ON FACTORIZATION OF SCHATTEN CLASS TYPE MAPPINGS

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Abstract. We present some results on factorization of multilinear mappings and polynomials of Schatten class type $\mathcal{S}_2$ through infinite dimensional Banach spaces, $\mathcal{L}_1$ and $\mathcal{L}_\infty$ spaces. We conclude this work with a factorization result for holomorphic mappings of Schatten class type $\mathcal{S}_2$.

From the linear theory, we know that Hilbert-Schmidt operators factor through an $\mathcal{L}_1$ space and an $\mathcal{L}_\infty$ space (theorem 1.3) and also through infinite dimensional Banach spaces (theorem 1.4). The converse is also true in both cases. When we work with non-linear Hilbert-Schmidt mappings, we notice that none of those linear results can be extended the same way [10]. The mappings of Schatten class type $\mathcal{S}_2$ are, in particular, Hilbert-Schmidt mappings for which it is possible to obtain an extension of the linear results cited above.

This work consists of 3 sections. In the first one, we remind some important definitions and results which will be used later. In section 2, we study the factorization results for multilinear mappings and polynomials and the third section is devoted to a factorization result for holomorphic mappings of Schatten class type $\mathcal{S}_2$.

1. Preliminaries

Throughout this paper, the symbol $\mathbb{K}$ represents the fields of real numbers and complex ones. The set of all positive integers is denoted by $\mathbb{N}$. $E, E_1, ..., E_n, F$ always represent Banach spaces and $H, H_1, ..., H_n, G$, Hilbert spaces over $\mathbb{K}$. B_E represents the closed unit ball of the space E. $\mathcal{L}(E_1, ..., E_n; F)$ denotes the space of the n-linear continuous mappings from $E_1 \times ... \times E_n$ into F. If $E_1 = ... = E_n = E$, we write $\mathcal{L}(^nE; F)$. The space of the n-homogeneous continuous polynomials from E into F is denoted by $\mathcal{P}(^nE; F)$. If $T \in \mathcal{L}(^nE; F)$, we write $\hat{T} \in \mathcal{P}(^nE; F)$ for the corresponding polynomial. $\check{P} \in \mathcal{L}(^nE; F)$ indicates the symmetric n-linear mapping which corresponds to $P \in \mathcal{P}(^nE; F)$ [11]. $W(K)$ denotes the set of all regular Borel probability measures on K.

We indicate the space of Schatten-von Neumman (linear) operators of order p from H into G by $\mathcal{S}_p(H; G)$ and the norm (p-norm if $0 < p < 1$), by $\sigma_p(.)$ [4].

The space formed by the absolutely $(r; s_1, ..., s_n)$-summing multilinear mappings from $E_1 \times ... \times E_n$ into F is indicated by $\mathcal{L}_{as,(r;s_1,...,s_n)}(E_1, ..., E_n; F)$ and the corresponding norm (or r-norm, if $r < 1$), by $\| \, . \, \|_{as,(r;s_1,...,s_n)}$. If $s_1 = ... = s_n = s$, we write $as, (r; s)$ in the place of $as, (r; s_1, ..., s_n)$ and if $r = s$, we just write as, r. We denote by $\mathcal{P}(^nE; F)$ the space of the absolutely $(r; s)$-summing polynomials and we write $\| \, . \, \|_{as,(r;s)}$ for the corresponding norm (or r-norm) [7], [1].

We indicate by $\mathcal{L}_{d,(s_1,...,s_n)}(E_1, ..., E_n; F)$ the space of all $(s_1, ..., s_n)$ - dominated multilinear mappings with the corresponding norm (or r-norm) $\| \, . \, \|_{d,(s_1,...,s_n)}$. If $s_1 = ... = s_n = s$, we just write d, s in the place of $d, (s_1, ..., s_n)$. Finally, we indicate by $\mathcal{P}_{d,r}(^nE; F)$ the space of the r-dominated n homogeneous polynomials from E to F and we write $\| \, . \, \|_{d,r}$ for the corresponding norm (or r-norm) [8], [1].

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For the holomorphic mappings, $\mathcal{H}(U;F)$ denotes the space of all holomorphic mappings from $U \subset E$ (a non-void open subset of E) into F (E and F are complex spaces). The n-th derivative of f in $x \in U$ is represented by $d^n f(x) \in \mathcal{L}(^nE;F)$ and the corresponding polynomial, by $\hat{d}^n f(x) \in \mathcal{P}(^nE;F)$.

The space of all sequences $(x_n)_n$ in E such that $\| (x_n)_n \|_p := \left(\sum_{n\in\mathbb{N}} \| x_n \|^p \right)^{\frac{1}{p}} < \infty$ is denoted by $l_p(E)$. $l_{p,w}(E)$ denotes the space of all sequences $(x_n)_n$ in E such that $(\langle x', x_n\rangle)_n$ is a sequence in $l_p = l_p(\mathbb{K})$ for all $x' \in E'$. A norm (p-norm if $p < 1$) is defined by $\| (x_n) \|_{p,w} := \sup_{x'\in B_{E'}} \left(\sum_{j=1}^{\infty} | x'(x_j) |^p \right)^{\frac{1}{p}}$.

Details about the linear theory and $\mathcal{L}_p$ spaces can be seen in [4]. For the holomorphic mappings theory, see [11].

The definition of Hilbert-Schmidt m-functionals is due to Dwyer [5]. These mappings were also studied by Matos in [9] for the vector-valued case.

Definition 1.1. *A multilinear mapping $T \in \mathcal{L}(H_1, ..., H_n; G)$ is a Hilbert-Schmidt mapping if, for each $k = 1, ..., n$, there exist an orthonormal basis $(h^k_{j_k})_{j_k \in J_k}$ of H_k such that $(\sum_{j_1,...,j_n} \|T(h^1_{j_1}, ..., h^n_{j_n})\|^2)^{\frac{1}{2}} < +\infty$. We denote by $\mathcal{L}_{HS}(H_1, ..., H_n; G)$ the space of such mappings.*

We can prove that, if $T \in \mathcal{L}_{HS}(H_1, ..., H_n; G)$, then $(\sum_{j_1,...,j_n} \|T(h^1_{j_1}, ..., h^n_{j_n})\|^2)^{\frac{1}{2}} < +\infty$ for all orthonormal bases of $H_1, ..., H_n$.

In the linear case, we have $\mathcal{S}_2(H;G) = \mathcal{L}_{HS}(H;G)$ and $\sigma_2(.) =\| \, . \, \|_{HS}$. Pelczynski has proved in [13] that $\mathcal{L}_{HS}(H;G) = \mathcal{L}_{as,p}(H;G)$ for all $p \geq 1$ and $B_p^{-1} \| \, . \, \|_{HS} \leq \| \, . \, \|_{as,p} \leq A_1^{-1} \| \, . \, \|_{HS}$ where A_1 and B_p are constants of the Khinchin's inequality [4].

Definition 1.2. *A polynomial $P \in \mathcal{P}(^nH;G)$ is a Hilbert-Schmidt polynomial if $\check{P} \in \mathcal{L}_{HS}(^nH;G)$. The space of such polynomials is indicated by $\mathcal{P}_{HS}(^nH;G)$ and a norm is defined by $\| P \|_{HS} =\| \check{P} \|_{HS}$.*

We present now the results on factorization of Hilbert-Schmidt linear operators.

Theorem 1.3. *(Lindenstrauss-Pelczynski,* [6]*) Let $u \in \mathcal{L}(H;G)$. The following are equivalent:*
(i) $u \in \mathcal{L}_{HS}(H;G)$.
(ii) u factors through an $\mathcal{L}_\infty$ space.
(iii) u factors through an $\mathcal{L}_1$ space.

A more recent result was proved by Diestel, Jarchow and Tonge. A proof can be seen in [4], 19.2.

Theorem 1.4. *$u \in \mathcal{L}_{HS}(H;G)$ if and only if, for any infinite dimensional Banach space Z, there are operators $v \in \mathcal{L}(Z;G)$ and $w \in \mathcal{L}(H;Z)$ such that $u = v \circ w$. Moreover, we can choose w to be compact and v, compact and 2-summing.*

About 1.4, write $u(h) = \sum_{s=1}^{\infty} \tau_s(h \mid h_s)g_s$, with $\tau = (\tau_s)_s \in l_2$ and $(h_s)_s$, $(g_s)_s$ orthonormal sequences in H and G, respectively. We will prove later (2.9) that $\tau_s = \alpha_s\sigma_s\beta_s$, where $\sigma = (\sigma_s)_s \in l_2$ and $\alpha = (\alpha_s)_s$, $\beta = (\beta_s)_s \in c_o$. On the proof of 1.4, we can notice that w and v can be chosen in such a way that $\| w \| \leq 8 \| \beta \|_\infty^{\frac{1}{4}}$ and $\| v \|_{as,2}$

$\leq 12C \parallel \alpha \parallel_\infty \parallel \beta \parallel_\infty^{\frac{1}{4}} \parallel \sigma \parallel_2$, where $1 \leq C \leq 4$ is a constant which depends on the space Z considered.

An important factorization result for dominated mappings is the following Pietsch's theorem. We present a multilinear version of the theorem. The polynomial version is analogous. For a proof, see [14], 3.17.

Theorem 1.5. *Let $r_1, ..., r_n \in [1, \infty)$, $T \in \mathcal{L}(E_1, ..., E_n; F)$ and $K_j \subset B_{E'_j}$ a weak star compact subset of $B_{E'_j}$ with the property $\|x_j\| = sup\{|x'_j(x_j)|; x'_j \in K_j\}$, $j = 1, ..., n$. The following conditions are equivalent:*
(i) T is $(r_1, ..., r_n)$-dominated.
(ii) For all $j = 1, ..., n$, there exist $\mu_j \in W(K_j)$, $X_j \subset L_{r_j}(\mu_j)$ a closed subspace, $j = 1, ..., n$ and $S \in (X_1, ..., X_n; F)$ such that $T = S \circ (J_{r_1} \circ i_{E_1}, ..., J_{r_n} \circ i_{E_n})$, where $i_{E_j} : E_j \longrightarrow C(K_j)$ is given by $i_{E_j}(x)(x') = \langle x', x\rangle$, $x' \in K_j$, $x \in E_j$ and $J_{r_j} : C(K_j) \longrightarrow L_{r_j}(\mu_j)$ is the formal inclusion, $j = 1, ..., n$. In addition, $\parallel S \parallel = \parallel T \parallel_{d,(r_1,...,r_n)}$.

2. Multilinear mappings and polynomials of Schatten class type

The Schatten class type mappings were studied by Braunss and Junek in [3].

Definition 2.1. *Let $0 < p < \infty$. A multilinear mapping $T \in \mathcal{L}(H_1, ..., H_n; F)$ is of Schatten class type $\mathcal{S}_p$ if , for each $i = 1, ..., n$, there exist a Hilbert space K_i, an operator $T_i \in \mathcal{S}_p(H_i; K_i)$ and $S \in \mathcal{L}(K_1, ..., K_n; F)$ such that $T = S \circ (T_1, ..., T_n)$. We denote the space of such mappings as $\mathcal{L}(\mathcal{S}_p)(H_1, ..., H_n; F)$. A norm (or $\frac{p}{n}$ norm if $p < 1$) for that space is $\parallel T \parallel_{\mathcal{S}_p} = \inf_{T=S\circ(T_1,...,T_n)} \parallel S \parallel \prod_{j=1}^{n} \sigma_p(T_j)$.*

In [2], Braunss gives the following definition for polynomials.

Definition 2.2. *Let $0 < p < \infty$. A polynomial $P \in \mathcal{P}(^nH; F)$ is of Schatten class type $\mathcal{S}_p$ if there exist a Hilbert space K, an operator $S \in \mathcal{S}_p(H; K)$ and $Q \in \mathcal{P}(^nK; F)$ such that $T = Q \circ S$. We denote the space of such polynomials by $\mathcal{P}(\mathcal{S}_p)(^nH; F)$ and also $\parallel P \parallel_{\mathcal{S}_p} = \inf_{P=Q\circ S} \parallel Q \parallel \sigma_p(S)^n$. Braunss [2] has proved that $\parallel . \parallel_{\mathcal{S}_p}$ is a q-norm, where $q = min\{1, p\}$ if $n = 1$ and F is a Hilbert space; $q = min\{1, \frac{2p}{2+p}\}$ if $n = 1$ and F is a Banach space (non Hilbert); $q = min\{\frac{2}{n}, \frac{p}{n}\}$ if $n \geq 2$.*

As in the linear case, we can prove the following

Proposition 2.3. *If $p \geq 2$ and $T \in \mathcal{L}(\mathcal{S}_p)(H_1, ..., H_n; F)$, then $\sum_{j_1,...,j_n} \parallel T(h^1_{j_1}, ..., h^n_{j_n}) \parallel^p < +\infty$ for all orthonormal bases $(h^i_{j_i})_{j_i \in J_i}$ of H_i, $i = 1, ..., n$.*

The proposition above can be proved using the definition and the properties of the Schatten linear operators (see [4], 4.7). As a consequence, we have

Corollary 2.4. *(i) $\mathcal{L}(\mathcal{S}_2)(H_1, ..., H_n; G) \subset \mathcal{L}_{HS}(H_1, ..., H_n; G)$. Moreover, $\parallel T \parallel_{HS} \leq \parallel T \parallel_{\mathcal{S}_2}$ for all $T \in \mathcal{L}(\mathcal{S}_2)(H_1, ..., H_n; G)$. The inclusion is strict in general.*
(ii) $\mathcal{P}(\mathcal{S}_2)(^nH; G) \subset \mathcal{P}_{HS}(^nH; G)$. Moreover, $\parallel P \parallel_{HS} \leq \frac{n^n}{n!} \parallel P \parallel_{\mathcal{S}_2}$ for all $P \in \mathcal{P}(\mathcal{S}_2)(^nH; G)$. The inclusion is strict in general.

The relationship between the norms $\parallel . \parallel_{HS}$ and $\parallel . \parallel_{\mathcal{S}_2}$ can be extracted from the proof of 2.3. To prove (ii) in 2.4, we use (i) of the same result and the fact that $P \in \mathcal{P}(\mathcal{S}_p)(^nH; F)$ if and only if there exist $T \in \mathcal{L}(\mathcal{S}_p)(^nH; F)$ such that $P = \hat{T}$.

The example below shows that the inclusion in 2.4 is strict in general.

Example 2.5. *$T \in \mathcal{L}(^2l_2;\mathbb{K})$, $T(x,y) = \sum_{j=1}^{\infty}\frac{1}{j}x_jy_j$. It is clear that $T \in \mathcal{L}_{HS}(^2l_2;\mathbb{K})$. If T was a Schatten class type $\mathcal{S}_2$ mapping, T would be written in the form $T = S\circ(T_1,T_2)$ as in 2.1. For $(x^k)_{k=1}^{\infty}$, $(y^k)_{k=1}^{\infty} \in l_{2,w}(l_2)$, we would have:*

$$\sum_{k=1}^{\infty} \| T(x^k,y^k) \| \leq \| S \| \left(\sum_{k=1}^{\infty} \| T_1x^k \|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \| T_2y^k \|^2\right)^{\frac{1}{2}} < +\infty.$$

Nevertheless, if e_k indicates the k-th element of l_2 usual basis ($(e_k)_k \in l_{2,w}(l_2)$), we have $\sum_{k=1}^{\infty} \| T(e_k,e_k) \| = \sum_{k=1}^{\infty}\frac{1}{k} = \infty$, a contradiction. Therefore, $T \notin \mathcal{L}(\mathcal{S}_2)(^2l_2;\mathbb{K})$.

There is an interesting relationship between dominated mappings and Schatten class type $\mathcal{S}_2$ mappings.

Proposition 2.6. *(i) For all $1 \leq p \leq 2$, we have $\mathcal{L}(\mathcal{S}_2)(H_1,...,H_n;F) = \mathcal{L}_{d,p}(H_1,...,H_n;F)$. Moreover $\| T \|_{\mathcal{S}_2} \leq \| T \|_{d,p} \leq (A_1^{-1})^n \| T \|_{\mathcal{S}_2}$, where A_1 is a constant of Khinchin's inequality* [4].
(ii) For all $1 \leq p \leq 2$, we have $\mathcal{P}(\mathcal{S}_2)(^nH;F) = \mathcal{L}_{d,p}(^nH;F)$. Moreover, $\| P \|_{\mathcal{S}_2} \leq \| P \|_{d,p} \leq (A_1^{-1})^n \| P \|_{\mathcal{S}_2}$.

Proof. We will prove (i). (ii) can be proved the same way, making use of the analogous theorems for polynomials.

If $T \in \mathcal{L}_{d,p}(H_1,...,H_n;F) \subset \mathcal{L}_{d,2}(H_1,...,H_n;F)$, $1 \leq p \leq 2$, then using theorem 1.5, we have $T = S\circ(J_2^1\circ i_{H_1},...,J_2^n\circ i_{H_n})$, with $\| S \| = \| T \|_{d,2}$ and $J_2^j\circ i_{H_j} \in \mathcal{L}_{as,2}(H_j;L_2(\mu_j))$ $= \mathcal{S}_2(H_j;L_2(\mu_j))$ ([4], 2.9). Therefore, $T \in \mathcal{L}(\mathcal{S}_2)(H_1,...,H_n;F)$. Moreover

$$\| T \|_{\mathcal{S}_2} \leq \| T \|_{d,2} \prod_{j=1}^{n} \| J_2^j \|_{as,2} \leq \| T \|_{d,p}$$

On the other hand, consider $T \in \mathcal{L}(\mathcal{S}_2)(H_1,...,H_n;F)$ and a decomposition for T, $T = R\circ(S_1,...,S_n)$, $S_j \in \mathcal{S}_2(H_j;K_j)$, K_j a Hilbert space $j = 1,...,n$ and $R \in \mathcal{L}(K_1,...,K_n;F)$. Using Pietsch's factorization result (1.5 for the linear case), we can write $S_j = w_j\circ v_j$, where $v_j \in \mathcal{L}_{as,p}(H_j;X_j^p)$, $v_j = J_p\circ i_{H_j}$, $\mu_j \in W(B_{H_j})$, X_j^p is a closed subspace of $L_p(\mu_j)$ and $w_j \in \mathcal{L}(X_j^p;K_j)$ is such that $\| w_j \| = \| S_j \|_{as,p}$, $j = 1,...,n$. Write $\tilde{T} = R\circ(w_1,...,w_n) \in \mathcal{L}(X_1^p,...,X_n^p;F)$. If $(h_k^j)_{k=1}^m \subset H_j$, $j = 1,...,n$, we have

$$\left(\sum_{k=1}^{m} \| T(h_k^1,...,h_k^n) \|^{\frac{p}{n}}\right)^{\frac{n}{p}} \leq \| \tilde{T} \| \left(\sum_{k=1}^{m} \| v_1h_k^1 \|^p\right)^{\frac{1}{p}} ... \left(\sum_{k=1}^{m} \| v_nh_k^n \|^p\right)^{\frac{1}{p}}$$
$$\leq \| \tilde{T} \| \prod_{j=1}^{n} \| v_j \|_{as,p} \| (h_k^j)_{k=1}^m \|_{w,p}.$$

Therefore, $T \in \mathcal{L}_{d,p}(H_1,...,H_n;F)$. In addition,

$$\| T \|_{d,p} \leq \| \tilde{T} \| \prod_{j=1}^{n} \| v_j \|_{as,p} = \| R \| \| w_1 \| ... \| w_n \| \prod_{j=1}^{n} \| J_p \|_{as,p} \| i_{H_j} \|$$
$$\leq \| R \| \prod_{j=1}^{n} \| S_j \|_{as,p} \leq A_1^{-1} \| R \| \prod_{j=1}^{n}\sigma_2(S_j).$$

As the decomposition taken is arbitrary, we conclude $\| T \|_{d,p} \leq A_1^{-1} \| T \|_{\mathcal{S}_2}$. □

We now present the factorization result for the Schatten class type mappings.

Theorem 2.7. *Let $T \in \mathcal{L}(H_1, ..., H_n; F)$. The following conditions are equivalent:*
(i) $T \in \mathcal{L}(\mathcal{S}_2)(H_1, ..., H_n; F)$.
(ii) For each $j = 1, ..., n$, there exist an $\mathcal{L}_1$ space Y_j, an operator $R_j \in \mathcal{L}(H_j; Y_j)$ and $R \in \mathcal{L}_{d,2}(Y_1, ..., Y_n; F)$ such that $T = R \circ (R_1, ..., R_n)$.
(iii) For each $j = 1, ..., n$, there exist an $\mathcal{L}_\infty$ space X_j, an operator $S_j \in \mathcal{L}(H_j; X_j)$ and $S \in \mathcal{L}_{d,2}(X_1, ..., X_n; F)$ such that $T = S \circ (S_1, ..., S_n)$.

Moreover, given $\epsilon > 0$, $R, R_1, ..., R_n$ can be taken in (ii) in such a way that $\| R_j \| = 1$, $j = 1, ..., n$ and $\| R \|_{d,2} \leq (1+\epsilon)(K_G)^n \| T \|_{\mathcal{S}_2}$ and $S, S_1, ..., S_n$ in (iii) such that $\| S_j \| = 1$, $j = 1, ..., n$ and $\| S \|_{d,2} \leq (1+\epsilon) \| T \|_{\mathcal{S}_2}$.

The result 2.7 is analogous to the linear factorization theorem due to Lindenstrauss and Pelczynski. For the Hilbert-Schmidt case, it is possible to prove that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) (without the condition that S and R are 2-dominated). The converse (i) $\Rightarrow$ (iii) is not true in general. For the $\mathcal{L}_1$ space case, we have no answer about the converse [10]. We also have a multilinear version for the Diestel-Jarchow-Tonge result.

Theorem 2.8. *A multilinear mapping $T \in \mathcal{L}(H_1, ..., H_n; F)$ is of Schatten class type $\mathcal{S}_2$ if and only if, given infinite Banach spaces $Z_1, ..., Z_n$, there exist $S_j \in \mathcal{L}(H_j; Z_j)$, $j = 1, ..., n$ and $S \in \mathcal{L}_{d,2}(Z_1, ..., Z_n; F)$ such that $T = S \circ (S_1, ..., S_n)$.*

Moreover, given $\epsilon > 0$ and $\delta > 0$, S and $S_1, ..., S_n$ can be chosen in such a way that $\| S \|_{d,2} \leq (1+\epsilon) \| T \|_{\mathcal{S}_2} \left(\frac{48A}{n^{5+\frac{5\delta}{5}}} \right)^n$, $A > 0$ is a constant, and $\| S_j \|_{as,2} \leq \frac{8}{n^{1+\frac{\delta}{8}}}$.

We will prove theorem 2.8. The proofs of (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) in 2.7 are almost the same, using 1.3 in the place of 1.4.

Before the proof of 2.8, we prove the following lemma.

Lemma 2.9. *Given $n \in \mathbb{N}$, $\delta > 0$ and $\tau = (\tau_s)_s \in l_2$, there exist $\gamma = (\gamma_s)_s \in c_o$ and $\sigma = (\sigma_s)_s \in l_2$ such that $\| \gamma \|_\infty = \frac{1}{n^{8+\delta}}$, $\| \sigma \|_2 \leq A$, $A > 0$ does not depend on the choice of $n \in \mathbb{N}$ and $\tau_s = \gamma_s \sigma_s$ for each $s \in \mathbb{N}$.*

Proof. With no loss of generality, we suppose that $\| \tau \|_2 = 1$ and $\tau_s \geq 0$ for all $s \in \mathbb{N}$.Write $N_o = 0$. We can inductively define a sequence of positive integers $N_1 < N_2 < ... < N_k < ...$ such that, for each $k \in \mathbb{N}$, $\tau_1^2 + ... + \tau_{N_k}^2 \geq \frac{2^{n+k}-1}{2^{n+k}}$ and N_k is the smallest positive integer with this property. Then, we write

$$\gamma_{N_{k-1}+1} = ... = \gamma_{N_k} = \frac{1}{(n+k-1)^{8+\delta}}$$
$$\sigma_{N_{k-1}+1} = (n+k-1)^{8+\delta} \tau_{N_{k-1}+1} \ , \ ... \ , \sigma_{N_k} = (n+k-1)^{8+\delta} \tau_{N_k}.$$

In this way, we can define the two sequences $\gamma = (\gamma_s)_s \in c_o$ with $\| \gamma \|_\infty = \frac{1}{n^{8+\delta}}$ and $\sigma = (\sigma_s)_s$ such that $\sigma_s \gamma_s = \tau_s$. We have to verify that $\sigma \in l_2$.

$$\sum_{s=1}^{\infty} \sigma_s^2 = \sum_{k=1}^{\infty} \sum_{s=N_{k-1}+1}^{N_k} (n+k-1)^{2(8+\delta)} \tau_s^2 \leq \sum_{k=1}^{\infty} (n+k-1)^{2(8+\delta)} \left(1 - \frac{2^{n+k-1}-1}{2^{n+k-1}} \right)$$

$$= \sum_{k=1}^{\infty} \frac{(n+k-1)^{2(8+\delta)}}{2^{n+k-1}} = \sum_{l=n+1}^{\infty} \frac{(l-1)^{2(8+\delta)}}{2^{l-1}} \leq \sum_{l=1}^{\infty} \frac{(l-1)^{2(8+\delta)}}{2^{l-1}}$$

Using the ratio test, we can prove that $A := \sum_{l=1}^{\infty} \frac{(l-1)^{2(8+\delta)}}{2^{l-1}} < +\infty$. $\square$

Proof. (2.8) Given $\epsilon > 0$ and $T \in \mathcal{L}(\mathcal{S}_2)(H_1, ..., H_n; F)$, for each $j = 1, ..., n$, there exist a Hilbert space K_j, an operator $u_j \in \mathcal{S}_2(H_j; K_j) = \mathcal{L}_{as,2}(H_j; K_j)$, $j = 1, ..., n$ and $L \in \mathcal{L}(K_1, ..., K_n; F)$ such that $T = L \circ (u_1, ..., u_n)$, with $\sigma_2(u_j) = 1$, $j = 1, ..., n$ and $\| L \| \leq (1 + \epsilon) \| T \|_{\mathcal{S}_2}$.

Write $u_j(h) = \sum_{s=1}^{\infty} \tau_s^{(j)} (h \mid h_s^{(j)}) k_s^{(j)}$, where $\tau^{(j)} = (\tau_s^{(j)})_s \in l_2$, $\| t^{(j)} \|_2 \ \sigma_2(u_j) = 1$, $(h_s^{(j)})_s$ is an orthonormal sequence in H and $(k_s^{(j)})_s$, an orthonormal sequence in K_j.

By the lemma 2.9, given $\delta > 0$, we can write $\tau_s^{(j)} = \alpha_s^{(j)} \sigma_s^{(j)} \beta_s^{(j)}$ for each $s \in \mathbb{N}$, where $\sigma^{(j)} = (\sigma_s^{(j)})_s \in l_2$, $\| \sigma^{(j)} \|_2 \leq A$, $\alpha_s^{(j)} = \beta_s^{(j)} = \sqrt{\gamma_s^{(j)}}$, $\alpha^{(j)} = (\alpha_s^{(j)})_s$ and $\beta^{(j)} = (\beta_s^{(j)})_s$ $\in c_o$, with $\| \alpha^{(j)} \|_\infty = \| \beta^{(j)} \|_\infty = \dfrac{1}{n^{4+\frac{\delta}{2}}}$.

Using theorem 1.4, for each $j = 1, ..., n$, we have $u_j = v_j \circ w_j$, where $w_j \in \mathcal{L}(H; Z_j)$, $v_j \in \mathcal{L}_{as,2}(Z_j; K_j)$, with $\| w_j \| \leq \dfrac{8}{n^{1+\frac{\delta}{8}}}$ and $\| v_j \|_{as,2} \leq \dfrac{48A}{n^{5+\frac{5\delta}{8}}}$.

If $S = L \circ (v_1, ..., v_n) \in \mathcal{L}(Z_1, ..., Z_n; F)$, for $(z_i^j)_{i=1}^m \subset Z_j$, $j = 1, ..., n$, we can write

$$\left(\sum_{i=1}^{m} \| S(z_i^1, ..., z_i^n) \|^{\frac{2}{n}} \right)^{\frac{n}{2}} \leq \| L \| \prod_{j=1}^{m} \left(\sum_{i=1}^{m} \| v_j z_i^j \|^2 \right)^{\frac{1}{2}}$$
$$\leq \| L \| \prod_{j=1}^{m} \| v_j \|_{as,2} \| (z_i^j)_{i=1}^m \|_{w,2}$$

Then, $\| S \|_{d,2} \leq \| L \| \prod_{j=1}^{m} \| v_j \|_{as,2} \leq (1 + \epsilon) \| T \|_{\mathcal{S}_2} \left(\dfrac{48A}{n^{5+\frac{5\delta}{8}}} \right)^n$, and also, $T = S \circ (S_1, ..., S_n)$ if $S_j = v_j$, $j = 1, ..., n$.

Suppose now that $T \in \mathcal{L}(H_1, ..., H_n; F)$ can be decomposed as described in 2.8. We can use theorem 1.5 to get a decomposition for $S \in \mathcal{L}_{d,2}(Z_1, ..., Z_n; F)$ say, $S = \tilde{S} \circ (w_1, ..., w_n)$ with $w_j \in \mathcal{L}_{as,2}(H_j; L_2(\mu_j)) = \mathcal{S}_2(H_j; L_2(\mu_j))$, $\mu_j \in W(B_{Z_j'})$, $j = 1, ..., n$ and $\tilde{S} \in \mathcal{L}(L_2(\mu_1), ..., L_2(\mu_n); F)$. If we call $v_j = w_j \circ S_j$, we have the decomposition $T = \tilde{S} \circ (v_1, ..., v_n)$ and we conclude that T is a multilinear Schatten class type mapping $\mathcal{S}_2$. □

For polynomials, the result is analogous.

Theorem 2.10. *Let $P \in \mathcal{P}(^n H; F)$. The following conditions are equivalent:*
(i) $P \in \mathcal{P}(\mathcal{S}_2)(^n H; F)$.
(ii) There exist an $\mathcal{L}_1$ space Y, an operator $R \in \mathcal{L}(H; Y)$ and $Q \in \mathcal{P}_{d,2}(Y; F)$ such that $P = Q \circ R$.
(iii) There exist an $\mathcal{L}_\infty$ space X, an operator $S \in \mathcal{L}(H; X)$ and $Q \in \mathcal{P}_{d,2}(X; F)$ such that $P = Q \circ S$.

Moreover, given $\epsilon > 0$, Q, R can be taken in (ii) in such a way that $\| R \| = 1$ and $\| Q \|_{d,2} \leq (1 + \epsilon)(K_G)^n \| P \|_{\mathcal{S}_2}$ and Q, S in (iii) such that $\| S \| = 1$, $j = 1, ..., n$ and $\| Q \|_{d,2} \leq (1 + \epsilon) \| P \|_{\mathcal{S}_2}$.

Theorem 2.11. *A polynomial $P \in \mathcal{P}(^n H; F)$ is of Schatten class type $\mathcal{S}_2$ if and only if, given an infinite dimensional Banach space Z, there exist $S \in \mathcal{L}(H; Z)$ and $Q \in \mathcal{P}_{d,2}(Z; F)$ such that $P = Q \circ S$.*

Moreover, given $\epsilon > 0$ and $\delta > 0$, S and Q can be chosen in such a way that $\| Q \|_{d,2}$ $\leq (1 + \epsilon) \| P \|_{\mathcal{S}_2} \left(\dfrac{48A}{n^{5+\frac{5\delta}{5}}} \right)^n$, $A > 0$ is a constant, and $\| S \| \leq \dfrac{8}{n^{1+\frac{\delta}{8}}}$.

It is important to say that there are other ways to choose the sequences in 2.9 and consequently, in 2.8 and 2.11. The choice made is (one of) the best for the proof of the factorization result for holomorphic mappings.

3. Holomorphic mappings of Schatten class type

The main purpose of this section is to present a factorization result for holomorphic mappings of Schatten class type $\mathcal{S}_2$. The spaces considered in this section are complex.

The holomorphic mappings has been already studied by Braunss in [2].

Definition 3.1. *Let $0 < p < \infty$ and $f \in \mathcal{H}(U;F)$, where $U \subset G$ is a non-void open subset of H. f is a mapping of Schatten class type $\mathcal{S}_p$ in $h \in H$ if $\hat{d}^n f(h) \in \mathcal{P}(\mathcal{S}_p)(^nH;F)$ for all $n \in \mathbb{N}_o$ and there exist real numbers $C \geq 0$ and $c \geq 0$ such that $\| \frac{1}{n!}\hat{d}^n f(h) \|_{\mathcal{S}_p} \leq Cc^n$ for all $n \in \mathbb{N}_o$.*

If f is of Schatten class type in all $h \in U$, we say that f is of Schatten class type on U. We denote the class of such mappings by $\mathcal{H}(\mathcal{S}_p)(U;F)$.

The factorization result is

Theorem 3.2. *Let $U \subset H$ be an open subset of H, $f \in \mathcal{H}(U;F)$ and $h_o \in U$. Then, f is of Schatten class type $\mathcal{S}_2$ in h_o if and only if, there exist an $\mathcal{L}_\infty$ space X, an operator $S \in \mathcal{L}(H;X)$ and $g \in \mathcal{H}(X;F)$ of 2-dominated type (see [8], 3.2) in $x_o = Sh_o$, such that $f = g \circ S$ in U_o, where U_o is a neighborhood of h_o.*

Proof. We call $P_n = \frac{1}{n!}\hat{d}^n f(h_o) \in \mathcal{P}(\mathcal{S}_2)(^nH;F)$ for each $n \in \mathbb{N}$. Using 2.11, we can write $P_n = Q_n \circ S_n$, where $S_n \in \mathcal{L}(H;l_\infty)$, $\| S_n \| \leq \frac{8}{n^{1+\frac{\delta}{8}}}$ and $Q_n \in \mathcal{P}_{d,2}(^nl_\infty;F)$, with $\| Q_n \|_{d,2} \leq (1+\epsilon) \| P_n \|_{\mathcal{S}_2} \left(\frac{48A}{n^{5+\frac{5\delta}{8}}}\right)^n$. (*)

We denote $X = l_\infty(l_\infty)$, $i_n : l_\infty \to X$ the n-th inclusion and $\pi_n : X \to l_\infty$ the n-th projection, $n \in \mathbb{N}$. Observe that $\| i_n \circ S_n \| \leq \frac{8}{n^{1+\frac{\delta}{8}}}$ and by the comparison test, we have that $\sum_{n=1}^{\infty} \| i_n \circ S_n \| < +\infty$. So , we define $S \in \mathcal{L}(H;X)$, $S(h) = \sum_{n=1}^{\infty} i_n \circ w_n(h)$.

If we define $g(x) = f(h_o) + \sum_{n=1}^{\infty} Q_n \circ \pi_n(x - x_o)$, we have $g \in \mathcal{H}(X;F)$, because $\limsup_{n\to\infty} \| Q_n \circ \pi_n \|^{\frac{1}{n}} = 0$. Moreover, if U_o is a neighborhood of h_o where $f(h) = f(h_o) + \sum_{n=1}^{\infty} P_n(h - h_o)$ for all $h \in U_o$, we have

$$(g \circ S)(h) = f(h_o) + \sum_{n=1}^{\infty} Q_n \circ \pi_n(Sx - Sx_o) = f(h_o) + \sum_{n=1}^{\infty} Q_n \circ \pi_n \left(\sum_{k=1}^{\infty} i_k \circ w_k(h - h_o)\right)$$

$$= f(h_o) + \sum_{n=1}^{\infty} Q_n \circ w_n(h - h_o) = f(h_o) + \sum_{n=1}^{\infty} R_n \circ v_n \circ w_n(h - h_o)$$

$$= f(h_o) + \sum_{n=1}^{\infty} R_n \circ u_n(h - h_o) = f(h_o) + \sum_{n=1}^{\infty} P_n(h - h_o) = f(h).$$

g is 2-dominated in $x_o = Sh_o \in X$. In fact, we know that $\frac{1}{n!}\hat{d}^n g(x_o) = Q_n \circ \pi_n \in \mathcal{P}_{d,2}(X;F)$ for all $n \in \mathbb{N}$. Using (*) and the fact that $\| P_n \|_{\mathcal{S}_2} \leq Cc^n$ for all $n \in \mathbb{N}_o$, we reach the desired conclusion.

(ii)⇒ (i) For $n \in \mathbb{N}$, $\dfrac{1}{n!}\hat{d}^n f(h_o) = \dfrac{1}{n!}\hat{d}^n g(Sh_o) \circ S \in \mathcal{P}_{d,2}(^n H; F) = \mathcal{P}(\mathcal{S}_2)(^n H; F)$ (see 2.6). □

We can show the result above for an $\mathcal{L}_1$ space. We only use l_1 in the place of l_∞ and define $X = l_1(l_1)$.

Remark 3.3. *Using the same notation as in 3.2, if U is a h_o-balanced set, then we have* $f(h) = f(h_o) + \sum_{n=1}^{\infty} P_n(h - h_o)$ *for all $h \in U$ and consequently, $f = g \circ S$ in U (see* [11]*, 8.4).*

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