Invariant nearly-Kähler structures

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Abstract

This paper considers invariant almost Hermitian structures on a flag manifold $G/P = U/K$ where G is a complex semi-simple Lie group, P is a parabolic subgroup of G, U is a compact real form of G and $K = U \cap P$ is the centralizer of a torus. The main result shows that there are nearly-Kähler structures in G/P which are not Kähler if and only if G/P has height three. This proves for the flag manifolds a conjecture by J.A. Wolf and A. Gray.

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1 Introduction

In this paper we study nearly-Kähler invariant almost Hermitian structures on the flag manifolds of a complex semi-simple Lie group G. The objective is to identify those flag manifolds that admit invariant nearly-Kähler structures which are not Kähler. Invariance here is taken with respect to a maximal compact subgroup U of G, that is, a compact real form.

This problem was considered in the 1968 paper by Wolf-Gray [10], where relationships are established between the geometric structures on the homogeneous spaces and the automorphisms fixing the isotropy subgroup or subalgebra. In particular, in [10] it is given evidence to the following conjecture: *Let U/K be a homogeneous space of a compact Lie group U, which is not Hermitian symmetric and such that the isotropy K has maximal rank in U. Then there are invariant almost Hermitian structures on U/K which are nearly-Kähler but not Kähler if and only if the isotropy subalgebra is the fixed point set of an automorphism of order three* (see [10], Conjecture 9.28).

In this paper we confirm this conjecture for the class of homogeneous spaces formed by the flag manifolds. This class covers most of the possible homogeneous spaces, in such a way that it remains open only two exceptional cases.

In order to explain the state of art let U/K be a homogeneous space with U compact and K having the same rank as U. There are two possibilities, namely K is the centralizer of a torus of U or not. In the former case U/K is a flag manifold of some complex Lie group, and hence is covered by the results of the present paper.

The second possibility occurs only in exceptional cases, classified by Wolf-Gray [9]: There are only twelve homogeneous spaces U/K admitting invariant almost complex structures and such that (i) U is compact; (ii) K is closed and connected; (iii) K is of maximal rank in U and (iv) K is not the centralizer of a torus (see [9], Theorem 4.11 and the table at pages 103-4). For the twelve homogeneous spaces obtained, U is always an exceptional Lie group. We list them below, indicating the conclusions of [10], concerning the existence of nearly-Kähler metrics. We note that by Corollary 9.5 of [10] none of these homogeneous spaces admit an invariant Kähler metric.

1. Every invariant metric is nearly-Kähler: G_2/A_2; F_4/A_2A_2; $E_6/A_2A_2A_2$; E_7/A_2A_5; E_8/A_8; E_8/A_2A_6 (see [10], page 157).

2. There are no nearly-Kähler invariant metrics: E_8/A_4A_4; $E_7/A_2A_2A_2T^1$;

$E_8/A_2A_2A_2A_2$; $E_8/A_2A_5T^1$ (see [10], Propositions 9.20, 9.21, 9.22 and 9.23, respectively).

3. Open: $E_8/A_2A_2A_2A_1T^1$; $E_8/A_2A_2A_2T^2$.

Of course these homogeneous spaces confirm the conjecture stated above, that is, only the isotropy of those spaces admitting nearly-Kähler metrics are fixed points of order three automorphisms (see [10], page 157).

The invariant almost Hermitian structures on a maximal flag manifold G/P, where P is a Borel subgroup, were studied recently in [8] (see also [1] and [2]). In these works a key role is played by the class of $(1,2)$-symplectic structures, in which the nearly-Kähler ones are included. As shown in [8] we can describe the $(1,2)$-symplectic structures in terms of abelian ideals of a Borel subalgebra (in the sense of Kostant [6]) as well as in terms of the alcoves of the corresponding affine Lie algebras. These descriptions provide decisive information on the $(1,2)$-symplectic structures, so that they can be classified up to equivalence under the Weyl group, paving the way to understand, among the invariant ones, the sixteen classes of almost Hermitian structures identified by Gray-Hervella [3]. In particular, it is proved in [8] that the conjecture of Wolf-Gray holds for the maximal flag manifolds, since A_2 ($=\mathfrak{sl}(3,\mathbb{C})$) is the only Lie algebra whose maximal flag manifold admits invariant nearly-Kähler structures that are not Kähler.

The result of this paper form a partial extension of those of [8], in the sense that we consider here only nearly-Kähler structures and do not attempt a classification like in [8]. The point is that when dealing with flag manifolds apart from the maximal ones there are less room for equivalences, since we must consider subgroups of the Weyl group leaving invariant the isotropy subgroup. This makes the classifications less feasible then in maximal flag manifold.

2 Flag manifolds

The purpose of this section is to fix notations and state general results related to flag manifolds. We shall work with simple Lie algebras and groups only. The results in the semi-simple case are easily obtained by piecing together the simple components.

Thus let $\mathfrak{g}$ be a complex simple Lie algebra and G a connected Lie group with Lie algebra $\mathfrak{g}$. Fix once and for all a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and denote by Π the set of roots of the pair $(\mathfrak{g}, \mathfrak{h})$. Put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \forall H \in \mathfrak{h},\, [H, X] = \alpha(H)X\}$$

for the root space corresponding to α.

The Cartan-Killing form of $\mathfrak{g}$ is denoted by $\langle \cdot, \cdot \rangle$ and for a root α let $H_\alpha \in \mathfrak{h}$ be defined by $\alpha(\cdot) = \langle H_\alpha, \cdot \rangle$. In what follows we keep fixed a set of Weyl elements $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Pi$. These elements satisfy $[X_\alpha, X_{-\alpha}] = H_\alpha$ (or equivalently $\langle X_\alpha, X_{-\alpha} \rangle = 1$) and $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ with $m_{\alpha,\beta} \in \mathbb{R}$ and $m_{\alpha,\beta} = m_{-\alpha,-\beta}$, so that $m_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root (see e.g. Helgason [4] and [7]).

Let $\Pi^+ \subset \Pi$ be a choice of positive roots, denote by Σ the corresponding simple system of roots and put $\Pi^- = -\Pi^+$.

Given a subset $\Theta \subset \Sigma$ let $\langle \Theta \rangle$ be the set of roots spanned over $\mathbb{Z}$ by Θ and put $\langle \Theta \rangle^\pm = \langle \Theta \rangle \cap \Pi^\pm$. The standard parabolic subalgebra of $\mathfrak{g}$ determined by Θ is defined by

$$\mathfrak{p}_\Theta = \mathfrak{h} \oplus \sum_{\alpha \in \langle \Theta \rangle} \mathfrak{g}_\alpha \oplus \sum_{\beta \in \Pi^+ \setminus \langle \Theta \rangle^+} \mathfrak{g}_\beta.$$

The corresponding parabolic subgroup P_Θ is the normalizer of $\mathfrak{p}_\Theta$ in G. Forming the coset space we obtain the flag manifold defined by Θ:

$$\mathbb{F}_\Theta = G/P_\Theta.$$

We take as compact real form of $\mathfrak{g}$ the real subalgebra

$$\mathfrak{u} = \operatorname{span}_\mathbb{R}\{i\mathfrak{h}_\mathbb{R}, A_\alpha, iS_\alpha : \alpha \in \Pi\}$$

where $A_\alpha = X_\alpha - X_{-\alpha}$ and $S_\alpha = X_\alpha + X_{-\alpha}$. Denote by $U = \exp \mathfrak{u}$ the corresponding compact real form of G and write $K_\Theta = P_\Theta \cap U$. It is well known that $K_\Theta \subset U$ is the centralizer of a torus and since U acts transitively on each $\mathbb{F}_\Theta$, it follows that

$$\mathbb{F}_\Theta = G/P_\Theta = U/K_\Theta.$$

Let $\mathfrak{k}_\Theta$ be the Lie algebra of K_Θ and write $\mathfrak{k}_\Theta^\mathbb{C}$ for its complexification. We have $\mathfrak{k}_\Theta = \mathfrak{u} \cap \mathfrak{p}_\Theta$ and

$$\mathfrak{k}_\Theta^\mathbb{C} = \mathfrak{h} \oplus \sum_{\alpha \in \langle \Theta \rangle} \mathfrak{g}_\alpha.$$

Denote by x_Θ the origin of $\mathbb{F}_\Theta$. The tangent space $T_{x_\Theta}\mathbb{F}_\Theta$ can be identified with the orthogonal complement of $\mathfrak{k}_\Theta$ in $\mathfrak{u}$, namely

$$T_{x_\Theta}\mathbb{F}_\Theta \approx \eta_\Theta = \mathrm{span}_\mathbb{R}\{A_\alpha, iS_\alpha : \alpha \notin \langle\Theta\rangle\} = \sum_{\alpha\in\Pi\setminus\langle\Theta\rangle} \mathfrak{u}_\alpha,$$

where $\mathfrak{u}_\alpha = (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{u} = \mathrm{span}_\mathbb{R}\{A_\alpha, iS_\alpha\}$. By complexifying η_Θ we obtain the complex tangent space of $T^\mathbb{C}_{x_\Theta}\mathbb{F}_\Theta$, which can be identified with

$$\mathfrak{q}_\Theta = \sum_{\beta\in\Pi\setminus\langle\Theta\rangle} \mathfrak{g}_\beta.$$

The adjoint representations of $\mathfrak{k}_\Theta$ and K_Θ leave η_Θ invariant, so that we get a well defined representation of both $\mathfrak{k}_\Theta$ and K_Θ in η_Θ. Analogously the complex tangent space $\mathfrak{q}_\Theta$ is invariant under the adjoint representation of $\mathfrak{k}^\mathbb{C}_\Theta$. This representation is semi-simple, so that we can decompose $\mathfrak{q}_\Theta$ into irreducible components

$$\mathfrak{q}_\Theta = V_1 \oplus \cdots \oplus V_s$$

Since the Cartan subalgebra $\mathfrak{h}$ is contained in $\mathfrak{k}_\Theta$, it follows that each irreducible component V_i is a direct sum of root spaces. Thus for each $i = 1, \ldots, s$, there exists a subset $A(i) \subset \Pi \setminus \langle\Theta\rangle$ such that

$$V_i = \sum_{\alpha\in A(i)} \mathfrak{g}_\alpha.$$

In the sequel we abuse notation and say that two roots $\alpha, \beta \in \Pi \setminus \langle\Theta\rangle$ belong to the same irreducible component in case the corresponding root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are contained in some component V_i.

It is a standard fact that the roots $A(i)$ in an irreducible component are either all positive or all negative, because both Π^+ and Π^- are invariant under $\mathfrak{k}^\mathbb{C}_\Theta$. We also note that if $\alpha \in \langle\Theta\rangle$ and $\beta \in \Pi \setminus \langle\Theta\rangle$ are roots such that $\alpha+\beta$ is also a root then β and $\alpha+\beta$ are in the same irreducible components.

We conclude this section with the statement of the following well known facts (see e.g. [8], Lemma 4.11).

Lemma 2.1 *Let α and β be positive roots that $\alpha + \beta$ is a root.*

1. *Suppose that $\beta = \beta_1 + \beta_2$ with β_1 and β_2 roots. Then $\alpha + \beta_1$ or $\alpha + \beta_2$ is a root.*

2. *There are simple roots $\alpha_1, \ldots, \alpha_s$ such that $\beta = \alpha_1 + \cdots + \alpha_s$ and all the intermediate sums $\alpha + \alpha_1 + \cdots + \alpha_k$, $k = 1, \ldots, s$, are roots.*

3 Invariant almost Hermitian structures

A U-invariant Riemannian metric on a flag manifold $\mathbb{F}_\Theta$ is completely determined by its value at origin x_Θ, namely an inner product $(\cdot,\cdot)$ in η_Θ, which is invariant under the adjoint action of K_Θ. Such an inner product has the form $(X,Y)_\Lambda = -\langle \Lambda(X), Y\rangle$ with $\Lambda : \eta_\Theta \to \eta_\Theta$ positive-definite with respect to the Cartan-Killing form. The inner product $(\cdot,\cdot)_\Lambda$ admits a natural extension to a symmetric bilinear form on the complexification $\mathfrak{q}_\Theta$ of η_Θ. These complexified objects are denoted by the same letters as the real ones.

The K_Θ-invariance of $(\cdot,\cdot)_\Lambda$ implies that the elements of the standard basis A_α, iS_α, $\alpha \in \Pi \setminus \langle\Theta\rangle$, are eigenvectors of Λ, with the same eigenvalue. Thus, in the complex tangent space we have $\Lambda(X_\alpha) = \lambda_\alpha X_\alpha$ with $\lambda_\alpha = \lambda_{-\alpha} > 0$. Furthermore, λ_α is constant along the irreducible components of the adjoint action of K_Θ in η_Θ. In short:

- An invariant Riemannian metric in $\mathbb{F}_\Theta$ is given by a set $\Lambda = \{\lambda_\alpha, \alpha \in \Pi \setminus \langle\Theta\rangle\}$ such that (i) $\lambda_\alpha > 0$; (ii) $\lambda_{-\alpha} = \lambda_\alpha$ and (iii) $\lambda_\alpha = \lambda_\beta$ if α and β are in the same irreducible component.

In the sequel we also denote by $(\cdot,\cdot)_\Lambda$ the metric on $\mathbb{F}_\Theta$ associated to Λ and abuse notation and say that a set of positive numbers $\Lambda = \{\lambda_\alpha\}$ is an invariant metric in $\mathbb{F}_\Theta$.

Regarding invariant almost complex structures, the situation is analogous. In fact, such a structure is completely determined by its value $J : \eta_\Theta \to \eta_\Theta$ on the tangent space at the origin. The map J satisfies $J^2 = -1$ and commutes with the adjoint action of K_Θ on η_Θ. We denote by the same letter the real valued structure J and its complexification to $\mathfrak{q}_\mathbb{C}$. The invariance of J entails that $J(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$ for all $\alpha \in \Pi \setminus \langle\Theta\rangle$. The eigenvalues of J are $\pm i$ and the eigenvectors in $\mathfrak{q}_\Theta$ are X_α, $\alpha \in \Pi \setminus \langle\Theta\rangle$. Hence $J(X_\alpha) = i\varepsilon_\alpha X_\alpha$ with $\varepsilon_\alpha = \pm 1$ satisfying $\varepsilon_\alpha = -\varepsilon_{-\alpha}$. As usual the eigenvectors associated to $+i$ are said to be of type $(1,0)$ while the $-i$-eigenvectors are of type $(0,1)$. Thus the $(1,0)$ vectors are linear combinations of X_α, $\varepsilon_\alpha = +1$, and the $(0,1)$ vectors are spanned by X_α, $\varepsilon_\alpha = -1$. It summarizes as follows:

- An invariant almost complex structure on $\mathbb{F}_\Theta$ is given by a set $\{\varepsilon_\alpha, \alpha \in \Pi \setminus \langle\Theta\rangle\}$ such that (i) $\varepsilon_\alpha = \pm 1$; (ii) $\varepsilon_{-\alpha} = -\varepsilon_{-\alpha}$ and (ii) $\varepsilon_\alpha = \varepsilon_\beta$ if α and β are in the same irreducible component.

Again we abuse of notation and say that an invariant structure on $\mathbb{F}_\Theta$ is the set $J = \{\varepsilon_\alpha\}$. Accordingly we say that a pair (J, Λ) is an invariant almost Hermitian structure on $\mathbb{F}_\Theta$.

It is well known (and easy to prove) that any invariant metric $(\cdot,\cdot)_\Lambda$ is almost Hermitian with respect to an invariant J, that is, $(JX, JY)_\Lambda = (X, Y)_\Lambda$. Let $\Omega = \Omega_{J,\Lambda}$ be the corresponding Kähler form

$$\Omega(X, Y) = (X, JY)_\Lambda = -\langle \Lambda X, JY \rangle.$$

It follows at once that $\Omega(X_\alpha, X_\beta) = -i\lambda_\alpha \varepsilon_\beta \langle X_\alpha, X_\beta \rangle$, so that $\Omega(X_\alpha, X_\beta) = 0$ if $\alpha + \beta \neq 0$ and $\Omega(X_\alpha, X_{-\alpha}) = i\lambda_\alpha \varepsilon_\alpha$. Also, an easy computation shows that $d\Omega(X_\alpha, X_\beta, X_\gamma) = 0$ unless $\alpha + \beta + \gamma = 0$, and in this case

$$3d\Omega(X_\alpha, X_\beta, X_\gamma) = im_{\alpha,\beta}(\varepsilon_\alpha\lambda_\alpha + \varepsilon_\beta\lambda_\beta + \varepsilon_\gamma\lambda_\gamma) \tag{1}$$

(cf. [8], Proposition 2.1).

Recall that an almost Hermitian structure is said to be $(1,2)$-symplectic (or quasi-Kähler) if its Kähler form, say ω, satisfies $d\omega(u, v, w) = 0$ if one of the complex vectors u, v, w is a $(1,0)$-vector while the other two are $(0,1)$-vectors. The nearly-Kähler structures form a subclass of the $(1,2)$-symplectic ones and is characterized by the condition $\nabla_X(J)(X) = 0$, where ∇ is Levi-Civita connection of the metric (see Gray-Hervella [3]).

For the invariant structures on the flag manifolds it can be proved that a pair (J, Λ) is nearly-Kähler if and only if it is $(1,2)$-symplectic and satisfies $(N(X, Y), X)_\Lambda \equiv 0$ for all $X, Y \in \eta_\Theta$, where N is the Nijenhuis tensor of J (this follows from the classification in [3]; see also [8], Section 7). In the sequel we use these conditions expressed in terms of roots in $\Pi \setminus \langle\Theta\rangle$. The precise statement requires the following terminology.

Definition 3.1 *Let $J = \{\varepsilon_\alpha\}$ be an invariant almost complex structure on $\mathbb{F}_\Theta$. The triple of roots $\alpha, \beta, \gamma \in \Pi \setminus \langle\Theta\rangle$ with $\alpha + \beta + \gamma = 0$ is said to be of J-type $\{0, 3\}$ (or a $\{0, 3\}$-triple, for short) if $\varepsilon_\alpha = \varepsilon_\beta = \varepsilon_\gamma$. It is of J-type $\{1, 2\}$ (or a $\{1, 2\}$-triple) otherwise.*

Note that the triples $\{\alpha, \beta, \gamma\}$ and $\{-\alpha, -\beta, -\gamma\}$ are of the same J-type, that is, either both are $\{0, 3\}$-triples or both are $\{1, 2\}$-triples.

The next statement gives a characterization of nearly-Kähler pairs in terms of triples of roots.

Proposition 3.2 *The invariant pair* $(J = \{\varepsilon_\alpha\}, \Lambda = \{\lambda_\alpha\})$ *is nearly-Kähler in* F_Θ *if, and only if, the following two conditions hold:*

1. $\lambda_\alpha = \lambda_\beta = \lambda_\gamma$ *if* $\{\alpha, \beta, \gamma\}$ *is a* $\{0, 3\}$*-triple, and*

2. $\lambda_\alpha = \lambda_\beta + \lambda_\gamma$ *if* $\{\alpha, \beta, \gamma\}$ *is a* $\{1, 2\}$*-triple with* $\varepsilon_\beta = \varepsilon_\gamma$*.*

Proof: See [8], Section 7, and [10], Theorem 9.17 (iii). □

We note that the first condition is equivalent to the annihilation of the tensor $(N(X,Y), X)_\Lambda$ whereas the second one holds if and only if the pair (J, Λ) is $(1, 2)$-symplectic, as follows by formula (1) for $d\Omega$ (see [8], Proposition 2.3, for details). That formula also shows that if a structure is Kähler then there are no $\{0, 3\}$-triples, and conversely, if there no $\{0, 3\}$-triples and the structure is $(1, 2)$-symplectic then it is Kähler. For later reference we state this fact with nearly-Kähler in place of $(1, 2)$-symplectic.

Lemma 3.3 *If* J *admits only* $\{1, 2\}$*-triples and* (J, Λ) *is nearly-Kähler then the structure is Kähler.*

4 Triples for nearly-Kähler structures

This section is devoted to the proof of the following theorem which shows that the nearly-Kähler condition is very restrictive for the invariant structures on the flag manifolds. This theorem will be used in the next section to relate invariant nearly-Kähler structures to fixed point sets of order three automorphisms, obtaining a proof of the conjecture stated in the introduction.

Theorem 4.1 *Let* (J, Λ) *be an invariant nearly-Kähler structure on a flag manifold* $\mathbb{F}_\Theta$*. Then the triples of roots are all of the same* J*-type, either* $\{1, 2\}$ *or* $\{0, 3\}$*.*

The proof of this theorem will be accomplished in two steps, namely

1. if there are triples of both J-types, then there exists at least one root β in the intersection of a $\{1, 2\}$-triple with a $\{0, 3\}$-triple.

2. If (J, Λ) is nearly-Kähler then $\{1,2\}$-triples cannot intercept $\{0,3\}$-triples.

The first step does not require the nearly-Kähler condition and uses only general facts about root systems. On the other hand the second part uses extensively Proposition 3.2, which characterize nearly-Kähler structures in terms of roots.

We start by introducing the following useful notation.

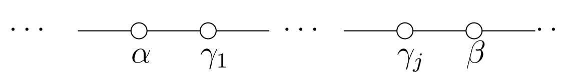
Definition 4.2 *Let $\Theta \subset \Sigma$ be a subset of the simple system of roots. Let $\{a,b\}$ be either $\{1,2\}$ or $\{0,3\}$ and denote by $T^{\Theta}_{\{a,b\}}$ (or simply $T_{\{a,b\}}$) the set of the roots $\alpha \in \Pi \setminus \langle\Theta\rangle$ such that there are roots $\beta, \gamma \in \Pi \setminus \langle\Theta\rangle$ such that $\{\alpha, \beta, \gamma\}$ is a $\{a,b\}$-triple.*

Note that we may have $T_{\{1,2\}} \cap T_{\{0,3\}} \neq \emptyset$, that is, a root α may belong to $\{1,2\}$-triples as well as to $\{0,3\}$-triples. In fact, we will show in the next few lemmas that if J admits $\{1,2\}$-triples as well as $\{0,3\}$-triples then $T_{\{1,2\}} \cap T_{\{0,3\}} \neq \emptyset$.

Lemma 4.3 *Suppose that $T_{\{0,3\}} \cap T_{\{1,2\}} = \emptyset$. Then either $T_{\{0,3\}} \cap (\Sigma \setminus \Theta) = \emptyset$ or $T_{\{1,2\}} \cap (\Sigma \setminus \Theta) = \emptyset$.*

Proof: Take $\alpha, \beta \in \Sigma \setminus \Theta$. We shall exhibit two triples of roots containing respectively α and β and such that they have non-empty intersection. Clearly, this implies the lemma.

Now, note that we can find a subdiagram $\gamma_1, \ldots, \gamma_j$ of the Dynkin diagram linking α to β:



Since the sum of roots in any Dynkin diagram is also a root, it follows that $\alpha + \gamma_1 + \cdots + \gamma_j$, $\gamma_1 + \cdots + \gamma_j + \beta$ and $\alpha + \gamma_1 + \cdots + \gamma_j + \beta$ are roots. Therefore we have the following triples of roots outside $\langle\Theta\rangle$:

$$\{\alpha, (\gamma_1 + \cdots + \gamma_j + \beta), -(\alpha + \gamma_1 + \cdots + \gamma_j + \beta)\}$$
$$\{(\alpha + \gamma_1 + \cdots + \gamma_j), \beta, -(\alpha + \gamma_1 + \cdots + \gamma_j + \beta)\},$$

which satisfy the desired conditions. $\square$

Lemma 4.4 *Let u, v be positive roots such that $u+v$ is a root. Denote γ the lowest root in the irreducible component of $u+v$. Then there are roots α and β in the irreducible components of u and v, respectively, such that $\alpha+\beta=\gamma$. The same result holds for the highest root in place of lowest one.*

Proof: If $u+v$ is not the lowest root then there exists a simple root $w \in \Theta$ such that $u+v-w$ is a root. This implies that $[\mathfrak{g}_{-w}, \mathfrak{g}_{u+v}] \neq \{0\}$. But

$$[\mathfrak{g}_{-w}, \mathfrak{g}_{u+v}] = [[\mathfrak{g}_{-w}, \mathfrak{g}_{u}], \mathfrak{g}_{v}] + [\mathfrak{g}_{u}, [\mathfrak{g}_{-w}, \mathfrak{g}_{v}]],$$

so that one of the terms in the right hand side is $\neq 0$. This implies that $u-w$ or $v-w$ is a root. Hence, the smaller root $(u+v)-w$ can be written as a sum of roots either as $(u-w)+v$ or as $u+(v-w)$. Clearly, by subtracting w we do not leave the irreducible components. Therefore, we can proceed by induction until we reach γ, which is a sum of roots in the irreducible components of u and v, respectively. The same argument holds for the highest root in the irreducible components. $\square$

Corollary 4.5 *Suppose that $\{u, v, -(u+v)\}$ is a triple of J-type $\{a, b\}$ ($=\{0,3\}$ or $\{1,2\}$). Then there exists a triple $\{\alpha, \beta, \gamma\}$ of J-type $\{a, b\}$ such that γ is the lowest (respectively highest) root of its irreducible component.*

Proof: Follows from the lemma after observing that triples having roots in the same irreducible components have the same J-type, because ε_α is constant along an irreducible component. $\square$

Now we can conclude the first part of the proof of Theorem 4.1.

Proposition 4.6 *Suppose that $T_{\{0,3\}} \cap T_{\{1,2\}} = \emptyset$. Then $T_{\{1,2\}} = \emptyset$ or $T_{\{0,3\}} = \emptyset$.*

Proof: The assumption $T_{\{0,3\}} \cap T_{\{1,2\}} = \emptyset$ combined with Lemma 4.3 ensures that all the triples containing simple roots are of the same J-type, say $\{a, b\}$. Suppose that there exists a triple of J-type different from $\{a, b\}$. Then by Corollary 4.5 there exists a triple $\{\alpha, \beta, \gamma\}$ of J-type different from $\{a, b\}$ such that $\gamma > 0$ is minimal in its irreducible component. If γ is a simple root then $\gamma \notin \Theta$ and there is nothing to prove. On the other hand, there exists $u \in \Theta$ such that $\gamma - u$ is a root. Then $\{u, \gamma - u, \gamma\}$ is a triple that contains the simple root u, so that it is of J-type $\{a,b\}$. Therefore $\gamma \in T_{\{0,3\}} \cap T_{\{1,2\}}$, contradicting the assumption. $\square$

We turn now to the second part of the proof of Theorem 4.1. It will be a consequence of the following lemmas.

Lemma 4.7 *Let $(J = \{\varepsilon_\alpha\}, \Lambda = \{\lambda_\alpha\})$ be a nearly-Kähler structure. Then J does not admit a $\{0,3\}$-triple $\{\alpha, \beta, -(\alpha+\beta)\}$ together with a $\{1,2\}$-triple $\{\beta, \gamma_1, \gamma_2\}$ such that $\varepsilon_{\gamma_1} = \varepsilon_{\gamma_2}$.*

Proof: Suppose by contradiction that there are triples as in the statement. It will be convenient to take $\beta_i = -\gamma_i$, so that $\{-\beta, \beta_1, \beta_2\}$ is a $\{1,2\}$-triple with $\varepsilon_{-\beta} \neq \varepsilon_{\beta_1} = \varepsilon_{\beta_2}$. Then by Proposition 3.2 we have $\lambda_\beta = \lambda_{\beta_1} + \lambda_{\beta_2}$, so that

$$\lambda_\beta > \lambda_{\beta_1}, \lambda_{\beta_2}. \tag{2}$$

Also by Proposition 3.2 and the fact that $\varepsilon_\alpha = \varepsilon_\beta = \varepsilon_{-(\alpha+\beta)}$, it follows that

$$\lambda_\alpha = \lambda_\beta = \lambda_{-(\alpha+\beta)}. \tag{3}$$

Now, since $\alpha+\beta$ is a root Lemma 2.1 implies that either $\alpha+\beta_1$ or $\alpha+\beta_2$ is root. We can assume without loss of generality that $\alpha+\beta_1$ is root.

In this case we claim that $\{-(\alpha+\beta_1), \alpha, \beta_1\}$ is a $\{1,2\}$-triple with $\varepsilon_\alpha = \varepsilon_{\beta_1}$ (that is, $\varepsilon_{\alpha+\beta_1} = \varepsilon_\alpha = \varepsilon_{\beta_1}$). To see this note first that $\varepsilon_\alpha = \varepsilon_{\beta_1}$ because $\{\alpha, \beta, -(\alpha+\beta)\}$ is $\{0,3\}$-triple (so that $\varepsilon_\alpha = \varepsilon_\beta \neq \varepsilon_{-\beta}$) and $\varepsilon_{-\beta} \neq \varepsilon_{\beta_1}$, hence $\varepsilon_{\beta_1} = \varepsilon_\beta = \varepsilon_\alpha$. Now, suppose by contradiction that $\{-(\alpha+\beta_1), \alpha, \beta_1\}$ is $\{0,3\}$-triple. Then, by Proposition 3.2, we would have $\lambda_\alpha = \lambda_{\beta_1} = \lambda_{-(\alpha+\beta_1)}$. But $\lambda_\alpha = \lambda_\beta$ (by (3)) and $\lambda_\beta > \lambda_{\beta_1}$ (by (2)), so that $\lambda_\alpha > \lambda_{\beta_1}$ which is a contradiction, proving the claim.

Therefore, applying Proposition 3.2 to the $\{1,2\}$-triple $\{-(\alpha+\beta_1), \alpha, \beta_1\}$ we have

$$\lambda_{-(\alpha+\beta_1)} = \lambda_\alpha + \lambda_{\beta_1}. \tag{4}$$

Combining the claim (that is, $\varepsilon_{\alpha+\beta_1} = \varepsilon_\alpha = \varepsilon_{\beta_1}$) with the assumptions ($\varepsilon_{-(\alpha+\beta)} = \varepsilon_\alpha = \varepsilon_\beta \neq \varepsilon_{-\beta} \neq \varepsilon_{\beta_2}$), we conclude that $\{\alpha+\beta_1, \beta_2, -(\alpha+\beta)\}$ is a $\{0,3\}$-triple. Hence by Proposition 3.2 we have

$$\lambda_{\alpha+\beta_1} = \lambda_{\beta_2} = \lambda_{-(\alpha+\beta)}. \tag{5}$$

But this leads to a contradiction. In fact, $\lambda_\alpha = \lambda_\beta$ (by (3)), $\lambda_\beta > \lambda_{\beta_2}$ (by (2)) and $\lambda_{\alpha+\beta_1} = \lambda_{\beta_2}$ (by (5)). Therefore, $\lambda_\alpha > \lambda_{\beta_2} = \lambda_{\alpha+\beta_1} = \lambda_{-(\alpha+\beta_1)}$

contradicting (4) and concluding the proof. $\square$

Lemma 4.8 *If* (J, Λ) *is nearly-Kähler then* J *does not admit a* $\{0,3\}$*-triple* $\{\alpha, \beta, -(\alpha+\beta)\}$ *together with a* $\{1,2\}$*-triple* $\{-\beta, \beta_1, \beta_2\}$ *such that* $\varepsilon_{-\beta} = \varepsilon_{\beta_1} \neq \varepsilon_{\beta_2}$.

Proof: First we have by Proposition 3.2 that

$$\lambda_\alpha = \lambda_\beta = \lambda_{-(\alpha+\beta)}. \tag{6}$$

Also, since $\{-\beta, \beta_1, \beta_2\}$ is a $\{1,2\}$-triple with $\varepsilon_{-\beta} = \varepsilon_{\beta_1}$, it follows by Proposition 3.2 that $\lambda_{\beta_2} = \lambda_{\beta_1} + \lambda_{-\beta}$, so that

$$\lambda_{\beta_2} > \lambda_\beta, \lambda_{\beta_1}. \tag{7}$$

Now, by Lemma 2.1 either $\alpha + \beta_1$ or $\alpha + \beta_2$ is root. In the rest of the proof we consider these possibilities separately (note the lack of symmetry between β_1 and β_2).

Suppose that $\alpha+\beta_2$ is root. We claim that $\{-(\alpha+\beta_2), \alpha, \beta_2\}$ is a $\{1,2\}$-triple with $\varepsilon_\alpha = \varepsilon_{\beta_2}$ (that is $\varepsilon_{\alpha+\beta_2} = \varepsilon_\alpha = \varepsilon_{\beta_2}$). In fact, we have first $\varepsilon_\alpha = \varepsilon_{\beta_2}$ because $\{\alpha, \beta, -(\alpha+\beta)\}$ is $\{0,3\}$-triple, implying that $\varepsilon_\alpha = \varepsilon_\beta \neq \varepsilon_{-\beta}$. But $\varepsilon_{-\beta} \neq \varepsilon_{\beta_2}$, so that $\varepsilon_{\beta_2} = \varepsilon_\beta = \varepsilon_\alpha$. Now, suppose to the contrary that $\{-(\alpha+\beta_2), \alpha, \beta_2\}$ is a $\{0,3\}$-triple. Then $\lambda_{-(\alpha+\beta_2)} = \lambda_\alpha = \lambda_{\beta_2}$ (by Proposition 3.2). However, $\lambda_\beta = \lambda_\alpha$ (by (6)) and $\lambda_{\beta_2} > \lambda_\beta$ (by (7)), so that $\lambda_{\beta_2} > \lambda_\alpha$ which is a contradiction, proving the claim.

Therefore, Proposition 3.2 implies that

$$\lambda_{-(\alpha+\beta_2)} > \lambda_\alpha, \lambda_{\beta_2}. \tag{8}$$

On the other hand $\{\alpha+\beta_2, \beta_1, -(\alpha+\beta)\}$ is a $\{1,2\}$-triple with $\varepsilon_{\alpha+\beta_2} = \varepsilon_{-(\alpha+\beta)}$, because $\varepsilon_{\alpha+\beta_2} = \varepsilon_\alpha = \varepsilon_{-(\alpha+\beta)}$ and $\varepsilon_{\beta_1} = \varepsilon_{-\beta} \neq \varepsilon_{-(\alpha+\beta)}$. Hence,

$$\lambda_{\beta_1} > \lambda_{\alpha+\beta_2}, \lambda_{-(\alpha+\beta)}. \tag{9}$$

Thus we have got $\lambda_{\beta_2} > \lambda_{\beta_1}$ (by (7)), $\lambda_{\beta_1} > \lambda_{\alpha+\beta_2}$ (by (9)) and $\lambda_{\alpha+\beta_2} > \lambda_{\beta_2}$ (by (8)). But this implies $\lambda_{\beta_2} > \lambda_{\beta_1} > \lambda_{\alpha+\beta_2} > \lambda_{\beta_2}$, which is absurd.

Assume now that $\alpha+\beta_1$ is a root. Then we claim that $\{-(\alpha+\beta_1), \beta_1, \alpha\}$ is a $\{0,2\}$-triple with $\varepsilon_{-(\alpha+\beta_1)} = \varepsilon_{\beta_1}$ (that is, $\varepsilon_{\alpha+\beta_1} = \varepsilon_\alpha \neq \varepsilon_{\beta_1}$). In fact, note first that $\varepsilon_\alpha = \varepsilon_\beta$ and $\varepsilon_{-\beta} = \varepsilon_{\beta_1}$, so that $\varepsilon_\alpha \neq \varepsilon_{\beta_1}$. Now, suppose by contradiction that $\{-(\alpha+\beta_1),\beta_1,\alpha\}$ is a $\{0,2\}$-triple with $\varepsilon_{-(\alpha+\beta_1)}=\varepsilon_\alpha$. Then we would have $\varepsilon_{\alpha+\beta_1}\neq\varepsilon_\alpha=\varepsilon_\beta=\varepsilon_{\beta_2}$ and $\varepsilon_{\alpha+\beta_1}\neq\varepsilon_\alpha=\varepsilon_{-(\alpha+\beta)}$, so that $\{\alpha+\beta_1,\beta_2,-(\alpha+\beta)\}$ is a $\{1,2\}$-triple with $\varepsilon_{\beta_2}=\varepsilon_{-(\alpha+\beta)}$. Applying Proposition 3.2 to the triples $\{-(\alpha+\beta_1),\beta_1,\alpha\}$ and $\{\alpha+\beta_1,\beta_2,-(\alpha+\beta)\}$ we get $\lambda_{\beta_1}=\lambda_\alpha+\lambda_{-(\alpha+\beta_1)}$ and $\lambda_{\alpha+\beta_1}=\lambda_{\beta_2}+\lambda_{-(\alpha+\beta)}$. But this implies $\lambda_{\alpha+\beta_1}<\lambda_{\beta_1}$ and $\lambda_{\beta_2}<\lambda_{\alpha+\beta_1}$, that is, $\lambda_{\beta_2}<\lambda_{\beta_1}$ which contradicts (7). Thus the claim follows.

Therefore, $\alpha+\beta_1=\varepsilon_\alpha=\varepsilon_{-(\alpha+\beta)}=\varepsilon_{\beta_2}$, so that $\{\alpha+\beta_1,\beta_2,-(\alpha+\beta)\}$ is a $\{0,3\}$-triple. Hence, by Proposition 3.2 we get

$$\lambda_{\alpha+\beta_1}=\lambda_{\beta_2}=\lambda_{-(\alpha+\beta)}. \tag{10}$$

Finally we have by (7) that $\lambda_{\beta_2}>\lambda_\beta$ and by (6) that $\lambda_\beta=\lambda_{-(\alpha+\beta)}$. But this contradicts (10), showing that there are not triples as in the statement. $\square$

End of the proof of Theorem 4.1: Suppose by contradiction that the nearly-Kähler structure $(J=\{\varepsilon_\alpha\},\Lambda=\{\lambda_\alpha\})$ has $\{0,3\}$-triples as well as $\{1,2\}$-triples. By Proposition 4.6 we can find a $\{0,3\}$-triple $\{\alpha,\beta,-(\alpha+\beta)\}$ and a $\{1,2\}$-triple $\{\beta,\gamma_1,\gamma_2\}$, having in common the root β. For the $\{1,2\}$-triple there are two possibilities:

1. $\varepsilon_{\gamma_1}=\varepsilon_{\gamma_2}\neq\varepsilon_\beta$, and
2. $\varepsilon_\beta=\varepsilon_{\gamma_1}$ or $\varepsilon_\beta=\varepsilon_{\gamma_2}$ (and hence $\varepsilon_{\gamma_1}\neq\varepsilon_{\gamma_2}$).

The first possibility is ruled out by Lemma 4.7 while the second one is not possible by Lemma 4.8. Since both possibilities lead to a contradiction, this concludes the proof.

5 Order 3 automorphisms

In this section we apply Theorem 4.1 to get conditions in terms of order three automorphisms for an invariant pair (J,Λ) to be nearly-Kähler. This discussion requires the height of a root with respect to Θ in the following sense:

Definition 5.1 *The height $h_\Theta(\alpha)$ of $\alpha\in\Pi\backslash\langle\Theta\rangle$ with respect to Θ is given by $\sum|a_i|$ with a_i running through the coefficients of α with respect to the* *simple roots in $\Sigma \backslash \Theta$. The height of $\Pi \backslash \langle \Theta \rangle$ is $h_\Theta = \max_{\alpha \in \Pi \backslash \langle \Theta \rangle} h_\Theta(\alpha)$ or equivalently $h_\Theta = h_\Theta(\mu)$ where μ is the highest root.*

We shall prove below that nearly-Kähler invariant structures on $\mathbb{F}_\Theta$ are Kähler if $h_\Theta \geq 3$. On the other hand $h_\Theta < 3$ is a necessary condition for the isotropy subalgebra of $\mathbb{F}_\Theta$ to be the fixed point set of an automorphism of order 3.

Lemma 5.2 *Let Θ be such that $h_\Theta \geq 3$. Then there are positive roots $\alpha, \beta, \gamma, \beta_1$ and β_2 outside $\langle \Theta \rangle$ such that $\gamma = \alpha + \beta$ and $\beta = \beta_1 + \beta_2$.*

Proof: Let β be a root such that $h_\Theta(\beta) = 2$ and β is the highest root in its irreducible component. Since $h_\Theta(\beta) < h_\Theta$, it follows that β is not the highest root of Π. Hence there exists a simple root α such that $\gamma = \alpha + \beta$ is a root. We have $\alpha \notin \Theta$ because β is the highest root of its irreducible component. Hence to conclude the proof it remains to write β as a sum of two roots. This follows by Lemma 4.4 as soon as we decompose some root in the same irreducible component as β. Thus let β_l be the lowest root in the irreducible component of β. Clearly, $h_\Theta(\beta_l) = h_\Theta(\beta) = 2$, so that there exists a simple root $u \notin \langle \Theta \rangle$ such that $\beta_l - u$ is a root. Hence, $\beta_l = (\beta_l - u) + u$ implying that β is a sum of roots outside $\langle \Theta \rangle$ as well. $\square$

Lemma 5.3 *Let J be an almost complex structure on the flag manifold $\mathbb{F}_\Theta$ with $h_\Theta \geq 3$. Then J admits $\{1,2\}$-triples.*

Proof: Suppose by contradiction that all triples are $\{0,3\}$ and let $\gamma = \alpha + \beta$ and $\beta = \beta_1 + \beta_2$ be the roots given by the above lemma. Since $\{-\gamma, \alpha, \beta\}$ and $\{-\beta, \beta_1, \beta_2\}$ are $\{0,3\}$-triples, it follows that $\varepsilon_{\beta_1} = \varepsilon_{\beta_2} = \varepsilon_{-\beta} \neq \varepsilon_\beta = \varepsilon_\alpha = \varepsilon_{-\gamma}$. On the other hand, by Lemma 2.1, we can assume that $\alpha + \beta_1$ is root. Then $\{\alpha + \beta_1, \beta_2, -\gamma\}$ is a $\{0,3\}$-triple, so that $\varepsilon_{\beta_2} = \varepsilon_{-\gamma}$, which is contradiction. $\square$

Combining this lemma with Theorem 4.1 we conclude that if (J, Λ) is nearly-Kähler in $\mathbb{F}_\Theta$ with $h_\Theta \geq 3$ then J admits only $\{1,2\}$-triples. A fortiori Lemma 3.3 implies that the structure is Kähler. Thus we get our main result regarding the relationship between the height h_Θ and nearly-Kähler structures on $\mathbb{F}_\Theta$.

Theorem 5.4 *The flag manifold $\mathbb{F}_\Theta$ admits an invariant structure (J, Λ) which is nearly-Kähler but not Kähler if and only if $h_\Theta = 2$.*

Proof: If (J, Λ) is nearly-Kähler and $h_\Theta \geq 3$ then J has only $\{1, 2\}$-triples, so that (J, Λ) is Kähler. On the other hand if $h_\Theta = 1$ then there are no triples at all, so that any invariant structure is Kähler. Now suppose that $h_\Theta = 2$ and define $J = \{\varepsilon_\alpha\}$ by $\varepsilon_\alpha = +1$ if $h_\Theta(\alpha) = 1$ and $\varepsilon_\alpha = -1$ if $h_\Theta(\alpha) = 2$. Then there are only $\{0, 3\}$-triples, so that if we take $\Lambda = \{\lambda_\alpha\}$ with $\lambda_\alpha > 0$ independent of α, it follows that (J, Λ) is nearly-Kähler. Also, (J, Λ) is not Kähler because there are $\{0, 3\}$-triples. $\square$

Now we shall rephrase the above theorem in terms of order three automorphisms of $\mathfrak{g}$. This yields the result conjectured by Wolf-Gray [10]. The key point is the following lemma which relates the height h_Θ of $\mathbb{F}_\Theta$ with order three automorphisms.

Lemma 5.5 *Let $\mathbb{F}_\Theta$ be a flag manifold such that the isotropy subalgebra $\mathfrak{k}_\Theta$ is the fixed point set of an automorphism of order three. Then $h_\Theta < 3$.*

Proof: Let ϕ be as in the statement. The eigenvalues of ϕ have the form ζ^i, $i = 0, 1, 2$, where ζ is a primitive third root of unity. Since the Cartan subalgebra $\mathfrak{h}$ is pointwise fixed by ϕ, it follows that the eigenspaces of ϕ are sums of root spaces. Hence we can write

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Pi_0} \mathfrak{g}_\alpha \oplus \sum_{\beta \in \Pi_1} \mathfrak{g}_\beta \oplus \sum_{\gamma \in \Pi_2} \mathfrak{g}_\gamma,$$

where $\Pi_i = \{\alpha : \phi(X_\alpha) = \zeta^i X_\alpha\}$, $i = 0, 1, 2$. Note that the assumption that the isotropy is the fixed point set of ϕ amounts to $\Pi_0 = \langle\Theta\rangle$. Also, if $\alpha \in \Pi_{\zeta^i}$ and $\beta \in \Pi_{\zeta^j}$ are roots such that $\alpha + \beta$ is a root then

$$m_{\alpha,\beta}\phi(X_{\alpha+\beta}) = \phi[X_\alpha, X_\beta] = [\phi X_\alpha, \phi X_\beta] = [\zeta^i X_\alpha, \zeta^j X_\beta] = m_{\alpha,\beta}\zeta^{i+j} X_{\alpha+\beta}$$

that is, $\alpha + \beta \in \Pi_k$ with $i + j = k \pmod 3$. Combining this with the fact that $\Pi_0 = \langle\Theta\rangle$ we see that if $h_\Theta(\alpha) > 0$ and $h_\Theta(\beta) > 0$ then $i = j$, for otherwise we would have $h_\Theta(\alpha + \beta) = 0$.

Now, suppose by contradiction that $h_\Theta \geq 3$ and let $\alpha, \beta, \gamma, \beta_1$ and β_2 be roots outside $\langle\Theta\rangle$ with $\gamma = \alpha + \beta$ and $\beta = \beta_1 + \beta_2$, as ensured by Lemma 5.2. Then $\mathfrak{g}_{\beta_1}$ and $\mathfrak{g}_{\beta_2}$ are contained in the same ϕ-eigenspace, say $\beta_1, \beta_2 \in \Pi_i$.

This implies that $\beta \in \Pi_j$ with $j = 2i \pmod 3$ and hence $j \neq i$. Since $\alpha + \beta$ is a root, it follows that $\alpha \in \Pi_j$. But by Lemma 2.1, $\alpha + \beta_1$ or $\alpha + \beta_2$ is a root, yielding to a contradiction because the sum of positive roots in different ϕ-eigenspaces is not a root. $\square$

Theorem 5.6 *The flag manifold* $\mathbb{F}_\Theta$ *admits an invariant structure* (J, Λ) *which is nearly-Kähler but not Kähler if and only if*

1. *the isotropy subalgebra* $\mathfrak{k}_\Theta$ *is the fixed point set of an automorphism* ϕ *of order three and*

2. $\mathbb{F}_\Theta$ *is not Hermitian symmetric.*

Proof: In view of Theorem 5.4 it must be checked that the two conditions together are equivalent to $h_\Theta = 2$. By the above lemma the first condition implies that $h_\Theta = 1$ or 2. But if $h_\Theta = 1$ then $\mathfrak{k}_\Theta$ is the fixed point of the order two automorphism ψ which is the identity in $\mathfrak{h}$ and $\psi(X_\alpha) = i^{h_\Theta(\alpha)} X_\alpha$. This implies that $\mathbb{F}_\Theta$ is Hermitian symmetric. Thus $h_\Theta = 2$.

Conversely, let $h_\Theta = 2$ and ζ a primitive third root of unity. Then the automorphism η which is the identity in $\mathfrak{h}$ and $\eta(X_\alpha) = \zeta^{h_\Theta(\alpha)} X_\alpha$ has order three and has $\mathfrak{k}_\Theta$ as fixed point set. $\square$

References

[1] N. Cohen, C. J. C. Negreiros and L.A.B. San Martin: $(1,2)$-*Symplectic metrics on flag manifolds and tournaments.* Bull. London Math. Soc., **34** (2002), 641-649.

[2] N. Cohen, C. J. C. Negreiros and L.A.B. San Martin: *A rank-three condition for invariant* $(1,2)$-*symplectic almost Hermitian structures on flag manifolds.* Bull. Bras. Math. Soc., New Series **33** (2002), 49-73.

[3] A. Gray and L. M. Hervella: *The sixteen classes of almost Hermitian manifolds and their linear invariants.* Ann. Mat. Pura Appl., **123** (1980), 35-58.

[4] S. Helgason: Differential geometry, Lie groups and symmetric spaces. Academic Press (1978).

[5] S. Kobayashi and K. Nomizu: Foundations of differential geometry. Interscience Publishers, vol 2 (1969).

[6] B. Kostant: *The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations*. Internat. Math. Res. Notices, **5** (1998), 225-252.

[7] L. A. B. San Martin: Álgebras de Lie. Ed. Unicamp (1999).

[8] L. A. B. San Martin and C. J. C. Negreiros: *Invariant almost Hermitian structures on flag manifolds*. Advances in Math., **178** (2003), 277-310.

[9] J. A. Wolf and A. Gray: *Homogeneous spaces defined by Lie group automorphisms I*. J. Diff. Geom., **2** (1968), 77-114.

[10] J. A. Wolf and A. Gray: *Homogeneous spaces defined by Lie group automorphisms II*. J. Diff. Geom., **2** (1968), 115-159.