Again on Lorentz Force and Minkowski Geometry

R. A. Mosna[(1)¤] and M. A. F. Rosa[(2)y]

(1) Instituto de Física Gleb Wataghin,

Universidade Estadual de Campinas,

CP 6165, 13083-970, Campinas, SP, Brazil.

(2) Departamento de Matemática Aplicada,

Universidade Estadual de Campinas,

CP 6065, 13081-970, Campinas, SP, Brazil.

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Abstract

Here the problem of associating a Lorentz transformation valued function or a Lorentz moving frame to a given massive particle's worldline, describing the evolution of its four-velocity, will be discussed. This is equivalent to the problem of …nding a Lorentz Force …eld that, acting on the massive particle, would produce its given worldline. We emphasise the geometrical character of Lorentz Force, its naturalness in Special Relativity and in other space-time theories having the Minkowskian one as its typical tangent space.

The problem can also be viewed as a gauge choice in a $SO(3)$ principal …ber bundle over the particle's worldline. This is a modern view of the approach to the problem given by Walker 1932 and uni…es all the possible Lorentz force laws describing the same curve, since the magnetic parts of the …elds associated to such forces are the local expressions, under the gauge choice, of a unique global $SO(3)$ connection while the electric part of these …elds are local expressions of a section of an associated bundle.

These discussions, as they are presented here, have the didatic value of linking many disciplines as Special and General Relativity, Di¤erential Geometry, Manifolds, Gauge Theories and Connections, in a precise but very comprehensible way. Therefore they could be a stimulating complement for a course in Special Relativity, followed by advanced undergraduate students.

1 Introduction

Our main objective is to discuss the problem of associating a Lorentz transformation valued function or a Lorentz moving frame to a given massive particle's worldline, describing the evolution of its four-velocity, this shall be done in part 2 and 3 of this presentation and is equivalent to the problem of …nding a Lorentz Force …eld that, acting on the massive particle, would produce its given worldline.

¤E-mail address: mosna@i….unicamp.br

yE-mail address: marcio@ime.unicamp.br

This problem can also be put as a gauge choice in a $SO(3)$ principal …ber bundle over the particle's worldline, a modern view of the approach to the problem given by Walker 1932 that uni…es all the possible Lorentz force laws describing the same curve, since the magnetic part of the …eld associated to these forces will be local expressions, under gauge choices, of a unique global $SO(3)$ connection in that bundle, while the electric part of these …elds, local expressions of a section of an associated bundle.

There we shall see that the Minkowskian Frenet-Serret moving frame, studied by Synge 1967, is a solution to this probem. Another solution is the Fermi-Walker transported moving frame, de…ned by Fermi 1922, studied by Levi-Civita 1926 and Walker 1932.

Despite of its study since the initial developements of Relativity theory, this problem continues up to date, as it can be seen in Bini, Felice and Jentzen 1999, Rodrigues, Vaz and Pavsic 1996, Bini and Jentzen 2002.

Buitrago (1995, in EJP) has discussed this problem and also the related problem of the geometrical character of Lorentz Force, he was followed by Argyris et al (1998, in Foundations of Physics Letters). Here we complete some points in these presentations, on the problem of de…ning those moving frames. We begin by discussing in a more precise way the geometrical character of Lorentz Force, emphasising its naturalness in Special Relativity and in space-time theories having the Minkowskian one as its typical tangent space.

These discussions, as they are presented here, have the didatic value of linking many disciplines as Special and General Relativity, Di¤erential Geometry, Manifolds, Gauge Theories and Connections, in a precise but very comprehensible way. Therefore they could be a stimulating complement for a course in Special Relativity, followed by advanced undergraduate students.

Lets begin by the concept of force...

... on the problem of …nding a generalization of force to the relativistic theories, Taylor and Wheeler 1966 have observed that the best way to understand the concept of force is to study the consequences of its abscence. There follows from the assumed homogeneity and isotropy of Minkowsky space-time that a free particle has to follow a straight worldline.

This a contrario senso interpretation identi…es forces with the causes of the non straightness of such worldlines.

The concept of momentum, as Taylor and Wheeler 1963, has to be generalized to Special Relativity in such a way that it be constant for the free particles following straight worldlines, characterizing these states of motion. In that presentation are employed gedankenexperiments and symmetry arguments to show that $m_0 u$ is the unique generalization of momentum to a moving massive particle in the Special Relativity, where u is the particle's unitary four velocity and m_0 its (rest) mass.

Therefore, being force understood as the cause of any change in the particle's state of motion, the following equation is not a speculation but a de…niton of the force acting in such massive particle,

$$\mathsf{F} = \frac{d}{d\tau}(m_0 u) . \tag{1}$$

This force law (where τ is the particle's proper time) was many times proposed, as in Barut 1980 and Rindler 1989. That reference assumes the constancy of the rest mass and this discuss its possible variation by e¤ect of the applied force, employing to the force the adjective pure if not.

Lorentz force is the unique "pure" force compatible with Special Relativity which is linear in the test particle four velocity. There are arguments given for this a¢rmation in many presentations, as in Barut

1964, Rindler 1982 and Buitrago 1995.

This last one also discuss the related problem of associating a variable Lorentz transformation to a given particle's worldline, describing the evolution of the particle's four-velocity, a similar discussion had been done in Walker 1932. From what follows we deduce that this reference was not known in Buitrago 1995 and in Argyris et al 1998.

We reinforce the necessity of certain precision and coherence in the presentation of this delicate theme and call the attention for the following point of Buitrago's exposition, then, in page 114, when considering a massive test particle with 4-velocity parameterised by its proper time, it was written the equation

$$u(\tau + d\tau) = A(\tau)\, u(\tau) \tag{2}$$

relating the 4-velocity in two close points of the trajectory, and after the conclusion that $A(\tau)$ is a Lorentz transformation, it was stated that

> " we shall, henceforth, regard $A(\tau)$ as a linear operator relating $u(\tau + d\tau)$ and $u(\tau)$ in the same frame of reference..."

It should be remembered that this Lorentz transformation is not uniquely determined by equation (2), since for each element of 3-dimensional Euclidean rotation group $SO(3)$ we can …nd a di¤erent Lorentz transformation relating the two given timelike vectors.

That statement was followed by Argyris et al 1998, where Buitrago was taken as starting point and in page 278 of such exposition... on that varying Lorentz transfomation...

> "The transformation (2) can be called an active Lorentz transformation since we observe here a mapping of the four vector $u(\tau)$ de…ned at a point x, into the four vector $u(\tau + d\tau)$, de…ned in the point $x + dx$, both points being in the space time of a single observer." (Argyris et al 1998)."

There a series of heuristic arguments arrives to a conclusion changing the usual concept for the states of polarization of a free photon (for usual we understand that concept stablished since Weinberg 1964).

In part 2, after some de…nitions and and a short digression on Lorentz force, we establish the problem of associating a variable Lorentz transformation to a massive particle´s worldline. We point that this interesting problem is equivalent to consider an arbitrary worldline... and look for an acting Lorentz force which would make the particle to follow the worldline given. Examples of solutions for this problem are given, showing its non uniqueness.

In part 3 it will be noted that this could be put as the problem of making a gauge choice in a $SO(3)$ bundle over the particle's trajectory, this approach could be seen as a modern version for Walker 1932.We point two natural solutions for it, one following a method similar to Frenet-Serret moving frame construction, which directly links the Lorentz force to the worldline invariants as a geometric curve in space time.The other one by employing the more physical idea of Fermi-Walker transportation, corresponding this last procedure, from the point of view of a hypothetical comoving traveller, to a minimum energy Lorentz force choice. Any two solutions are related by an appropriate gauge transformation.

We make a brief observation, in the end of part 2, that the force problem can be discussed similarly in the context of geometric spacetime theories having Minkowsky spacetime as the typical tangent space for their con…guration manifolds. In fact the Lorentz force will appear as the natural force law in the four velocities corresponding to the Machian substitutes of the inertial fictitious forces in the linearized general relativity, which is a model for geometrized weak field theories. The Lorentz force also appears in space-time theories of Kaluza-Klein type, which are the classical model for gauge theories, as a contraction between the gauge field and the particle's charge.

This implies that Lorentz force is not private to electromagnetic interaction appearing naturally in many theories.

>From part 3, if a worldline is given, it is always possible to find a Lorentz force field such that, if it was acting in the particle, this would follow the given worldline... but the physical reality of the force field defined only for a curve, even making one of the natural choices as Frenet-Serret or Fermi-Walker, shouldn't go beyond the perception of a hypothetical traveller's following such trajectory. Our hypothetical traveller can decompose such hypothetical Lorentz field into its eletric and magnetic parts (for some local inertial frame). Of course the electric and magnetic parts of those fields have to be interpreted by him as the generators of boosts and space rotations.

We have noticed such decomposition's appearence from studies of Linear General Relativity (page 191 of Rindler 1977) to Black Holes (Bini, Carini and Jentzen 1997) and Classical Gauge and Kaluza-Klein theories (page 146 of Bleecker 1981), but this is a kind of fundamental problem, such that, each time we turn to it, we obtain a...

> "New insight into the nature of the electric and magnetic fields" (Buitrago 1995).

...since...

> "An accelerated motion of a test particle (generally, on a curved trajectory) can be considered as a succession of infinitesimal active Lorentz boosts and rotations ,e.g. Buitrago,..."(Argyris et al 1998).

2 Minkowsky space-time, Lorentz Lie group and algebra, Lorentz force

Here we consider $M = \mathbb{R}^{1,3}$, as a four dimensional vector space together the Minkowsky inner product,

$$\langle v, w\rangle = v_0 w_0 - v_1 w_1 - v_2 w_2 - v_3 w_3.$$

This non degenerate product isn't positive definite, dividing the vectors in three classes, the spacelike, the timelike and the lightlike vectors (if their squared norm are respectively negative, positive or null). We shall employ matrices and column vector notation, that is,

$$v^T = (v_0, v_1, v_2, v_3) \ , w^T = (w_0, w_1, w_2, w_3) ,$$

and we can write the inner product as

$$\langle v, w\rangle = v^T \eta w \, , \eta = diag\,(1, -1, -1, -1) \, .$$

The linear transformations $A : M \to M$ are identified with 4×4 matrices, $A \in M(4 \times 4)$,and the isomorphism

$$A = [A_{\mu\nu}] \mapsto (A_{00}, A_{01}, ..., A_{23}, A_{33}),$$

also identifies these with $\mathbb{R}^{16}$ from which they inherit topology and metric.

The group of homogeneous isometries of Minkowsky space defined as

$$O(1,3) = \{A : M \to M, \langle v, w\rangle = \langle Av, Aw\rangle, \ \ for\,all \ \ v, w \in M\}$$

or

$$O(1,3) = \left\{A \in M(4 \times 4), \ \ s.t. A^T \eta A = \eta\right\}$$

is called the full Lorentz group. As $M(4 \times 4) \cong \mathbb{R}^{16}$, with its Euclidean metric and inner product (this can be written as $trA^T B$ for a pair of matrices), that group is a six dimensional manifold inside $\mathbb{R}^{16}$ defined by the ten quadratic equations $A^T \eta A = \eta$, which also imply for $A \in O(1,3)$ that $\det A = \pm 1$, and if $A = [A_{\mu\nu}]$, $A_{00} \geq 1$ or $A_{00} \leq -1$.

The applications $f, g : M(4 \times 4) \to \mathbb{R}$, given by $f(A) = \det A$, $g(A) = A_{00}$, are continuous and therefore split $O(1,3)$ in four connected components, being the Lorentz group

$$\mathcal{L} = \{A \in O(1,3) \ \ s.t. \det A = 1 \text{ and } A_{00} \geq 1\},$$

the component connected to the identity element. As a group and a manifold for which product and inversion operations are differentiable, the Lorentz group is a Lie Group.

We can define the Lie Algebra of a Lie group as its tangent space to the identity element, that is the vector space formed by the vectors tangent to curves through the identity matrix, $LieG = T_I G$ (Curtis 199Z). To determine the Lorentz group Lie Algebra we consider a curve $t \mapsto A(t) \in \mathcal{L}$ through the identity, with $A(0) = I, A'(0) = B$ and derive the relation $\langle A(t)v, A(t)w\rangle = \langle v, w\rangle$ to obtain $\langle Bv, w\rangle + \langle v, Bw\rangle = 0$, then

$$Lie\mathcal{L} = \{B : M \to M \ \ s.t. \langle Bv, w\rangle = -\langle v, Bw\rangle \ \ for\ all\ v, w \in M\}$$

or

$$Lie\mathcal{L} = \left\{B \in M(4 \times 4) \ \ s.t. \ B^T \eta = -\eta B\right\}.$$

The elements of this Lie Algebra are sometimes called by infinitesimal Lorentz transformations, since if these are very close to the identy, their difference to it is an element of the Lie algebra. This can be seen by taking an appropriate curve, like the $t \mapsto A(t)$ above and making its Taylor expansion centered in the identity element to obtain $A \approx I + tB$. Lorentz group Lie algebra can be seen as a six dimensional linear manifold through the origin of $M(4 \times 4) = \mathbb{R}^{16}$, obtained by the translation to the origin of what could be called the "tangent plane to the surface, which is the Lorentz group, through its identity element". Such linear manifold is defined by the ten linear equations $B^T \eta = -\eta B$.

>>From these, if $B = [B_{\mu\nu}]$, $B_{\mu\mu} = 0$, we also have $B_{0j} = B_{j0}$, $B_{jk} = -B_{kj}$ for $1 \leq j, k \leq 3$. That is, always Lorentz Lie algebra matrices can be written as

$$B = \begin{bmatrix} 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ \varepsilon_1 & 0 & \beta_3 & -\beta_2 \\ \varepsilon_2 & -\beta_3 & 0 & \beta_1 \\ \varepsilon_3 & \beta_2 & -\beta_1 & 0 \end{bmatrix}, \ \ \varepsilon_j, \beta_k \in \mathbb{R}. \tag{3}$$

Now we pass from Minkowsky space to Special Relativity by declaring the canonical frame of $M = \mathbb{R}^{1,3}$ an inertial frame and by identifying M with the set of all possible events, being the other inertial frames obtained by the application of Lorentz transformations and translations to the canonical frame (all these transformations generate the so called Poincaré group).

The massive particles are described by their worldlines, and (rest) masses,

$$p \mapsto \gamma(p) \in M, \ \langle \gamma', \gamma' \rangle > 0, \langle \gamma', e_0 \rangle > 0, m_0 > 0,$$

with future pointed (e_0 is the unitary vector in the direction of the time axis for the fixed canonical frame) timelike velocities in all inertial frames, which agree on this and also on the particle's proper time, defined as...

$$\tau(p) = \frac{1}{c} \int_0^p \langle \gamma'(q), \gamma'(q) \rangle^{1/2} dq.$$

If we employ the proper lenght $s = c\tau$ as the parameter, particle's four velocity $u(s) = \gamma'(s)$ has unitary Minkowsky norm, i.e.,

$$\langle u(s), u(s) \rangle = 1. \tag{4}$$

Therefore the derivative of (4) gives us that

$$\left\langle \frac{du}{ds}, u \right\rangle = 0 \tag{5}$$

and the four acceleration has to be orthogonal, in Minkowsky sense, to the velocity.

$>>$From equation (1), a pure force law is given by

$$\mathrm{F} = m_0 \frac{du}{d\tau}, \tag{6}$$

What can be shown as we shall see in what follows, is that given a worldline, its is possibe to find a force law like (6) with a linear dependence on the four velocity, that would imply in it. Anyway, assuming (6), we have from (5) the Minkowsky orthogonality between the force and the four velocity,

$$\langle \mathrm{F}, u \rangle = 0. \tag{7}$$

This implies that the 4-force has to depend on the 4-velocity. The simplest choice of force therefore would be the one having a linear dependence on the 4-velocity. We note that, since the velocity is unitary, this cannot be thought as a linear approximation of any force depending on the 4-velocity. Despite of this observation we shall see above that for a large class of non-linear field theories, the weak field approximation corresponds to a linear dependence in the 4-velocity. Under this hypothesis we write $\mathrm{F} = Fu$, where the associated field $F \in M(4 \times 4)$ is a linear transformation in the Minkowsky spacetime. From (7),

$$\langle Fu, u \rangle = 0 \tag{8}$$

and this should be valid for all timelike vectors, therefore, by linearity, for all vectors in space time.

But this is the same as saying that $F \in Lie\,\mathrm{L}$, since if we substitute $u = v + w$ in (8) we see that this is the same as $\langle Fv, w \rangle = -\langle v, Fw \rangle$. Then we can write this matrix as in (3), and the force law

$$\frac{du}{ds} = Fu \tag{9}$$

can be written as

$$\frac{d}{ds}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ \varepsilon_1 & 0 & \beta_3 & -\beta_2 \\ \varepsilon_2 & -\beta_3 & 0 & \beta_1 \\ \varepsilon_3 & \beta_2 & -\beta_1 & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}. \tag{10}$$

Putting $u^T = (u_0, u_1, u_2, u_3) = (u_0, \mathbf{u})$ we obtain a pair of equations equivalent to (10),

$$\frac{d}{ds}(m_0 u_0) = \varepsilon \cdot \mathbf{u} \quad (16), \tag{11a}$$

$$\frac{d}{ds}(m_0 \mathbf{u}) = u_0 \varepsilon + \mathbf{u} \times \beta \quad (17) \tag{11b}$$

in our canonical frame (we have also put $\beta = (\beta_1, \beta_2, \beta_3)$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{R}^3$ and applied the usual definitions of dot and cross product for triples of real numbers). If we want to employ the coordinate time $t = x_0/c$, from $ds^2 = c^2 dt^2 - (dx_1)^2 - (dx_2)^2 - (dx_3)^2$ we obtain

$$\frac{ds}{cdt} = \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma}, \quad v^2 = |\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}, \quad \mathbf{v} = (v_1, v_2, v_3)$$

where $v_k = \frac{dx^k}{dt} = \gamma u^k$, $0 \leq k \leq 3$, $u^0 = \gamma c$ and the above pair of equations, (11a) and (11b), becomes, if we define $\mathbf{E} = (c/\alpha)\varepsilon$, $\mathbf{B} = (c/\alpha)\beta$, as

$$\frac{d}{dt}\left(mc^2\right) = \alpha \mathbf{E} \cdot \mathbf{v}, \quad m = \gamma m_0$$

$$\frac{d}{dt}(m\mathbf{u}) = \alpha \mathbf{E} + \alpha \frac{\mathbf{v}}{c} \times \mathbf{B}$$

This can be recognized as Lorentz force law in the canonical frame. Its 4-dimensional covariant version is equation (9) and solving this by matrix exponentiation in some particular cases, we see that the canonical frame defined electric and magnetic fields respectively generate boosts and rotations in such frame.

Lets stop this exposition for a (heuristic) moment. We've assumed equation (6), for which (9) is the linear version. Are these the most general pure force laws? We again return to the a contrario senso interpretation of force from Taylor and Wheeler 1963, in fact, a free particled has to follow straight worldlines, being the forces the causes of non straightness, and we have at least the Parsimony's Principle saying that we don't need so many force laws...

Turning to the theory of space curves, we can see there aren't so many curves in that case (therefore not so many ways of changing them) since then there is a theorem of existence and uniqueness (Guggenheimer 1977, Spivak 1975) saying that for given torsion and curvature (as functions of the curve parameter) an analytic curve is uniquely determined. This theory employs the Frenet-Serret frame and as seen in those references,can be generalized to higher dimensions and Riemannian Manifolds. This method can be adaptated to the Minkowsky case when we conclude that a worldline is uniquely defined by its curvature, torsion and hypertorsion. This, as poited by Pauri and Vallisnery 2000, had been done in Synge 1967, who studied some solutions to the corresponding Frenet-Serret equations giving a classification for the timelike helices in space-time.

The massive particle's trajectories in space-time are not much more numerous nor more promiscuous than the space curves... there aren't so many of them... nor so many ways of changing them.

The Frenet-Serret moving frame construction shall correspond to make a choice of a varying Lorentz transformation along the worldline, or equivalently, to the choice of a Lorentz force fitting this curve. The Fermi-Walker transported moving frame, also discussed above, will be another of such choices.

The discussion we are following on the force law and curves could be generalized from Special Relativity to a large class of space-time theories, having the Minkowsky one as their tangent space.

In fact, if we consider the Linearized General Relativity, it's known since Thirring 1918 and well shown in (pages 190-192 of) Rindler 1977, that equation (9) is followed by a test particle in a local comoving frame, where the field derives from a potential obtained from the Christoffel symbols of the Levi-Civita connection. Then the Lorentz force will appear as the natural force law in the four velocities corresponding to the Machian substitutes of the inertial fictitious forces.

The generalized Kaluza-Klein theories are the classical setting for gauge theories, there the space-time structure corresponds to a principal fiber bundle over a base that is identified to General Relativity space-time.We note that it's then possible to show the projection theorem, which says (see page 144 of Bleecker 1981) that a free particle following a geodesic in the total space of the fiber bundle has its trajectory projected onto a forced curve in basis Space-Time. Furthermore, that this curve differs from a geodesic by a force which is linear in the particle's four velocity, being this Lorentz force given by the gauge invariant contraction between the the gauge field and the the gauge valued particle's charge (this one corresponds to the momentum in the internal degrees of freedom).

After this comment we turn to the Minkowsky setting and for what follows lets first to establish the problem of finding a Lorentz transfomation valued function describing the evolution of the four velocity for a given massive particle. In the approach to the problem by Buitrago that corresponds to equation (2). Here we consider an equivalent version,

$$u(s) = A(s)\,u(0)\,,\ A(0) = \mathrm{I},\ A(s)\,\eta A(s) = \eta, \tag{12}$$

then $u(0) = e_0 = (1,0,0,0)^T$ is just the timelike vector of our canonical frame basis. If the the initial acceleration $u'(0)$ is not zero we also suppose that it gives the direction of $e_1 = (0,0,0,1)^T$ of our canonical frame.

It will be usefull to define the group of spatial rotations in the canonical frame

$$SO(3) = \{R \in \mathrm{L},\ R\,e_0 = e_0\}$$

Any of these transformations leaves invariant that timelike vector and has a matrix like

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{bmatrix}$$

where the 3×3 submatrix is a rotation in the three dimensional linear manifold orthogonal to the timelike vector e_0.

Now we observe that there are infinitely many Lorentz valued curves $s \mapsto A(s)$ such that $u(s) = A(s)\,e_0$ satisfying (12), since...

...(i) if $s \mapsto R(s)$ is a rotation valued curve and

$$B(s) = A(s)\,R(s)\,, \tag{13}$$

we get $B(s)\,e_0 = A(s)\,e_0$, furthermore...

...(ii) if $B(s)e_0 = A(s)e_0$, $A^{-1}(s)\,B(s)\,e_0 = e_0$, then $s \mapsto R(s) = A^{-1}(s)\,B(s)$ is a rotation valued curve and $B(s) = A(s)\,R(s)$.

By employing this idea we construct three Lorentz valued curves,

$$A_1(s) = \begin{bmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_2(s) = \begin{bmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & \cos s & -\sin s \\ 0 & 0 & \sin s & \cos s \end{bmatrix},$$

$$A_3(s) = \begin{bmatrix} \cosh s & \sinh s \cos s & \sinh s \sin s \cos s & \sinh s \sin^2 s \\ \sinh s & \cosh s \cos s & \cosh s \sin s \cos s & \cosh s \sin^2 s \\ 0 & -\sin s & \cos^2 s & \cos s \sin s \\ 0 & 0 & -\sin s & \cos s \end{bmatrix},$$

such that all these transform the canonical basis timelike vector $e_0 = (1,0,0,0)^T$ into the first column vector common to the three matrices, satisfying (12) for the same four velocity, $u(s) = (\cosh s, 0, \sinh s, 0)^T$.

It is intesting that the matrix multiplication by rotation in the left can be substituted by an equivalent in the right side of the Lorentz matrix and vice-versa, since

$$A(s)\,e_0 = A(s)\,R(s)\,e_0 = \left(A(s)\,R(s)\,A^{-1}(s)\right)A(s)\,e_0.$$

This action in the left is made by a Lorentz transformation valued curve leaving invariant, for each s, the worldline's velocity $u(s) = A(s)\,e_0$, corresponding to a rotation in a linear three dimensional space manifold orthogonal to it. This corresponds to a curve with values in a group isomorphic to $SO(3)$,

$$s \mapsto A(s)\,SO(3)\,A^{-1}(s).$$

Despite of the non uniqueness of the Lorentz valued curve, in the next part we shall see that this is only an apparent tragedy since the Lorentz matrix of equation (12) can be determined if we fix a certain choice procedure. We shall see two of these wich can be seen as very natural choices. Before this, lets see that for each Lorentz curve $s \mapsto A(s) \in \mathsf{L}$ it corresponds to a Lorentz force choice $s \mapsto F(s) \in Lie\mathsf{L}$ ...

...(i) suppose that equation (12) is valid and derive $A^T(s)\,\eta\, A(s) = \eta$ to obtain

$$A'^T(s)\,\eta A(s) + A^T(s)\,\eta A'(s) = 0,\; A'^T(s)\,\eta + A^T(s)\,\eta A'(s)\,A(s)^{-1} = 0$$

and since $\left(A^T(s)\right)^{-1} A(s)^T = \left\{A'(s)\,A^{-1}(s)\right\}^T$, we have

$$\left(A'(s)\,A^{-1}(s)\right)^T \eta = -\eta A'(s)\,A^{-1}(s),$$

therefore, we define, for a Lorentz valued curve $s \mapsto A(s)$, its logarithmical derivative $F(s) = A'(s)\,A^{-1}(s)$ satisfying

$$A'(s) = F(s)\,A(s) \tag{14}$$

and this defines a Lorentz Lie algebra curve $s \mapsto F(s)$, since from the manipulations above we have $F^T(s)\,\eta = -\eta F(s)$ and applying (14) to e_0 we get Lorentz law of force (9) for $u(s) = A(s)\,e_0$...

(ii)...and conversely, if we assume equation (9), the existence and uniqueness theorem for differential equations implies in equation (12) where

$$A(s) = \exp\left\{ P \int_0^s F(\sigma)\,d\sigma \right\}.$$

3 Moving frames in Minkowsky space-time

Given a set of four vectors in Minkowsky space, we associate to this a frame matrix by

$$(u_0, u_1, u_2, u_3) \mapsto U = \begin{bmatrix} | & | & | & | \\ u_0 & u_1 & u_2 & u_3 \\ | & | & | & | \end{bmatrix}. \tag{15}$$

If someone wants to consider frame matrices as linear transformations acting in Minkowsky spacetime, the corresponding frames are the images of the elements of the canonical orthonormal basis. The canonical basis has $U = I = diag(1,1,1,1)$ as its frame matrix. If $\det U = 0$ the frame is called singular, if not, regular. The regular frames satisfying $U^T \eta U = \eta$, $U_{00} > 1$ and $\det U = 1$ shall be called by Lorentz frames, being the set of Lorentz frames, when considered as linear transformations, identical to the Lorentz group.

With this identification a Lorentz curve as those in the end of the last part is a Lorentz moving frame. There we had associated such Lorentz curve $s \mapsto A(s)$, from now on a Lorentz moving frame, to a uniquely a Lorentz force law, or a Lorentz Lie algebra valued curve $s \mapsto F(s)$, it is also useful to associate, for each s, the so called Cartan matrix of the moving frame, given by

$$s \mapsto \mathcal{F}(s) = A^{-1}(s)\,F(s)\,A(s)$$

and from equation (14) we have

$$\frac{dA}{ds} = A(s)\,\mathcal{F}(s) \quad (29) \tag{16}$$

and this differential equation, as (12) defines $s \mapsto A(s)$ uniquely from the initial condition $A(0) = I$ and from the Lorentz Lie valued curve $s \mapsto \mathcal{F}(s)$ which gives the Cartan matrix evolution. Equation (16) shows that the columns of the cartan matrix give us the Fourier coefficients of the derivatives of the frame vectors with respect to the basis formed by these frame vectors, that is

$$\frac{d}{ds}u_m = \sum_{n=1} \mathcal{F}_m^n\, u_n, \quad \mathcal{F}_m^n = \frac{\langle u_m', u_n\rangle}{\langle u_n, u_n\rangle} \tag{17}$$

where the frame vectors are given as in (15).

Now lets consider a massive particle's worldline parameterised by its proper lenght with timelike velocity $u(s)$, that is $\langle u(s), u(s)\rangle = 1, u_0 > 1$. It is possible to associate a moving frame $s \mapsto U(s)$ to this curve, analogously to the Frenet-Serret moving frame employed in the study of space curves...

...(i) we begin by taking $u_0(s) = u(s)$ as the first column of our frame matrix...

...(ii) and follow with $u_1(s)$, defined by $du_0/ds = ku_1(s)$, the unitary vector in the direction of the spacelike acceleration $u_0'(s)$ (we take $u_1 = 0$ when $u_0' = 0$)...

...(iii) since $\langle u_1, u_0\rangle = 0$ and $\langle u_1, u_1\rangle = -1$ (or 0), we respectively have that $\langle du_1/ds, u_0\rangle = -\langle u_1, du_0/ds\rangle = -k\langle u_1, u_1\rangle = k$ and $\langle du_1/ds, u_1\rangle = 0$, therefore it is possible to write $du_1/ds = ku_0 + v_2$ where v_2 is orthogonal to $span\,\{u_0, u_1\}$ and we define the third column u_2 as the unitary vector in the direction of the spacelike v_2, by $v_2 = -\tau u_2$, $\tau = \langle v_2, u_2\rangle$ (if $v_2 = 0$, we take also $u_2 = 0$)...

...(iv) now $\langle u_2, u_0\rangle = 0$, $\langle u_2, u_1\rangle = 0$ and $\langle u_2, u_2\rangle = -1$ (or 0), we respectively have that $\langle du_2/ds, u_0\rangle = -\langle u_2, du_0/ds\rangle = 0$, $\langle du_2/ds, u_1\rangle = -\langle u_2, du_1/ds\rangle = -\langle u_2, v_2\rangle = -\tau$ and $\langle du_2/ds, u_2\rangle = 0$, therefore it is possible to write $du_2/ds = \tau u_1 + v_3$ where v_3 is orthogonal to $span\,\{u_0, u_1, u_2\}$ and we define the fourth column u_3 as the unitary vector in the direction of the spacelike v_3 by $v_3 = -h u_3$, $h = \langle v_3, u_3\rangle$ (if $v_3 = 0$, we take also $u_3 = 0$).

Someone could say that would be enough to have applied the Gram-Shimidt to, or have scaloned, the frame matrix $[U_{\mu\nu}] = [(d/ds)^{\nu}\, u_{\mu}]$. In fact this is true, but the above detailed construction has an advantage, for this lets make a last consideration...

...(v) from the above $\langle du_3/ds, u_0\rangle = 0$, $\langle du_3/ds, u_1\rangle = -\langle u_0, du_1/ds\rangle = -\langle u_3, hu_0 - \tau v_2\rangle = 0$, $\langle du_3/ds, u_2\rangle = -\langle u_3, du_2/ds\rangle = -\langle u_3, \tau u_1 + v_3\rangle = -\langle u_3, v_3\rangle = -h$, $\langle du_3/ds, u_3\rangle = 0$ and therefore $du_3/ds = hu_2$.

And from these five considerations we get the Minkowskian Frenet-Serret equations for our constructed moving frame,

$$\frac{du_0}{ds} = ku_1,\ \frac{du_1}{ds} = ku_0 - \tau u_2,\ \frac{du_2}{ds} = \tau u_1 - hu_3,\ \frac{du_3}{ds} = hu_2,$$

which can be writen in matrix form,

$$\frac{d}{ds}\begin{bmatrix} | & | & | & | \\ u_0 & u_1 & u_2 & u_3 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ u_0 & u_1 & u_2 & u_3 \\ | & | & | & | \end{bmatrix}\begin{bmatrix} 0 & k & 0 & 0 \\ k & 0 & \tau & 0 \\ 0 & -\tau & 0 & h \\ 0 & 0 & -h & 0 \end{bmatrix} \tag{18}$$

Then, analogously to the space curves, k, τ, h can be interpreted as the curvature, torsion and hypertorsion of the worldline (this last would measure the failure of some curve to belong locally to a tridimensional manifold of Minkowski space).

It is necessary to be carefull to analyse the meaning and the validity of the above equation, since from the way the Frenet-Serret moving frame was constructed, it is not even a continuous function of the proper lenght, the problem is that the frame's dimension can change abruptly. If the constructed moving frame $s \to U(s)$ was a continuous function its rank would depend continuously on the proper lenght and being an integer valued function this rank wouldn't depend on s.

Lets suppose that these dimension changes have occured in a set $\{s_1, s_2, ..., s_3\}$ of values of the proper lenght (put also $s_0 = 0$). For each interval $[s_i, s_j]$ the equation (18) gives us a unique solution $s \to U(s)$ from the initial condition $U(s_i)$. Through the interval (s_i, s_j), the construction was made in such a way that

$$U^T \eta U = \xi,$$

where, depending on the rank of U we have four possibilities for ξ,

$$\xi = \eta,\ \xi = diag(1, -1, -1, 0)$$
$$\xi = diag(1, -1, 0, 0),\ \xi = diag(1, 0, 0, 0).$$

In the four possibilities we have

$$\frac{d}{ds}U^T\eta U = 0,$$

we note then that the rank of the frame $U(s)$ is constant along each open interval (s_i, s_j).

If we look for a continuous frame in the whole curve, its rank is constant and determined by the rank of the initial condition $U(0)$. Therefore the frame constructed by solving the differential equation (31) with the initial condition $A(0) = \mathbf{I}$ will be non singular, defining a Lorentz moving frame $s \to A(s)$ which agrees with the Frenet-Serret moving frame up to its null space (this from the uniqueness of solutions of differential equations). In this way the Frenet-Serret construction determines a Lorentz moving frame and if we don't explicitly state the opposite is this last one what we shall understand by Frenet-Serret moving frame in what follows.

Now it arises a question, why would be this choice for the Lorentz moving frame so special, lets compare for this the three moving Lorentz frames A_1, A_2, A_3 from the last part, no one of them is obtained straightly from Frenet-Serret construction since if we had done it from $u(s) = (\cosh s, \sinh s, 0, 0)^T$, we would obtain

$$U(s) = \begin{bmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and if we calculated the associated field we would get

$$F(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which, integrated and exponentiated would give us

$$A(s) = \begin{bmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

this is matrix $A_1(s)$ and is what we have called by the Frenet-Serret Lorentz moving frame. In our canonical frame, the simplest of those three ways of generating the worldline corresponding to $s \mapsto u(s)$, with a unique boost, if you want, generated by a unitary electric-like field in the x direction.

Lets see the fields of force for the other two. Making the logarithmic derivative,

$$F_2(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$F_3(s) = \begin{bmatrix} 0 & 1 & \sinh s & \sinh s \\ 1 & 0 & \cosh s & \cosh s \\ \sinh s & -\cosh s & 0 & \cos s \\ \sinh s & \cosh s & -\cos s & 0 \end{bmatrix}$$

In the second case the associated frame would have the axis y and z rotating continuously, and a magnetic part in the x direction, not useful for the purpose of constructing the worldline, since this magnetic field is parallel to our particle's velocity. The third case's field would be much more crazy and expensive yet, generating the same curve as the other two. It seems that the Lorentz curve obtained from Frenet-Serret is the simplest way of producing the required worldline and in fact this procedure is very simple way of doing that, but in other cases, we shall see, will be difficult to decide what is the simplest way, if this or of the Fermi-Walker construction.

This has first appeared in Fermi 1922 and as pointed in Bini and Jentzen 2001, that bright scientist inteded to find a rule $s \mapsto n(s)$ to transport a vector (given at $s = 0$) along the worldline in such a way this vector be always orthogonal to the particle's velocity $u(s)$. From the orthogonality relation $\langle n(s), u(s)\rangle = 0$, we have the necessary condition for that

$$\langle n'(s), u(s)\rangle = -\langle n(s), u'(s)\rangle, \tag{19}$$

which becomes also sufficient if the initial condition given is admissible, that is, if $\langle n(0), u(0)\rangle = 0$.

Fermi observed then that a simple way of satisfying this equation is to impose the following differential equation to $n(s)$,

$$\frac{dn}{ds} = -\langle n(s), u'(s)\rangle\, u(s) \tag{20}$$

since, given the initial condition, this defines $n(s)$ uniquely and implies (39). As pointed by Bini and Jentzen 2002, Walker 1932 extended Fermi's transport for not only orthogonal vectors, helding the parallel component of $n(s)$ and imposing (40) to the normal. For a so called Fermi-Walker transported vector $s \mapsto v(s)$ we write

$$p(s) = \langle v(s), u(s)\rangle, \quad v(s) = p(s)\, u(s) + n(s),$$

then impose that $p(s)$ be constant and equation (20) to $n(s)$, what defines $v(s)$ uniquely from the initial condition. This gives us

$$\frac{dv}{ds} = p(s)\frac{du}{ds}(s) + \frac{dn}{ds} = \langle v(s), u(s)\rangle\, a(s) - \langle v(s), a(s)\rangle\, u(s) \tag{21}$$

where $a(s) = u'(s)$. Fermi and Walker where thinking only in geodesic worldlines, Levi-Civita 1938 generalized the above rule for nongeodesic ones, nowadays it's common to say that any vector or tensor field is Fermi-Walker transported along a worldline when it has zero Fermi-Walker derivative with respect to it, being this last one defined by

$$D_{FW} = \frac{d}{ds} + u(s) \otimes \langle a(s)| - a(s) \otimes \langle u(s)|$$

$$\text{that is...} D_{FW} = \frac{d}{ds} + (u(s) \wedge a(s))\cdot \tag{22}$$

where we employed Dirac's bra notation for the contraction with a given vector, the usual definition of wedge product (as in Spivak 1965) and the second term in the right side of (22) acts by contraction in the right.

Lets consider (21) for a given s. If we write this equation in the inertial frame having the two first canonical vectors given by $u(s) = e_0 = (1,0,0,0)^T$ and $a(s) = e_1 = \alpha(0,1,0,0)^T$, we obtain for $v = (v_0, v_1, v_2, v_3)$

that $\langle dv/ds, a(s)\rangle = -\alpha v_0$ and $\langle dv/ds, u(s)\rangle = v_1$, then

$$d\begin{bmatrix} | \\ v \\ | \end{bmatrix} = \begin{bmatrix} | \\ v \\ | \end{bmatrix}\begin{bmatrix} 0 & \alpha ds & 0 & 0 \\ \alpha ds & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where we see that from the point of view of a hypothetical traveller following the worldline, who would employ in that moment such frame, the transport is made by an infinitesimal boost, without rotations. Therefore the Fermi-Walker is the transport rule followed by the axis of any gyroscope transported by our traveller.

We also note that, if v_1, v_2 are Fermi-Walker transported, from (21)

$$\left\langle v_1, \frac{d}{ds}v_2 \right\rangle = \langle v_2, u\rangle \langle v_1, a\rangle - \langle v_2, a\rangle \langle v_1, u\rangle$$

and therefore

$$\frac{d}{ds}\langle v_1, v_2\rangle = \left\langle \frac{d}{ds}v_1, v_2 \right\rangle + \left\langle v_1, \frac{d}{ds}v_2 \right\rangle = 0,$$

being preserved the angles between two of such gyroscopes.

Since our task is to construct moving frames it is a good idea that our traveller carries three of these gyroscopes, these together its four-velocity will define a Fermi-Walker transported moving frame. Initially we suppose that the traveller four velocity and the giroscopes moving frame are aligned with the axis of our canonical frame. In what follows all they obey equation (21). Then we obtain

$$\frac{d}{ds}\begin{bmatrix} | & | & | & | \\ u & v_1 & v_2 & v_3 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ u & v_1 & v_2 & v_3 \\ | & | & | & | \end{bmatrix}\begin{bmatrix} 0 & h_1 & h_2 & h_3 \\ h_1 & 0 & 0 & 0 \\ h_2 & 0 & 0 & 0 \\ h_3 & 0 & 0 & 0 \end{bmatrix}$$

straight from (17), (19) and (20), with $h_j = \langle dv_j/ds, u\rangle$ for $j = 1, 2, 3$.

The equations (18) and (20) together the initial condition of (12), which imposes that the moving frame agree with the canonical frame for $s = 0$,

$$\frac{d}{ds}A = A\,\mathcal{P}\,,\ U(0) = I$$

characterize this moving frame uniquely from its Cartan matrix. As pointed in Jentzen 2002, this characterization was taken in Walker 1932, this author has elected Fermi-Walker transport as the simplest moving frame, this was not a selfish act, since such frame had not its nowadays denomination. As commented by Jentzen ...

> "..., Walker notes that the simplest choice of W amounts to setting B = 0, which corresponds to Fermi-Walker transport of the frame along the curve. For a geodesic, he notes that it is natural to pick the orthonormal frame $\{e_\alpha\}$ to contain its tangent u, and for an accelerated curve, the Frenet-Serret frame (containing u) is suggested, in orther to construct the family of Riemann normal coordinates..."

Mutatis mutandis, Walker was studying the moving frames to construct from them normal coordinates adaptated to a given worldline and what Jentzen means by B = 0, is that the magnetic part of the Cartan matrix, with respect to the particle's four velocity as time direction, is zero for the Fermi-Walker solution, where we have only boosts. We must understand this affirmation as the magnetic part from the point of view of a co-moving traveller, from the point of view of our canonical frame the Lorentz force $F = A\tilde{F}A^{-1}$ is not purely magnetic for this choice.

>> From our discussion in part 2, two moving frames $s \mapsto A(s)$ and $s \mapsto B(s)$, differ by a $SO(3)$ valued function $s \mapsto R(s)$, by equation (13), $B(s) = A(s)R(s)$. We can relate the Cartan matrix fields $\tilde{G} = B^{-1}B'$ and $\tilde{F} = A^{-1}A$ associated to these fields, obtaining

$$\begin{aligned}\tilde{G}(s) &= B^{-1}(s)B'(s) = (A(s)R(s))^{-1}(A(s)R(s))' = \\ &= R^{-1}(s)A^{-1}(s)\{A'(s)R(s) + A(s)R'(s)\}\end{aligned}$$

that is

$$\tilde{G}(s) = R^{-1}(s)\tilde{F}(s)R(s) + R^{-1}(s)R(s) \tag{23}$$

where $s \mapsto \omega(s) = R^{-1}(s)R(s)$ is a spatial rotation group Lie algebra valued curve and can be written as

$$\omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_3 & \omega_2 \\ 0 & \omega_3 & 0 & -\omega_1 \\ 0 & -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

>> From the above transformation rule it can be shown that, from the point view of a traveller following the worldline, the Fermi-Walker is the less energetic choice for the Cartan matrix. We have the inner product $(A, B) = trA^TB$, for $A, B \in M(4 \times 4) = \mathbb{R}^{16}$ and if we write the Cartan matrix as

$$\tilde{F} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

we have that $\left\|\tilde{F}\right\|^2 = \left(\tilde{F}, \tilde{F}\right) = \|E\|^2 + \|B\|^2$, is representing its hypothetical density of electromagnetic energy.

From (23) and taking as $\tilde{F}$ the Fermi-Walker Cartan matrix and as $\tilde{G}$ any other, we get

$$\left\|\tilde{G}\right\|^2 = \left\|R^{-1}\tilde{F}R\right\|^2 + 2tr\left(\tilde{F}^T\omega\right) + \|\omega\|^2 = \left\|\tilde{F}\right\|^2 + \|\omega\|^2 \geq \left\|\tilde{F}\right\|^2,$$

since the similarities by the rotation matrix preserves the inner product and $\left(\tilde{F}^T, \omega\right) = 0$, since these two matrices are orthogonal in $\mathbb{R}^{16} = M(4 \times 4)$.

The transformation rule (23) can be associated with the gauge transformation of a $SO(3)$ potential given by the the magnetic part of the Cartan matrix together the transformation of an associated section given by its eletric part as we show in what follows.

First, following the methods from Bleecker 1981, we define a principal fiber bundle with the particle's curve as the base space, having $SO(3)$ as its structural group.

For each point $\gamma(s)$ of the particle's worldline $\gamma : \mathbb{R} \to M$, the four velocity $u(s) = \gamma'(s)$ defines, as a normal vector, a 3-dimensional subspace $\mathbb{R}^3(s) \subset TM(s)$ of the space tangent to the Minkowsky space-time in the point $\gamma(s)$. We then construct a principal fiber bundle over particle's worldline $L = \mathrm{Im}\,\gamma \subset M$ as follows...

...(i) for each s we define the fiber $\Phi(s)$ over $\gamma(s)$ the set of all Lorentz frames frames in $TM(s)$ having $u(s)$ as its timelike vector, each of these frames is defined by an application $\phi : \mathbb{R}^3 \to \mathbb{R}^3(s)$ thorough

$$A = \begin{bmatrix} | & | & | & | \\ u(s) & i(\phi(E_1)) & i(\phi(E_2)) & i(\phi(E_3)) \\ | & | & | & | \end{bmatrix}$$

where E_1, E_2, E_3 is the canonical basis of $\mathbb{R}^3$ and $i : \mathbb{R}^3(s) \to TM(s)$ is the inclusion of vector spaces...

...(ii) and as a total space for our principal fiber bundle we take the union of all these fibers, $\Phi = \cup_s \Phi(s)$, and the effective and free right action $\Phi \times SO(3) \mapsto \Phi$, given by $(\phi, R) \mapsto \phi \circ R$, where $R \in SO(3)$, $R : \mathbb{R}^3 \to \mathbb{R}^3$ is a spatial rotation...

...(iii) obtaining in this way a principal fiber bundle $\pi : \Phi \to L$ with structural group $SO(3)$.

The above constructed bundle is a trivial one and the above studied Lorentz moving frames $s \to A(s)$ can be seen as global sections $A : L \to \Phi$ in this bundle.

Decomposing the Cartan matrices of equation (23) into electric and magnetic parts we obtain

$$F = \begin{bmatrix} 0 & \varepsilon^t \\ \varepsilon & \omega \end{bmatrix}, \quad \text{with } \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \text{ and } \Omega = \begin{bmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{bmatrix},$$

and taking primed quantities for $\tilde{G}$, we get from (23)

$$\begin{bmatrix} 0 & \varepsilon'^t \\ \varepsilon' & \Omega' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} 0 & \varepsilon^t \\ \varepsilon & \Omega \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{dR}{ds} \end{bmatrix}$$

and therefore

$$\begin{bmatrix} 0 & \varepsilon'^t \\ \varepsilon' & \Omega' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon^t R \\ R^t \varepsilon & R^{-1}\Omega R + R^{-1}\frac{dR}{ds} \end{bmatrix}$$

and the electric and magnetic parts are related by

$$\varepsilon' = R^{-1}\varepsilon \tag{24}$$

...and...

$$\Omega' = R^{-1}\Omega R + R^{-1}\frac{dR}{ds} \tag{25}$$

It follows from (25) that the set of possible magnetic parts of all possible Cartan matrices can be seen (Bleecker 1981) as the gauge potentials, or local expressions of a unique $SO(3)$ connection in the above bundle, each one defined by one of the local sections. All these are the pullbacks of a unique global connection wich would represent the moving orthogonal space to the massive's particle worldline.

Equation (24) permits us to identify the electric part of Cartan matrix with a global section $s \to \mathcal{E}(s)$ in the associated fiber bundle $\Phi \times_\rho \mathbb{R}^3$, where ρ is the fundamental representation of $SO(3)$. The section is given by

$$s \mapsto \mathcal{E}(s) = [(\phi(s), \varepsilon_\phi(s))].$$

Note that the above expressions give us, for each s, that

$$(\phi^{0}, \varepsilon_{\phi^{0}}) = (\phi \pm R, \rho(R^{i\ 1})\,\varepsilon_{\phi\pm\mathbf{R}}) \text{ » } (\phi, \varepsilon_{\phi})$$

and therefore (see Kobayashi and Nomizu 1963) E is well de…ned as a section of the associated bundle.

Seen in this way the “electric and magnectic parts” corresponding to di¤erent Lorentz force laws, or di¤erent moving frames or di¤erent Cartan matrices can be glued by the principal …ber bundle structure. We can then think in...

...(i) a unique globally de…ned magnetic …eld on ©, as a connection 1-form,

...(ii) an electric …eld globally de…ned on ©, as a section E of the associated …ber bundle © $\pounds_{\rho}$ R^3.

4 Conclusion

We have discussed the problem of associating a Lorentz transformation valued function or a Lorentz moving frame to a given massive particle’s worldline, describing the evolution of its four-velocity. This was seen to be equivalent to the problem of …nding a Lorentz Force …eld that, acting on the massive particle, would produce its given worldline.

This problem was also put as a gauge choice in a $SO(3)$ principal …ber bundle over the particle’s worldline, a modern view of the approach to the problem given by Walker 1932 that uni…es all the possible Lorentz force laws describing the same curve, since the magnetic part of the …eld associated to these forces are local expressions, under gauge choices, of a unique global $SO(3)$ connection in that bundle, while the electric part of these …elds, local expressions of a section of an associated bundle.

Any two solutions to the proposed problem are related by an appropriate gauge transformation. We have pointed two of them, one following a method similar to Frenet-Serret moving frame construction, which directly links the Lorentz force to the worldline invariants as a geometric curve in space time.The other one by employing the more physical idea of Fermi-Walker transportation, corresponding this last procedure, from the point of view of a hypothetical comoving traveller, to a minimum energy Lorentz force choice.

...but the physical reality of the force …eld de…ned only for a curve, even making one of the natural choices as Frenet-Serret or Fermi-Walker, shouldn’t go beyond the perception of a hypothetical traveller’s following such trajectory.

It’s now reasonable propose a realistic Lorentz force …eld, when the all the magnetic $SO(3)$ connections and electric sections, respectively de…ned on the principal and appropriated associated bundles over all worldlines of all test particles, are pullback of geometric objects de…ned in the whole space-time. The necessary restrictions and structures on such space-times, together consequences of this approach to the force law problem are interestig points to investigate...

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