfrac{1}{R} u_{2yy}(x,1,t). \end{cases} \tag{2}$$

It can be easily seen that $U(x,y) = (y,0)$, $P = \textit{constant}$ is a steady solution of problem (1). The vector field $U(x,y) = (y,0)$ is known as Couette flow.

Using the resolvent technique, one can prove and quantify asymptotic stability for this flow. By quantification we mean the derivation a number $M(R)$ such that disturbances of the flow with norm less than $M(R)$ will tend to zero as time t tends to infinity. In other words, deriving a lower bound for the norm of perturbations that can lead to turbulence. For general discussion about the resolvent technique, see [3] and [4].

This problem has been studied for the 3 spatial dimensions case in [5], and a threshold amplitude for perturbations was found to be of order $\mathcal{O}(R^{-\frac{21}{4}})$. The estimates of the resolvent of the linearized equations governing perturbations where those found numerically in [8] and [10], predicting the resolvent constant of the linear operator associated with the problem to be proportional to R^2. In [6], the resolvent technique was used again to prove the stability of the 3 dimensional problem but the estimates for the resolvent constant were those in [7]. By using modified norms, the authors achieve $M(R)$ of order $\mathcal{O}(R^{-3})$ for two of the components of the perturbation, and of order $\mathcal{O}(R^{-4})$ for the remaining component. Our approach uses again the resolvent technique, and we use the same norms as [5], with the obvious modifications for the 2 spatial dimensions case. We show that this approach leads, in our case, to a threshold amplitude of order $\mathcal{O}(R^{-3})$. We note that our argument is the same used in [5], with some minor differences. The only reason for the better exponent in our case is the better dependence of the resolvent constant on R for the 2 dimensional case. In this case, the resolvent constant is proportional to R, as found in [1]. We carry out the argument in details again only because it is important for our aim, which is to clarify a subtle point that has been overseen in previous works: In [5], it was said that one needs control over the sobolev norm H^2 of the perturbation to assure stability. Later on, in [6], the authors note that the H^2 is not enough, and claim that one needs control over the norm H^4. Actually, this is not enough yet, since in one of the directions, one needs control over six derivatives of the perturbation. This necessity is due to the pressure terms appearing in the problem. In section 5, we show in details estimates for these terms, and clarify the reason of this requirement. Moreover, our argument shows that derivatives of different orders of the perturbation scale differently with the Reynolds number. In other words, to assure decay of the perturbations via the resolvent method, one should require the perturbation to be small in some weighted norm involving six derivatives, where the weights depend on the Reynolds number R.

This work is divided in 4 sections: in section 3, we introduce some basic notation and derive the equations for perturbations of the Couette flow; in section 4 we derive estimates for the solution of the linearized equations for the perturbations; in 5, we use those estimates to prove asymptotic stability for the flow, and to derive the threshold amplitude $M(R)$. In section 6, we derive carefully the estimates for the pressure terms involved in the problem.

3 Notation and equations for the perturbations

We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the L_2 inner product and norm over $\Omega = [0,1] \times [0,1]$:

$$\langle u, w \rangle = \int_{\Omega} \overline{u} \cdot w \; dxdy \quad ; \quad \|u\|^2 = \langle u, u \rangle.$$

All the matrix norms that appear in this paper are the usual Frobenius norms. The usual sobolev norm of u over Ω is denoted by

$$\|u\|^2_{H^n(\Omega)} = \sum_{j=0}^{n} \|D^j u\|^2$$

where D^j denotes the j-th derivative of u with respect to the space variables. Unless stated otherwise, all norms in the space variables will be calculated over Ω and therefore we will write $\|\cdot\|_{H^n(\Omega)}$ as $\|\cdot\|_{H^n}$. We make use of a 2 dimensional version of the weighted norm $\|\cdot\|_{\widetilde{H}}$ used in Kreiss[5]:

$$\|u\|^2_{\widetilde{H}} = \|u\|^2 + \frac{1}{R}\|Du\|^2 + \frac{1}{R^2}\|u_{xy}\|^2. \tag{3}$$

We also define another weighted norm $\|\cdot\|_{H^6_m}$ by

$$\begin{aligned} \|u\|^2_{H^6_m} &= \|u\|^2_{H^2} + \frac{1}{R^2}\|D^3 u\|^2 + \frac{1}{R^2}\|D^4 u\|^2 \\ &\quad + \frac{1}{R^4}\|u_{2xxyyy}\|^2 + \frac{1}{R^4}\|u_{2yyyyy}\|^2 + \frac{1}{R^4}\|u_{2yyyyyy}\|^2, \end{aligned} \tag{4}$$

where $u = (u_1, u_2)$.

The maximum norm over Ω is denoted by $|\cdot|_\infty$. The norm $\|\cdot\|_{\widetilde{H}}$ is related with the maximum norm by the sobolev type inequality (see [2], Appendix 3, Theorem A.3.14)

$$|\cdot|^2_\infty \leq \widetilde{C} R \|\cdot\|^2_{\widetilde{H}}.$$

Since we are interested in functions which are also dependent on time, we use that

$$|u(\cdot,t)|^2_\infty \leq \widetilde{C} R \|u(\cdot,t)\|^2_{\widetilde{H}} \;,\; \forall\, t \geq 0, \tag{5}$$

where $\widetilde{C}$ is a constant independent of any of the parameters.

We are interested first in proving asymptotic stability for the Couette flow, which is a stationary solution of (1), that is, to prove that perturbations of the stationary solution that are small enough in some norm will tend to 0 as t tends to infinity. More specifically, we will show that perturbations having norm $\|\cdot\|_{H^6_m}$ of order R^{-3} decay with time.

To this end, let $U = U(x,y)$, $P = P(x,y)$ be a stationary solution of (1). We can obviously use the Couette flow, but we think that the structure of the argument is easier to be understood if one uses any stationary solution. This will not change the estimates we will prove. We derive the equations satisfied by perturbations of this base flow. Let $u(x,y,t)$, $p(x,y,t)$ be a solution of (1) with initial condition $f(x,y) = U(x,y) + \epsilon f'(x,y)$, where f' is divergence free and $\|f'\|_{H^6_m(\Omega)} = 1$. Then, ϵ defines a unique perturbation amplitude. Write $u(x,y,t) = U(x,y) + \epsilon u'(x,y,t)$ and $p(x,y,t) = P(x,y) + \epsilon p_1'(x,y,t) + \epsilon^2 p_2'(x,y,t)$. Then u',p_1',p_2' satisfy the system

$$\begin{cases} u'_t + (u'\cdot\nabla)U + (U\cdot\nabla)u' + \nabla p_1' + \epsilon(u'\cdot\nabla)u' + \epsilon\nabla p_2' = \dfrac{1}{R}\Delta u' \\ \nabla\cdot u' = 0 \\ u'(x,0,t) = (0,0) \\ u'(x,1,t) = (0,0) \\ u'(x,y,t) = u'(x+1,y,t) \\ u'(x,y,0) = f'(x,y). \end{cases}$$

The functions p_1' and p_2' are given in terms of u' by

$$\begin{cases} \Delta p_1' = -\nabla\cdot((u'\cdot\nabla)U) - \nabla\cdot((U\cdot\nabla)u') \\ p'_{1y}(x,0,t) = \dfrac{1}{R}u'_{2yy}(x,0,t) \\ p'_{1y}(x,1,t) = \dfrac{1}{R}u'_{2yy}(x,1,t) \end{cases}$$

and

$$\begin{cases} \Delta p_2' = -\nabla\cdot((u'\cdot\nabla)u') \\ p'_{2y}(x,0,t) = 0 \\ p'_{2y}(x,1,t) = 0. \end{cases}$$

As we show in section 6, the functions p_1' and p_2' can be estimated in terms of u' by

$$\begin{aligned} \|\nabla p_1'(\cdot,\cdot,t)\|^2 &\leq C\left(\|u'(\cdot,\cdot,t)\|^2_{H^1} + \frac{1}{R^2}\|u'_{2yy}(\cdot,\cdot,t)\| + \frac{1}{R^2}\|u'_{2yyy}(\cdot,\cdot,t)\|\right), \forall t \geq 0, \\ \|\nabla p_2'(\cdot,\cdot,t)\|^2 &\leq \|(u'\cdot\nabla)u'(\cdot,\cdot,t)\|^2, \forall t \geq 0. \end{aligned}$$

¿From now on, to simplify the notation, we drop the $'$ in the equations above, and just write u, p_1, p_2. With this notation, the equations above are

$$\begin{cases} u_t + (u\cdot\nabla)U + (U\cdot\nabla)u + \nabla p_1 + \epsilon(u\cdot\nabla)u + \epsilon\nabla p_2 = \dfrac{1}{R}\Delta u \\ \nabla\cdot u = 0 \\ u(x,0,t) = (0,0) \\ u(x,1,t) = (0,0) \\ u(x,y,t) = u(x+1,y,t) \\ u(x,y,0) = f(x,y), \end{cases} \tag{6}$$

$$\begin{cases} \Delta p_1 = -\nabla\cdot((u\cdot\nabla)U) - \nabla\cdot((U\cdot\nabla)u) \\ p_{1y}(x,0,t) = \dfrac{1}{R}u_{2yy}(x,0,t) \\ p_{1y}(x,1,t) = \dfrac{1}{R}u_{2yy}(x,1,t) \end{cases} \tag{7}$$

and

$$\begin{cases} \Delta p_2 = -\nabla \cdot ((u \cdot \nabla)u) \\ p_{2y}(x, 0, t) = 0 \\ p_{2y}(x, 1, t) = 0. \end{cases} \tag{8}$$

Note that p_1 depends linearly on u. Moreover, for all $t \geq 0$, we have

$$\|\nabla p_1(\cdot, \cdot, t)\|^2 \leq C\left(\|u(\cdot, \cdot, t)\|_{H^1}^2 + \frac{1}{R^2}\|u_{2yy}(\cdot, \cdot, t)\| + \frac{1}{R^2}\|u_{2yyy}(\cdot, \cdot, t)\|\right) \tag{9}$$

$$\|\nabla p_2(\cdot, \cdot, t)\|^2 \leq \|(u \cdot \nabla)u(\cdot, \cdot, t)\|^2. \tag{10}$$

When the initial data is divergence free and the terms of pressure are given by the equations (7) and (8) above, the solution u of problem (6) remains divergence free for all time t. Therefore, we drop the continuity equation and write problem (6) as

$$\begin{cases} u_t = \mathcal{L}u - \epsilon(u \cdot \nabla)u - \epsilon\nabla p_2 \\ u(x, 0, t) = (0, 0) \\ u(x, 1, t) = (0, 0) \\ u(x, y, t) = u(x + 1, y, t) \\ u(x, y, 0) = f(x, y), \end{cases} \tag{11}$$

where $\mathcal{L}$ is a linear operator depending on the parameter R, defined by

$$\mathcal{L}u = \frac{1}{R}\Delta u - (u \cdot \nabla)U - (U \cdot \nabla)u - \nabla p_1, \tag{12}$$

with p_1 given by (7). It is very important to note that this linear operator has also an integral part, which is the term ∇p_1. Moreover, as inequality (9) shows, to estimate $\|\nabla p_1\|$ one needs three space derivatives of the second component of u, at least in one of the directions.

This integral part of the operator $\mathcal{L}$ seems to be the point which was so far overseen. If one neglects that and consider only to the differential part of $\mathcal{L}$, one will be led to conclude that $\| \cdot \|_{H^2}$ will be enough to estimate the right hand side of (12).

We first apply the resolvent technique to prove stability of the stationary flow. For that end, it is convenient to have homogeneous initial conditions. Therefore, we transform the problem (11) to a similar problem with homogeneous initial condition by defining

$$v(x, y, t) := u(x, y, t) - e^{-t}f(x, y). \tag{13}$$

Note that v and u have the same behavior as $t \to \infty$. Moreover, v given by (13) satisfies

$$\begin{cases} v_t = \mathcal{L}v - \epsilon\{(v \cdot \nabla)v + e^{-t}(v \cdot \nabla)f + e^{-t}(f \cdot \nabla)v\} - \epsilon\nabla p_2 + F(x, y, t) \\ v(x, 0, t) = (0, 0) \\ v(x, 1, t) = (0, 0) \\ v(x, y, t) = v(x + 1, y, t) \\ v(x, y, 0) = (0, 0), \end{cases} \tag{14}$$

where $F(x, y, t) = e^{-t}((\mathcal{L}+I)f - \epsilon e^{-t}(f \cdot \nabla)f)$. Note that F, $F_t \in L_2([0, \infty); L_2(\Omega))$, that is, both $\|F(\cdot, \cdot, t)\|^2$ and $\|F_t(\cdot, \cdot, t)\|^2$ are integrable over $[0, \infty)$.

4 Linear Problem

We first consider the general linear problem

$$\begin{cases} v_t = \mathcal{L}v + F(x,y,t) \\ v(x,0,t) = (0,0) \\ v(x,1,t) = (0,0) \\ v(x,y,t) = v(x+1,y,t) \\ v(x,y,0) = (0,0), \end{cases} \tag{15}$$

where $\|F(\cdot,t)\|^2$ and $\|F_t(\cdot,t)\|^2$ integrable over the domain $[0,\infty)$:

$$\int_0^\infty \left(\|F(\cdot,t)\|^2 + \|F_t(\cdot,t)\|^2\right) dt < \infty.$$

In our case of two spatial dimensions, resolvent estimates were found in [1]:

$$\|\widetilde{v}(\cdot,s)\|^2 \le C_1 R^2 \|\widetilde{F}(\cdot,s)\|^2 \quad , \quad \mathrm{Re}s \ge 0, \tag{16}$$

where $\widetilde{}$ stands for the Laplace transform with respect to t , s is its variable and C_1 is an absolute constant, that is, it does not depend on any of the parameters or functions. One can prove, as in [5], Appendix A, that (16) implies

$$\|\widetilde{v}(\cdot,s)\|^2_{\widetilde{H}} \le CR^2 \|\tilde{F}(\cdot,s)\|^2 \tag{17}$$

where C depends on C_1 and on U and its first derivative. Since for our problem U is fixed as the Couette flow, C is an absolute constant as well. From now on, we will use C for any absolute constant, and replace its value as necessary keeping the notation C. No attempt is made to optimize those constants, since the most important result is the dependence of the threshold amplitude on the Reynolds number.

Using Parseval's relation, inequality (17) for the transformed functions is translated to the original functions as

$$\int_0^\infty \|v(\cdot,t)\|^2_{\widetilde{H}} dt \le CR^2 \int_0^\infty \|F(\cdot,t)\|^2 dt. \tag{18}$$

Obviously, $\int_0^T \|v(\cdot,t)\|^2_{\widetilde{H}} dt \le \int_0^\infty \|v(\cdot,t)\|^2_{\widetilde{H}} dt$, $\forall T \ge 0$. Moreover, since the solution of the equation up to time T does not depend on the forcing $F(x,y,t)$ for $t > T$, we have

$$\int_0^T \|v(\cdot,t)\|^2_{\widetilde{H}} dt \le CR^2 \int_0^T \|F(\cdot,t)\|^2 dt \quad , \quad \forall T \ge 0. \tag{19}$$

For our argument, we also need similar estimates for v_t. To this end, differentiate equation (15) to get

$$\begin{cases} v_{tt} = \mathcal{L}v_t + F_t(x,y,t) \\ v_t(x,0,t) = (0,0) \\ v_t(x,1,t) = (0,0) \\ v_t(x,y,t) = v_t(x+1,y,t) \\ v_t(x,y,0) = F(x,y,0) =: g(x,y), \end{cases} \tag{20}$$

that is, v_t satisfies an equation of the same type as (15), but with non-homogeneus initial conditions $g(x,y) = F(x,y,0)$. Performing the same type of initialization as before, that is, defining $\varphi := v_t - e^{-t}g$, we get a similar problem for φ, with homogeneus initial conditions and an extra forcing term. Using the estimates for the resolvent, and writing those in terms of v_t, we get

$$\begin{aligned}\int_0^T \|v_t(\cdot,t)\|_{\tilde{H}}^2 dt \quad &\leq \quad \|F(x,y,0)\|_{\tilde{H}}^2 + CR^2\|(\mathcal{L}+I)F(x,y,0)\|^2 \\ &\qquad + CR^2\int_0^T \|F_t(\cdot,t)\|^2 dt \quad , \quad \forall T \geq 0.\end{aligned} \tag{21}$$

Combining (19) and (21) gives, for v the solution of (15),

$$\begin{aligned}&\int_0^T \left(\|v(\cdot,t)\|_{\tilde{H}}^2 + \|v_t(\cdot,t)\|_{\tilde{H}}^2\right) dt \leq \|F(x,y,0)\|_{\tilde{H}}^2 + CR^2\|(\mathcal{L}+I)F(x,y,0)\|^2 \\ &+ CR^2\int_0^T \left(\|F(\cdot,t)\|^2 + \|F_t(\cdot,t)\|^2\right) dt \quad , \quad \forall T \geq 0.\end{aligned} \tag{22}$$

Now, using these estimates for the solution of the linear problem, we can prove a stability result for the nonlinear equation.

5 Stability for the Nonlinear Problem

The nonlinear problem (14) is

$$\begin{cases} v_t = \mathcal{L}v - \epsilon\{(v\cdot\nabla)v + e^{-t}(v\cdot\nabla)f + e^{-t}(f\cdot\nabla)v\} - \epsilon\nabla p_2 + F(x,y,t) \\ v(x,0,t) = (0,0) \\ v(x,1,t) = (0,0) \\ v(x,y,t) = v(x+1,y,t) \\ v(x,y,0) = (0,0), \end{cases} \tag{23}$$

where $F(x,y,t) = e^{-t}((\mathcal{L}+I)f - \epsilon e^{-t}(f\cdot\nabla)f)$. We prove the following:

THEOREM 5.1 *There exists $\epsilon_0 > 0$, $\epsilon_0 = \epsilon_0(R)$, such that if $0 \leq |\epsilon| < \epsilon_0$, then the solution $v(x,y,t)$ of (23) satisfies*

$$\lim_{t\to\infty} |v(\cdot,t)|_\infty = 0.$$

Moreover, $\epsilon_0 = \mathcal{O}(R^{-3})$.

Proof: We consider problem (23) as a linear problem with forcing

$$G(x,y,t) := F(x,y,t) - \epsilon\{(v\cdot\nabla)v + e^{-t}(v\cdot\nabla)f + e^{-t}(f\cdot\nabla)v\} - \epsilon\nabla p_2. \tag{24}$$

Applying inequality (22) with forcing term G gives

$$\begin{aligned}&\int_0^T \left(\|v(\cdot,t)\|_{\tilde{H}}^2 + \|v_t(\cdot,t)\|_{\tilde{H}}^2\right) dt \leq \|G(x,y,0)\|_{\tilde{H}}^2 + CR^2\|(\mathcal{L}+I)G(x,y,0)\|^2 + \\ &\qquad + CR^2\int_0^T \left(\|G(\cdot,t)\|^2 + \|G_t(\cdot,t)\|^2\right) dt \quad \forall T \geq 0.\end{aligned} \tag{25}$$

¿From the definition of G, we have

$$\int_0^T \left(\|v(\cdot,t)\|^2_{\widetilde{H}} + \|v_t(\cdot,t)\|^2_{\widetilde{H}}\right) dt \le 2\|F(x,y,0)\|^2_{\widetilde{H}} + 2\epsilon^2\|\nabla p_2(x,y,0)\|^2_{\widetilde{H}}$$
$$+CR^2\|(\mathcal{L}_R+\mathcal{I})F(x,y,0)\|^2 + CR^2\|(\mathcal{L}_R+\mathcal{I})p_2(x,y,0)\|^2$$
$$+ CR^2\int_0^T \left(\|F - \epsilon\{(v\cdot\nabla)v + e^{-t}(v\cdot\nabla)f + e^{-t}(f\cdot\nabla)v\} - \epsilon\nabla p_2\|^2\right) dt$$
$$+ CR^2\int_0^T \left(\|(F - \epsilon\{(v\cdot\nabla)v + e^{-t}(v\cdot\nabla)f + e^{-t}(f\cdot\nabla)v\} - \epsilon\nabla p_2)_t\|^2\right) dt. \tag{26}$$

Since p_2 is given by (8), we have (see section 6)

$$\|\nabla p_2\| \le \|(u\cdot\nabla)u\| \quad ; \quad \|(\nabla p_2)_t\| \le \|((u\cdot\nabla)u)_t\|.$$

Thus, using (13), we can estimate ∇p_2 by f and v. Moreover,

$$\|\nabla p_2(\cdot,\cdot,0)\|^2 \le \|(u\cdot\nabla)u(\cdot,\cdot,0)\|^2 = \|(f\cdot\nabla)f\|^2,$$

and since $\|f\|^2_{H^6_m} = 1$, inequality (26) gives

$$\int_0^T \left(\|v(\cdot,t)\|^2_{\widetilde{H}} + \|v_t(\cdot,t)\|^2_{\widetilde{H}}\right) dt \le \|F(x,y,0)\|^2_{\widetilde{H}} + CR^2\|(\mathcal{L}+I)F(x,y,0)\|^2$$
$$+ CR^2\int_0^\infty \left(\|F\|^2 + \|F_t\|^2\right) dt + CR^2\epsilon^2\int_0^T \left(\|(v\cdot\nabla)v\|^2 + \|(v_t\cdot\nabla)v\|^2 + \|(v\cdot\nabla)v_t\|^2\right) dt$$
$$+ CR^2\epsilon^2\int_0^T \left(\|e^{-t}(v\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v\|^2 + \|e^{-t}(v_t\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v_t\|^2\right) dt.$$

Since

$$F(x,y,t) = e^{-t}((\mathcal{L}+I)f - \epsilon e^{-t}(f\cdot\nabla)f),$$

we have $F(x,y,0) = (\mathcal{L}+I)f - \epsilon(f\cdot\nabla)f := \mathcal{P}f$. With this notation, the inequality above is

$$\int_0^T \left(\|v(\cdot,t)\|^2_{\widetilde{H}} + \|v_t(\cdot,t)\|^2_{\widetilde{H}}\right) dt \le \|\mathcal{P}f\|^2_{\widetilde{H}} + CR^2\|(\mathcal{L}+I)\mathcal{P}f\|^2 + CR^2\|(\mathcal{L}+I)f\|^2$$
$$+ CR^2\epsilon^2\|(f\cdot\nabla)f\|^2 + CR^2\epsilon^2\int_0^T \left(\|(v\cdot\nabla)v\|^2 + \|(v_t\cdot\nabla)v\|^2 + \|(v\cdot\nabla)v_t\|^2\right) dt$$
$$+ CR^2\epsilon^2\int_0^T \left(\|e^{-t}(v\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v\|^2 + \|e^{-t}(v_t\cdot\nabla)f\|^2 + +\|e^{-t}(f\cdot\nabla)v_t\|^2\right) dt. \tag{27}$$

To apply the resolvent method, one needs control over the terms depending on f of the right hand side of inequality (27). Its second term is

$$\|(\mathcal{L}+I)\mathcal{P}f\| = \|(\mathcal{L}+I)((\mathcal{L}+I)f - \epsilon(f\cdot\nabla)f)\|.$$

To estimate $\|\mathcal{L}^2 f\|$ it is necessary to have, at least in the y direction, control over six derivatives of f_2, the second component of f. The reason is that three derivatives of f_2 in the y direction are necessary to control $\mathcal{L}f$, due to the integral part of the operator $\mathcal{L}$. Moreover, derivatives of different orders of the perturbation may have different scales with respect to the Reynolds number. These facts are shown in details in section 6. As already mentioned, clarifying and showing these estimates in details is our aim, since it is a point that has been overseen in previous works, and lead to mistakes about the necessary assumptions on f.

We now continue the proof of stability. As mentioned above, since $\|f\|^2_{H^6_m} = 1$, we can replace all the terms depending on f by an absolute constant and write inequality (27) as

$$\begin{aligned}\int_0^T \left(\|v\|^2_{\widetilde{H}} + \|v_t\|^2_{\widetilde{H}}\right) dt \leq CR^2 + CR^2\epsilon^2 \int_0^T \|(v\cdot\nabla)v\|^2 dt \\ + CR^2\epsilon^2 \int_0^T \left(\|(v_t\cdot\nabla)v\|^2 + \|(v\cdot\nabla)v_t\|^2\right) dt \\ + CR^2\epsilon^2 \int_0^T \left(\|e^{-t}(v\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v\|^2\right) dt \\ + CR^2\epsilon^2 \int_0^T \left(\|e^{-t}(v_t\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v_t\|^2\right) dt.\end{aligned} \tag{28}$$

¿From now on, we fix the constant C. To finish the proof, we use the following Lemma, which is proved later:

Lemma 5.1 *There exists $\epsilon_0 > 0$, $\epsilon_0 = \mathcal{O}(R^{-3})$, such that if $0 \leq \epsilon < \epsilon_0$ then*

$$\int_0^T \left(\|v(\cdot,t)\|^2_{\widetilde{H}} + \|v_t(\cdot,t)\|^2_{\widetilde{H}}\right) dt < 2CR^2\ ,\ \forall\, T \geq 0. \tag{29}$$

Now, using (5) and a simple one dimensional sobolev inequality, we have

$$\max_{a\leq t\leq b} |v(\cdot,t)|^2_\infty \leq \widetilde{C}R \max_{a\leq t\leq b} \|v(\cdot,t)\|^2_{\widetilde{H}} \leq \widetilde{C}R\left(1+\frac{1}{b-a}\right)\int_a^b \|v(\cdot,t)\|^2_{\widetilde{H}}dt + \int_a^b \|v_t(\cdot,t)\|^2_{\widetilde{H}}dt.$$

This implies

$$\sup_{a\leq t} |v(\cdot,t)|^2_\infty \leq \widetilde{C}R \int_a^\infty \left(\|v(\cdot,t)\|^2_{\widetilde{H}} + \|v_t(\cdot,t)\|^2_{\widetilde{H}}\right) dt. \tag{30}$$

Note that in view of Lemma 5.1, the right hand side of inequality (30) is finite. Letting $a \to \infty$ in (30), we have that $\lim_{t\to\infty} |v(\cdot,t)|^2_\infty = 0$, which proves the theorem.

Proof of Lemma 5.1: First, note that for $T > 0$ small enough, inequality (29) obviously holds. Now, suppose it does not hold for all $T \geq 0$, that is, there exists $T_0 > 0$ such that

$$\int_0^{T_0} \left(\|v(\cdot,t)\|^2_{\widetilde{H}} + \|v_t(\cdot,t)\|^2_{\widetilde{H}}\right) dt = 2CR^2. \tag{31}$$

Using (28), we have:

$$2CR^2 = \int_0^{T_0} \left(\|v\|_{\widetilde{H}}^2 + \|v_t\|_{\widetilde{H}}^2 \right) dt \le CR^2$$
$$+ CR^2\epsilon^2 \int_0^{T_0} \left(\|(v\cdot\nabla)v\|^2 + \|(v_t\cdot\nabla)v\|^2 + \|(v\cdot\nabla)v_t\|^2 \right) dt$$
$$+ \int_0^{T_0} \left(\|e^{-t}(v\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v\|^2 + \|e^{-t}(v_t\cdot\nabla)f\|^2 + \|e^{-t}(f\cdot\nabla)v_t\|^2 \right) dt. \tag{32}$$

We now estimate the integrands on the right hand side of inequality (32) by the integral on its left hand side. To this end, we will use the inequalities (5) and

$$\max_{0\le t\le T_0} \|v(\cdot,t)\|_{\widetilde{H}}^2 \le \int_0^{T_0} \left(\|v(\cdot,t)\|_{\widetilde{H}}^2 + \|v_t(\cdot,t)\|_{\widetilde{H}}^2 \right) dt = 2CR^2. \tag{33}$$

Since $\|v\|_{\widetilde{H}}^2 = \|v\|^2 + \frac{1}{R}\|Dv\|^2 + \frac{1}{R^2}\|v_{xy}\|^2$, we have $\|Dv\|^2 \le R\|v\|_{\widetilde{H}}^2$. Therefore, for each $0 \le t \le T_0$:

$$\begin{aligned} \|\{(v\cdot\nabla)v\}(\cdot,t)\|^2 &\le |v(\cdot,t)|_\infty^2 \|Dv(\cdot,t)\|^2 \\ &\le \left(\widetilde{C}R\|v(\cdot,t)\|_{\widetilde{H}}^2\right)\left(R\|v(\cdot,t)\|_{\widetilde{H}}^2\right) \le 2\widetilde{C}CR^4\|v(\cdot,t)\|_{\widetilde{H}}^2, \end{aligned} \tag{34}$$

$$\begin{aligned} \|\{(v_t\cdot\nabla)v\}(\cdot,t)\|^2 &\le |v_t(\cdot,t)|_\infty^2 \|Dv(\cdot,t)\|^2 \\ &\le \left(\widetilde{C}R\|v_t(\cdot,t)\|_{\widetilde{H}}^2\right)\left(R\|v(\cdot,t)\|_{\widetilde{H}}^2\right) \le 2\widetilde{C}CR^4\|v_t(\cdot,t)\|_{\widetilde{H}}^2, \end{aligned} \tag{35}$$

$$\begin{aligned} \|\{(v\cdot\nabla)v_t\}(\cdot,t)\|^2 &\le |v(\cdot,t)|_\infty^2 \|Dv_t(\cdot,t)\|^2 \\ &\le \left(\widetilde{C}R\|v(\cdot,t)\|_{\widetilde{H}}^2\right)\left(R\|v_t(\cdot,t)\|_{\widetilde{H}}^2\right) \le 2\widetilde{C}CR^4\|v_t(\cdot,t)\|_{\widetilde{H}}^2, \end{aligned} \tag{36}$$

$$\|e^{-t}\{(v\cdot\nabla)f\}(\cdot,t)\|^2 \le e^{-2t}|v(\cdot,t)|_\infty^2\|Df\|^2 \le \widetilde{C}R\|v(\cdot,t)\|_{\widetilde{H}}^2, \tag{37}$$

$$\|e^{-t}\{(f\cdot\nabla)v\}(\cdot,t)\|^2 \le e^{-2t}|f|_\infty^2\|Dv(\cdot,t)\|^2 \le R\|v(\cdot,t)\|_{\widetilde{H}}^2, \tag{38}$$

$$\|e^{-t}\{(v_t\cdot\nabla)f\}(\cdot,t)\|^2 \le e^{-2t}|v_t(\cdot,t)|_\infty^2\|Df\|^2 \le \widetilde{C}R\|v_t(\cdot,t)\|_{\widetilde{H}}^2, \tag{39}$$

$$\|e^{-t}\{(f\cdot\nabla)v_t\}(\cdot,t)\|^2 \le e^{-2t}|f|_\infty^2\|Dv_t(\cdot,t)\|^2 \le R\|v_t(\cdot,t)\|_{\widetilde{H}}^2. \tag{40}$$

Applying (34), (35), (36), (37), (38), (39), (40) to (32) gives

$$\begin{aligned} &2CR^2 \le CR^2 + CR^2\epsilon^2\left\{6\widetilde{C}CR^4\int_0^{T_0}\left(\|v(\cdot,t)\|_{\widetilde{H}}^2 + \|v_t(\cdot,t)\|_{\widetilde{H}}^2\right)dt\right\} \\ &= CR^2 + CR^2\epsilon^2\left\{12\widetilde{C}C^2R^6\right\}. \end{aligned} \tag{41}$$

This implies

$$1 \le 12\widetilde{C}C^2R^6\epsilon^2, \tag{42}$$

which is equivalent to

$$\epsilon \geq \frac{1}{CR^3\sqrt{12\widetilde{C}}} = \frac{1}{KR^3} \tag{43}$$

where $K := C\sqrt{12\widetilde{C}}$. Therefore, if $\epsilon < \dfrac{1}{KR^3}$, equality (31) never holds. This proves the Lemma.

6 Estimates for the pressure terms

We now turn our attention to our main objective, which is to show why one needs to assume control over at least six derivatives of f_2 in the y direction to apply the method above. This necessity follows from estimates (9) and (10) for the pressure terms $p_1(x, y, t)$ and $p_2(x, y, t)$. So, we begin by showing these estimates.

THEOREM 6.1 *If $p_1(x, y, t)$, $p_2(x, y, t)$ are the solutions of*

$$\begin{cases} \Delta p_1 = -\nabla \cdot ((u \cdot \nabla)U) - \nabla \cdot ((U \cdot \nabla)u) \\ p_{1y}(x, 0, t) = \dfrac{1}{R} u_{2yy}(x, 0, t) \\ p_{1y}(x, 1, t) = \dfrac{1}{R} u_{2yy}(x, 1, t) \end{cases} \tag{44}$$

and

$$\begin{cases} \Delta p_2 = -\nabla \cdot ((u \cdot \nabla)u) \\ p_{2y}(x, 0, t) = 0 \\ p_{2y}(x, 1, t) = 0, \end{cases} \tag{45}$$

then

$$\|\nabla p_1(\cdot, \cdot, t)\|^2 \leq C\left(\|u(\cdot, \cdot, t)\|^2_{H^1} + \frac{1}{R^2}\|u_{2yy}(\cdot, \cdot, t)\| + \frac{1}{R^2}\|u_{2yyy}(\cdot, \cdot, t)\|\right) \tag{46}$$

$$\|\nabla p_2(\cdot, \cdot, t)\|^2 \leq \|(u \cdot \nabla)u(\cdot, \cdot, t)\|^2. \tag{47}$$

for all $t \geq 0$, where C is an absolute constant.

Note that the inequalities above are for norms with respect to the space variables x and y. Therefore, to simplify the notation, we prove them for functions depending only on these variables. As before, $\Omega = [0, 1] \times [0, 1]$, and all norms are over Ω. For clarity of the presentation, we separate the proof of Theorem 6.1 into two Lemmas.

LEMMA 6.1 *Let $g : \mathbb{R} \times [0, 1] \to \mathbb{R}^2$, $g(x, y) = (g_1(x, y), g_2(x, y))$, be a C^∞ function satisfying*

$$g(x, 1) = g(x, 0) = (0, 0) \tag{48}$$

$$g(x, y) = g(x + 1, y) \, \forall \, x \in \mathbb{R}. \tag{49}$$

If $h : \mathbb{R} \times [0, 1] \to \mathbb{R}$ is the solution of

$$\begin{cases} \Delta h = \nabla \cdot g \\ h_y(x, 0) = 0 \\ h_y(x, 1) = 0 \\ h(x, y) = h(x + 1, y), \end{cases} \tag{50}$$

then

$$\|\nabla h\|^2 \leq \|g\|^2. \tag{51}$$

Proof: If $\Delta h = \nabla \cdot g$, then

$$\int_\Omega (h_{xx} + h_{yy})h \, dxdy = \int_\Omega (g_{1x} + g_{2y})h \, dxdy.$$

Through integration by parts,

$$-\int_\Omega (h_x^2 + h_y^2) \, dxdy + \int_{\partial\Omega} h_x h \nu^x \, dS + \int_{\partial\Omega} h_y h \nu^y \, dS =$$
$$-\int_\Omega g_1 h_x \, dxdy - \int_\Omega g_2 h_y \, dxdy + \int_{\partial\Omega} g_1 h \nu^x \, dS + \int_{\partial\Omega} g_2 h \nu^y \, dS,$$

where ν^x and ν^y denote the components of the outer normal to $\partial\Omega$ in the x and y directions respectively. ¿From the conditions satisfied by h and g at the boundary, the boundary integrals above vanish. Then,

$$\|\nabla h\|^2 = \int_\Omega (h_x^2 + h_y^2) \, dxdy = \int_\Omega g_1 h_x \, dxdy + \int_\Omega g_2 h_y \, dxdy.$$

Using the Cauchy-Schwarz inequality,

$$\begin{aligned} \|\nabla h\|^2 \leq \|g_1\|\|h_x\| + \|g_2\|\|h_y\| &\leq \frac{1}{2}\|g_1\|^2 + \frac{1}{2}\|h_x\|^2 + \frac{1}{2}\|g_2\|^2 + \frac{1}{2}\|h_y\|^2 \\ &= \frac{1}{2}\|g\|^2 + \frac{1}{2}\|\nabla h\|^2. \end{aligned}$$

This implies the desired estimate

$$\|\nabla h\|^2 \leq \|g\|^2.$$

The Lemma above gives inequality (47) for $p_2(x, y, t)$, the solution of (45).

As mentioned before, the estimates to be proved do not depend on the variable t. Therefore, we write simply $u(x, y) = (u_1(x, y), u_2(x, y))$, for u the solution of problem (6). We remind the reader that $U(x, y) = (y, 0)$. We prove the following lemma, completing the proof of Theorem 6.1:

Lemma 6.2 *If* $h : \mathbb{R} \times [0, 1] \to \mathbb{R}$ *is the solution of*

$$\begin{cases} \Delta h = -\nabla \cdot (u \cdot \nabla)U - \nabla \cdot (U \cdot \nabla)u \\ h_y(x, 0) = \dfrac{1}{R} u_{2yy}(x, 0) \\ h_y(x, 1) = \dfrac{1}{R} u_{2yy}(x, 1) \\ h(x, y) = h(x + 1, y), \end{cases} \tag{52}$$

then

$$\|\nabla h\|^2 \leq C\left(\|u\|_{H^1}^2 + \frac{1}{R^2}\|u_{2yy}\|^2 + \frac{1}{R^2}\|u_{2yyy}\|^2\right). \tag{53}$$

Proof: We begin by noting that if h_1, h_2 are the solutions of

$$\begin{cases} \Delta h_1 = -\nabla \cdot (u \cdot \nabla) U - \nabla \cdot (U \cdot \nabla) u \\ h_{1y}(x,0) = 0 \\ h_{1y}(x,1) = 0 \\ h_1(x,y) = h_1(x+1,y), \end{cases} \tag{54}$$

and

$$\begin{cases} \Delta h_2 = 0 \\ h_{2y}(x,0) = \dfrac{1}{R} u_{2yy}(x,0) \\ h_{2y}(x,1) = \dfrac{1}{R} u_{2yy}(x,1) \\ h_2(x,y) = h_2(x+1,y), \end{cases} \tag{55}$$

then $h = h_1 + h_2$ is the solution of problem (52). Therefore, to prove (53), it is sufficient to prove estimates for h_1 and h_2, solutions of (54) and (55) respectively. For h_1, lemma 6.1 implies

$$\|\nabla h_1\|^2 \leq \|(u \cdot \nabla) U + (U \cdot \nabla) u\|^2 \leq 2\|u\|^2 + 2\|Du\|^2 = 2\|u\|^2_{H^1}. \tag{56}$$

To prove the estimates for h_2, expand in a Fourier series in the x direction. The Fourier coefficients $\widehat{h}_2(k,y)$ satisfy

$$-k^2 \widehat{h}_2 + \widehat{h}_2'' = 0 \tag{57}$$

$$\widehat{h}_{2y}(k,0) = \frac{1}{R} \widehat{u}_{2yy}(k,0) \tag{58}$$

$$\widehat{h}_{2y}(k,1) = \frac{1}{R} \widehat{u}_{2yy}(k,1), \tag{59}$$

where $'$ denotes the derivative with respect to y. Consider first the case $k \neq 0$. To simplify notation, let

$$\alpha_k := \frac{1}{R} \widehat{u}_{2yy}(k,0)$$

$$\beta_k := \frac{1}{R} \widehat{u}_{2yy}(k,1).$$

Using a one-dimensional sobolev type inequality, we estimate α_k and β_k by

$$|\alpha_k|^2 \leq \frac{1}{R^2} \max_{0 \leq y \leq 1} |\widehat{u}_{2yy}(k,y)|^2 \leq \frac{C}{R^2} \left(\|\widehat{u}_{2yy}(k,\cdot)\|^2 + \|\widehat{u}_{2yyy}(k,\cdot)\|^2 \right) \tag{60}$$

$$|\beta_k|^2 \leq \frac{1}{R^2} \max_{0 \leq y \leq 1} |\widehat{u}_{2yy}(k,y)|^2 \leq \frac{C}{R^2} \left(\|\widehat{u}_{2yy}(k,\cdot)\|^2 + \|\widehat{u}_{2yyy}(k,\cdot)\|^2 \right) \tag{61}$$

where C is an absolute constant. As before, we keep the notation simple by using C to represent any absolute constant, whose value can possibly change for different inequalities.
The general solution of the differential equation (57) is

$$\widehat{h}_2(k,y) = a_k e^{|k|(y-1)} + b_k e^{-|k|y}. \tag{62}$$

Imposing the boundary conditions (58) and (59), we determine the coefficients a_k and b_k:

$$a_k = \frac{e^{|k|}}{e^{|k|}-e^{-|k|}}\frac{\beta_k}{|k|} - \frac{1}{e^{|k|}-e^{-|k|}}\frac{\alpha_k}{|k|} \quad ; \quad b_k = \frac{1}{e^{|k|}-e^{-|k|}}\frac{\beta_k}{|k|} - \frac{e^{|k|}}{e^{|k|}-e^{-|k|}}\frac{\alpha_k}{|k|}.$$

Therefore,

$$|a_k|^2 \leq C\left(\frac{|\alpha_k^2|}{k^2} + \frac{|\beta_k|^2}{k^2}\right) \tag{63}$$

$$|b_k|^2 \leq C\left(\frac{|\alpha_k^2|}{k^2} + \frac{|\beta_k|^2}{k^2}\right). \tag{64}$$

Using (60), (61), (62), (63), (64), we have

$$\begin{aligned} k^2\|\widehat{h}_2(k,\cdot)\|^2 &\leq Ck^2(|a_k|^2\|e^{|k|(y-1)}\|^2 + |b_k|^2\|e^{-|k|y}\|^2) \leq \frac{C}{|k|}(|\alpha_k|^2+|\beta_k|^2) \\ &\leq C(|\alpha_k|^2+|\beta_k|^2) \leq \frac{C}{R^2}\left(\|\widehat{u}_{2yy}(k,\cdot)\|^2 + \|\widehat{u}_{2yyy}(k,\cdot)\|^2\right) \end{aligned} \tag{65}$$

and

$$\begin{aligned} \|\widehat{h}_{2y}(k,\cdot)\|^2 &\leq Ck^2(|a_k|^2\|e^{|k|(y-1)}\|^2 + |b_k|^2\|e^{-|k|y}\|^2) \leq \frac{C}{|k|}(|\alpha_k|^2+|\beta_k|^2) \\ &\leq C(|\alpha_k|^2+|\beta_k|^2) \leq \frac{C}{R^2}\left(\|\widehat{u}_{2yy}(k,\cdot)\|^2 + \|\widehat{u}_{2yyy}(k,\cdot)\|^2\right). \end{aligned} \tag{66}$$

For the $k=0$ mode, we solve equation (57) under boundary conditions (58), (59) directly. Note that the divergence free condition satisfied by u assures that this problem is solvable, as expected. Differentiating the solution with respect to y, we get

$$\widehat{h}_{2y}(0,y) = \frac{1}{R}\widehat{u}_{2yy}(0,1).$$

Then

$$\|\widehat{h}_{2y}(0,\cdot)\|^2 \leq \frac{1}{R^2}\max_{0\leq y\leq 1}|\widehat{u}_{2yy}(0,y)|^2 \leq \frac{C}{R^2}\left(\|\widehat{u}_{2yy}(0,\cdot)\|^2 + \|\widehat{u}_{2yyy}(0,\cdot)\|^2\right). \tag{67}$$

Using (65), (66) and (67),

$$\begin{aligned} \|\nabla h_2\|^2 &= \sum_{k\in\mathbb{Z}}\left(k^2\|\widehat{h}_2(k,\cdot)\|^2 + \|\widehat{h}_{2y}(k,\cdot)\|^2\right) \\ &\leq \frac{C}{R^2}\sum_{k\in\mathbb{Z}}\left(\|\widehat{u}_{2yy}(k,\cdot)\|^2 + \|\widehat{u}_{2yyy}(k,\cdot)\|^2\right) \\ &= \frac{C}{R^2}\left(\|u_{2yy}\|^2 + \|u_{2yyy}\|^2\right). \end{aligned} \tag{68}$$

Therefore, by (56) and (68), we have that $h = h_1 + h_2$, solution of (52), satisfies

$$\|\nabla h\|^2 \leq C\left(\|u\|_{H^1}^2 + \frac{1}{R^2}\|u_{2yy}\|^2 + \frac{1}{R^2}\|u_{2yyy}\|^2\right). \tag{69}$$

This finishes the proof of the Lemma. This Lemma applied to $p_1(x, y, t)$ completes the proof of theorem 6.1.

In section 5, we used that the terms depending only on f on the right hand side of (27) can be estimated by $C\|f\|^2_{H^6_m}$. This fact is a consequence of Theorem 6.1. In fact, it is easy to see that the term that requires highest derivatives of f to be estimated is $\|\mathcal{L}^2 f\|^2$. Therefore, we should bound this term in a sharp way, that is, estimate it using a norm for f that involves derivatives of lowest possible order. We show this norm to be $\|f\|_{H^6_m}$.

Recall that

$$\mathcal{L}f = \frac{1}{R}\Delta f - (f \cdot \nabla)U - (U \cdot \nabla)f - \nabla p, \tag{70}$$

where $U = (y, 0)$, and p is the solution of

$$\begin{cases} \Delta p = -\nabla \cdot ((f \cdot \nabla)U) - \nabla \cdot ((U \cdot \nabla)f) \\ p_y(x, 0) = \frac{1}{R} f_{2yy}(x, 0) \\ p_y(x, 1) = \frac{1}{R} f_{2yy}(x, 1). \end{cases} \tag{71}$$

Using Theorem 6.1, we get that

$$\|\mathcal{L}f\|^2 \le C\left(\|f\|^2_{H^1} + \frac{1}{R^2}\|f_{xx}\|^2 + \frac{1}{R^2}\|f_{yy}\|^2 + \frac{1}{R^2}\|f_{2yyy}\|^2\right). \tag{72}$$

Therefore,

$$\|\mathcal{L}^2 f\|^2 \le C\left(\|\mathcal{L}f\|^2_{H^1} + \frac{1}{R^2}\|\mathcal{L}f_{xx}\|^2 + \frac{1}{R^2}\|\mathcal{L}f_{yy}\|^2 + \frac{1}{R^2}\|(\mathcal{L}f)_{2yyy}\|^2\right). \tag{73}$$

Straightforward computations show that

$$\begin{aligned} \|\mathcal{L}f\|^2_{H^1} &\le C\left(\|f\|^2_{H^2} + \frac{1}{R^2}\|D^3 f\|^2 + \frac{1}{R^2}\|D^4 f\|^2\right) \\ \frac{1}{R^2}\|\mathcal{L}f_{xx}\|^2 &\le \frac{C}{R^2}\left(\|f\|^2_{H^2} + \|D^3 f\|^2 + \frac{1}{R^2}\|D^4 f\|^2 + \frac{1}{R^2}\|f_{2yyyxx}\|^2\right) \\ \frac{1}{R^2}\|\mathcal{L}f_{yy}\|^2 &\le \frac{C}{R^2}\left(\|f\|^2_{H^2} + \|D^3 f\|^2 + \frac{1}{R^2}\|D^4 f\|^2 + \frac{1}{R^2}\|f_{2yyyyy}\|^2\right) \\ \frac{1}{R^2}\|(\mathcal{L}f)_{2yyy}\|^2 &\le \frac{C}{R^2}\left(\|f_{2yyy}\|^2 + \|f_{2xyyy}\|^2\right) \\ &\quad + \frac{C}{R^4}\left(\|f_{2yyyxx}\|^2 + \|f_{2yyyyy}\|^2 + \|f_{2yyyyyy}\|^2\right). \end{aligned}$$

The inequalities above, together with (73), imply

$$\begin{aligned} \|\mathcal{L}^2 f\|^2 &\le C\left(\|f\|^2_{H^2} + \frac{1}{R^2}\|D^3 f\|^2 + \frac{1}{R^2}\|D^4 f\|^2 + \frac{1}{R^4}\|f_{2xxyyy}\|^2\right. \\ &\quad \left. + \frac{1}{R^4}\|f_{2yyyyy}\|^2 + \frac{1}{R^4}\|f_{2yyyyyy}\|^2\right) = C\|f\|^2_{H^6_m}. \end{aligned} \tag{74}$$

Note that this estimate is sharp in view of theorem 6.1 and the arguments above: one needs at least six derivatives of f_2 to be able to estimate $\|\mathcal{L}^2 f\|^2$.

7 conclusions

As showed in section 5, perturbations with norm $\|\cdot\|_{H^6_m}$ of order R^{-3} decay with time. Note that this also shows that even though one needs control over derivatives of high order of the perturbation, there are different scales for those derivatives. In fact, if

$$\begin{aligned} \|f\|^2_{H^6_m} &= \|f\|^2_{H^2} + \frac{1}{R^2}\|D^3 f\|^2 + \frac{1}{R^2}\|D^4 f\|^2 + \frac{1}{R^4}\|f_{2xxyyy}\|^2 \\ &\quad + \frac{1}{R^4}\|f_{2yyyyy}\|^2 + \frac{1}{R^4}\|f_{2yyyyyy}\|^2 = \mathcal{O}(R^{-6}), \end{aligned} \tag{75}$$

then $\|f\|_{H^2}$ is of order R^{-3}, $\|D^3 f\|$ and $\|D^4 f\|$ are of order R^{-2}, and $\|f_{2xxyyy}\|$, $\|f_{2yyyyy}\|$ and $\|f_{2yyyyyy}\|$ are of order R^{-1}.

In applications, control over this many derivatives is too restrictive. One possible way to avoid this requirement is to incorporate a smoothing property of the system to the argument. We hope to address this question in the future.

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