POSITIVE AND MULTIPLE SOLUTIONS FOR QUASILINEAR PROBLEMS

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Abstract. In this paper we establish the existence of positive and multiple solutions for the quasilinear elliptic problem

$$\begin{aligned} -\Delta_p u = g(x,u) & \quad \text{in} \quad \Omega \\ u = 0 & \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(x,0) = 0$ and which is asymptotically linear. We suppose that $g(x,t)/t$ tends to an L^r-function, $r > N/p$ if $1 < p \leq N$ and $r = 1$ if $p > N$, which can change sign. We consider both cases, resonant and nonresonant.

1. Introduction

Let us consider the problem

$$\begin{aligned} -\Delta_p u = g(x,u) & \quad \text{in} \quad \Omega \\ u = 0 & \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(x,0) = 0$, which implies that (1) possesses the trivial solution $u = 0$. We will be interested in nontrivial solutions. Here Δ_p denotes the p-Laplace operator, that is, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

Assume that g have a subcritical growth, that is,

$$|g(x,t)| \leq c(1 + |t|^{q-1}), \quad \text{a.e in} \quad \Omega, \quad t \in \mathbb{R}, \tag{2}$$

where $q \in [1, p^*[$, where $p^* = pN/(N-p)$ if $1 < p < N$ and $p^* = \infty$ if $1 < N \leq p$. The classical solutions of the problem (1) correspond to critical points of the functional F defined on $W_0^{1,p}(\Omega)$, by

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega G(x,u)dx, \quad u \in W_0^{1,p}(\Omega), \tag{3}$$

where $G(x,t) = \int_0^t g(x,s)ds$. Under the above assumptions $\Phi \in C^1$.

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Let $m(x)$ be a function in L^r, $r > N/p$ if $1 < p \leq N$ and $r = 1$ if $p > N$, which can change sign in Ω. Consider the eigenvalue problem

$$\begin{aligned} -\Delta_p u = \lambda m(x)|u|^{p-2}u \quad & \text{in} \quad \Omega \\ u = 0 \qquad\qquad & \text{on} \quad \partial\Omega. \end{aligned} \tag{4}$$

It is well known (see [4]) that, if $m(x) > 0$ on a subset of positive measure in Ω, the problem has a first eigenvalue $\mu_1(m) > 0$ which is simple, isolated in the spectrum and admits an eigenfunction φ_m which is positive in Ω. Moreover, $\mu_1(m)$ has the following variational characterization

$$\mu_1(m) = \inf\Big\{ \int_\Omega |\nabla u|^p dx \ ; \ u \in W_0^{1,p}(\Omega) \text{ and} \int_\Omega m(x)|u|^p dx = 1 \Big\}. \tag{5}$$

We define the second eigenvalue positive $\mu_2(m)$ as

$$\mu_2(m) = \min\{\lambda \in \mathbb{R} \ ; \ \lambda \text{ eigenvalue and} \lambda > \mu_1(m)\}.$$

We denote by $\lambda_k = \mu_k(1)$, i.e. $m \equiv 1$, $k = 1, 2$.

Moreover, we assume that the L^r-functions $k_\pm$ and $L_\pm$ defined by

$$k_\pm(x) = \liminf_{t\to\pm\infty} \frac{g(x,t)}{|t|^{p-2}t} \quad \text{and} \quad L_\pm(x) = \limsup_{t\to\pm 0} \frac{pG(x,t)}{|t|^p}$$

have nontrivial positive parts, and the limits are uniformly in $x \in \Omega$.

Theorem 1.1. *Assume that there exists a constant $c \in \mathbb{R}$ such that $|g(x,t)| \leq c|t|^{p-1}$. Suppose that either $\mu_1(k_+) < 1 < \mu_1(L_+)$ or $\mu_1(k_-) < 1 < \mu_1(L_-)$, then problem (1) has at least one nontrivial solution which is positive in the first case and negative in the second case.*

Remark 1.1. The existence of positive solution for the problem (1) with asymptotically linear nonlinearities has been studied by many authors. More recently, Zhou [13] studied the case $0 \leq L = L_+ = l_+,\ K = K_+ = k_+ \in L^\infty$ with $||L||_\infty < \lambda_1$. Magrone in her doctorate thesis [10] has considered the case L_+^+ and K^+ are non trivial. The cited authors used the Mountain Pass Theorem and where considered only the case $p = 2$. The case $p \neq 2$ was studied by Zhou [14] with the assumption $l = L_+ = l_+,\ k = K_+ = k_+$ $(l,\ k \in \mathbb{R})$ and $l < \lambda_1 < k$.

More generally consider the quasilinear eigenvalue problem

$$\begin{aligned} -\Delta_p u = \lambda[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}] \quad & \text{in} \quad \Omega \\ u = 0 \qquad\qquad & \text{on} \quad \partial\Omega. \end{aligned} \tag{6}$$

where $u^\pm = \max\{\pm u, 0\}$ and $m, n \in L^r$ with m^+ and n^+ nontrivial in Ω. Under this hypothesis Arias et al. [2] studied the eigenvalue problem (6) (for more references to this problem see [2]). In [2], it was proved that $\min\{\mu_1(m), \mu_1(n)\}$ and $\max\{\mu_1(m), \mu_1(n)\}$ are the first two positive eigenvalues of (6). Now we remark the construction of a nontrivial eigenvalue of (6) made in [2].

We will use a variational approach and consider the functionals

$$A(u) = \int_\Omega |\nabla u|^p dx,$$

$$B_{m,n}(u) = \int_\Omega \big(m(u^+)^p + n(u^-)^p\big) dx,$$

which are C^1-functionals on $W_0^{1,p}$. We are interested in the critical points of the restriction $\tilde{A}$ of A to the manifold

$$M_{m,n} = \{u \in W_o^{1,p} \ ; \ B_{m,n}(u) = 1\}.$$

By Lagrange's multiplier rule, $u \in M_{m,n}$ is a critical point of $\tilde{A}$ if and only if there exists $\lambda \in \mathbb{R}$ such that $A'(u) = \lambda B'_{m,n}(u)$, i.e.

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx = \lambda \int_\Omega \big(m(u^+)^{p-1} + n(u^-)^{p-1}\big) v dx, \tag{7}$$

for all $v \in W_0^{1,p}$. Taking $v = u$ in (7), one sees that its Lagrange multiplier λ is equal to the critical value $\tilde{A}(u)$. By the Proposition 2 in [2], we have that φ_m and $-\varphi_n$ are strict local minima of $\tilde{A}$, with corresponding critical values $\mu_1(m)$ and $mu_1(n)$. Consider

$$\Gamma = \{\gamma \in C([-1,1], M_{m,n}) \ ; \ \gamma(-1) = \varphi_m \ \text{ and } \ \gamma(1) = -\varphi_n\}.$$

Then, it was proved in [2] (Theorem 7)

$$c(m,n) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([-1,1])} \tilde{A}(u) \tag{8}$$

is a critical value of $\tilde{A}$, with $c(m,n) > \max\{\mu_1(m), \mu_1(n)\}$. Moreover, problem (6) does not admit any eigenvalue in $]\max\{\mu_1(m), \mu_1(n)\}, c(m,n)[$ ([2] Theorem 11) and the eigenfuctions associated with $c(m,n)$ change sign ([2] Corollary 19).

Now we start our results concerned with the multiplicity for the problems (1). We assume that the L^r-functions $l_\pm$ and $K_\pm$ defined by

$$l_\pm(x) = \liminf_{t \to \pm 0} \frac{pG(x,t)}{|t|^p} \quad \text{and} \quad K_\pm(x) = \limsup_{t \to \pm\infty} \frac{pG(x,t)}{|t|^p},$$

have nontrivial positive parts, and the limits are uniformly in $x \in \Omega$.

Theorem 1.2. *Assume that $c(L_+, L_-) > 1$ and $\mu_1(K_\pm) > 1$. Suppose that either*

(H1) $\mu_1(l_\pm) < 1$, *or*

(H2) *there is $\eta > 0$ such that*

$$\begin{aligned} l_+(x)|t|^p &\le pG(x,t) \quad \text{for} \quad 0 \le t < \eta, \quad \text{a.e.} \quad x \in \Omega; \\ l_-(x)|t|^p &\le pG(x,t) \quad \text{for} \quad 0 \le -t < \eta, \quad \text{a.e.} \quad x \in \Omega. \end{aligned}$$

Then problem (1) has at least two nontrivial solutions.

Theorem 1.3. *Assume that* $c(L_+, L_-) > 1$, $\min\{\mu_1(K_\pm)\} = 1$ *and that*

$$\lim_{|t|\to\infty} [tg(x,t) - pG(x,t)] = \infty.$$

Suppose that either (H1) *or* (H2), *then problem* (1) *has at least two nontrivial solutions.*

Remark 1.2. i) De Figueiredo and Massabò [6] studied the problem of existence when $p = 2$ and $\mu_1(K_\pm) > 1$; in this case the functional Φ is coercive. In [6] the authors also consider the resonant case and in this case they assume a kind of Landesmann-Lazer condition. Moreover, in [6] the authors also considered the resonant case $\mu_1(K_\pm) = 1$ using a kind of Saddle Point Theorem and a Landesmann-Lazer condition.

ii) The multiple solutions for the problem (1) was studied by Liu and Su [9] in the case $K = K_\pm < \lambda_1$ and $\lambda_1|t|^p \le pF(x,t) \le \hat{\lambda}|t|^p$ for t near 0, where $\hat{\lambda} < \overline{\lambda} \le \lambda_2$. Liu and Su [9] considered the resonant case, with $K \equiv \lambda_1$.

iii) Our results are new even for the case $p = 2$.

2. Proof of Theorem 1.1

We apply the Mountain Pass Theorem [1]. We proove the theorem for the case $\mu_1(k_+) < 1 < \mu_1(L_+)$, the case $\mu_1(k_-) < 1 < \mu_1(L_-)$ is analogous.

Set

$$f(x,t) = \begin{cases} g(x,t), & t \ge 0, \\ 0, & t \le 0, \end{cases}$$

and consider the problem

$$\begin{aligned} -\Delta_p u = f(x,u) \quad & \text{in} \quad \Omega \\ u = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned} \tag{9}$$

Define

$$\Psi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(x,u)dx, \quad u \in W_0^{1,p}(\Omega).$$

where $F(x,u) = \int_0^u f(x,t)dt$, and $\Psi \in C^1$.

Lemma 2.1. *Under the assumptions of Theorem 1.1 the functional* Ψ *satisfies the* (PS) *condition.*

Proof. Let $\{u_n\} \subset W_0^{1,p}$ be a sequence such that $\{\Psi(u_n)\}$ is bounded, and $||\Psi'(u_n)|| \to 0$ as $n \to \infty$ (i.e. $\{u_n\}$ is a (PS) sequence).

We need to show that $\{||u_n||\}$ is bounded. Since Ω is bounded and f is subcritical, then if $\{||u_n||\}$ is bounded, by the compactness of Sobolev embedding and by standard processes we know that there exists a subsequence of $\{u_n\}$ in $W_0^{1,p}$ which converges strongly, hence the Lemma will be proved.

Assume then by contradiction that $||u_n|| \to \infty$ as $n \to \infty$. Let $v_n = u_n/||u_n||$, then $||v_n|| = 1$. So we can assume that $v_n \to v$ weakly in $W_0^{1,p}$, strongly in L^p and a.e. in Ω.

Let us divide the proof in three steps.

Step 1) $v \ne 0$.

Arguing by contradiction, if $v = 0$, then $v_n \to 0$ in L^p, and

$$\frac{\Psi'(u_n)(u_n)}{||u_n||^p} \to 0,$$

since $||\Psi'(u_n)|| \to 0$ and $p > 1$. This means

$$\int_\Omega |\nabla v_n|^p dx - \int_\Omega \frac{f(x,u_n)}{|u_n|^{p-2}u_n}|v_n|^p dx \to 0,$$

i.e.,

$$1 = \lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{|u_n|^{p-2}u_n}|v_n|^p dx. \tag{10}$$

Since $\frac{f(x,u_n)}{|u_n|^{p-2}u_n}$ is bounded and $v_n \to 0$ in L^p, we have that the right side in (10) goes to 0, a contradiction. Hence we have $v \neq 0$.

Step 2) $v > 0$.

For any $\nu \in W_0^{1,p}$ we have

$$\frac{\Psi'(u_n)(\nu)}{||u_n||^{p-1}} \to 0.$$

So, since $f(x,0) = 0$ for $s \leq 0$,

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla \nu dx - \int_\Omega \frac{f(x,u_n^+)}{(u_n^+)^{p-1}} \frac{(u_n^+)^{p-1}}{||u_n||^{p-1}} \nu dx \to 0. \tag{11}$$

Since $\frac{f(x,u_n^+)}{(u_n^+)^{p-1}}$ is bounded, by the Alaoglu's Theorem $w_n = \frac{f(x,u_n^+)}{(u_n^+)^{p-1}}$ converges in L^∞, in the weak topology $* \ \sigma(L^\infty, L^1)$, to some function $\omega \in L^\infty$. Now $(v_n^+)^{p-1}\nu \in L^1$, by (11), we have

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \nu dx - \int_\Omega \omega(x)(v^+)^{p-1}\nu dx = 0, \quad \forall\ \nu \in W_0^{1,p}.$$

Using $\nu = v^-$, one gets

$$\int_\Omega |\nabla v^-|^p dx = 0,$$

which implies that $v \geq 0$ and satisfies the equation

$$-\Delta_p v = \omega(x) v^{p-2} v \ \text{ in } \ \Omega. \tag{12}$$

Then by a Harnack inequality proved in [12], we have that $v > 0$ in Ω. In particular $\mu_1(\omega) = 1$.

It is a contradiction with the hypotheses $\mu_1(k_+) < 1$. In fact, since $v > 0$, we have $u_n \to \infty$ a.e. in Ω, as $n \to \infty$. So

$$\liminf_{n\to\infty} \frac{f(x,u_n)}{u_n} = k_+(x) \ \text{ a.e. in } \ \Omega, \tag{13}$$

and

$$\lim_{n\to\infty} \frac{f(x,u_n)}{u_n} = \omega(x) \ \text{ in } \ * \ \sigma(L^\infty, L^1). \tag{14}$$

Given a function $u \in W_0^{1,p}$, by (13), Fatou's Lemma and (14), we have

$$\begin{aligned}\int_\Omega k_+(x)|u|^p dx &= \int_\Omega \liminf_{n\to\infty} \frac{f(x,u_n)}{u_n}|u|^p dx\\ &\leq \liminf_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{u_n}|u|^p dx\\ &= \int_\Omega \omega(x)|u|^p dx.\end{aligned}$$

So,

$$\frac{1}{\mu_1(\omega)} = \sup_{\substack{u\in W_0^{1,p}\\ u\neq 0}} \frac{\int_\Omega \omega(x)|u|^p dx}{\int_\Omega |\nabla u|^p dx} \geq \sup_{\substack{u\in W_0^{1,p}\\ u\neq 0}} \frac{\int_\Omega k_+(x)|u|^p dx}{\int_\Omega |\nabla u|^p dx} = \frac{1}{\mu_1(k_+)}\ ;$$

i.e., $\mu_1(\omega) \leq \mu_1(k_+) < 1$. Thus we have the contradiction, then $||u_n||$ is bounded. □

Now we prove that the functional Ψ has the mountain pass geometry.

We have, by the variational characterization of $\mu_1(L_+)$, see (5) ,

$$\frac{1}{\mu_1(L_+)} \geq \frac{\int_\Omega L_+|u|^p dx}{\int_\Omega |\nabla u|^p dx}, \quad \forall\, u \in W_0^{1,p} \setminus \{0\}. \tag{15}$$

Given $\epsilon > 0$ there exists $\delta > 0$ such that

$$pF(x,t) \leq L_+(x)t^p + \epsilon t^p, \quad \text{for} \quad 0 \leq t < \delta.$$

By (2), we have, for a constant c,

$$|F(x,t)| \leq c|t|^q + c, \quad p < q < p*\,.$$

Then

$$F(x,t) \leq \frac{1}{p}L_+(x)t^p + \frac{\epsilon}{p}t^p + c|t|^q, \quad \forall\, t \in \mathbb{R}.$$

Using this inequality, we have

$$\begin{aligned}\Psi(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(x,u)dx\\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p}\int_\Omega L_+(x)|u|^p - \frac{\epsilon}{p}\int_\Omega |u|^p - c\int_\Omega |u|^q\\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p\mu_1(L_+)}\int_\Omega |\nabla u|^p - \frac{\epsilon}{p\lambda_1}\int_\Omega |\nabla u|^p - c\int_\Omega |u|^q,\end{aligned}$$

where the last inequality follows from (15). Using the Sobolev inequality, we obtain

$$\Psi(u) \ \ \geq\ \ \frac{1}{p}\Big(1 - \frac{1}{\mu_1(L_+)} - \frac{\epsilon}{\lambda_1}\Big)||u||^p - c||u||^q.$$

Now, since $\mu_1(L_+) > 1$, we can choose ϵ small enough such that $(1 - \frac{1}{\mu_1(L_+)} - \frac{\epsilon}{\lambda_1}) > 0$. So, since $p < q$, there exist $a > 0$ and $\rho > 0$ such that if $||u|| = \rho$ then $\Psi(u) \geq a > 0$.

Let φ_{k_+} be the first eigenfunction associated to $\mu_1(k_+)$ such that $\varphi_{k_+} > 0$. We have, using the Fatou's Lemma,

$$\begin{aligned}\liminf_{t\to\infty}\frac{\Psi(t\varphi_{k_+})}{t^p} &\le \frac{1}{p}\int_\Omega|\nabla\varphi_{k_+}|^p dx - \int_\Omega \liminf_{t\to\infty}\frac{F(x,t\varphi_{k_+})}{(t\varphi_{k_+})^p}\varphi_{k_+}^p\\ &= \frac{1}{p}\int_\Omega|\nabla\varphi_{k_+}|^p dx - \frac{1}{p}\int_\Omega k_+(x)\varphi_{k_+}^p dx\\ &= \frac{1}{p}\int_\Omega|\nabla\varphi_{k_+}|^p dx - \frac{1}{p\mu_1(k_+)}\int_\Omega|\nabla\varphi_{k_+}|^p dx\\ &= \frac{1}{p}\Big(1-\frac{1}{\mu_{k_+}}\Big)||\varphi_{k_+}||^p < 0.\end{aligned}$$

Then there exists $t_0 > 0$ such that $\Psi(t_0\varphi_{k_+}) < 0$. So Ψ satisfies the assumptions of Mountain Pass Theorem, then there exists $u \in W_0^{1,p}\setminus\{0\}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi dx = \int_\Omega f(x,u)\phi dx, \quad \forall\ \phi\in W_0^{1,p}.$$

Taking $\phi = u^-$, and since $f(x,t) = 0$ for $t \le 0$, we get

$$\int_\Omega |\nabla u^-|^p = 0.$$

Therefore $u \ge 0$, so u is a solution of problem (1). □

3. Proofs of Theorems 1.2 and 1.3

Local linking. In this subsection we started some results that we will use in the proof of Theorems 1.2 and 1.3; their proofs can be found in [11] and [9].

The next definition can be found in [11] and generalizes the notion of local linking introduced by Li and Liu in [8].

Let J be a real C^1-functional defined on a Banach space X.

Definition 3.1. *Assume that* 0 *is an isolated critical point of* J *with* $J(0) = 0$ *and let* n, β *be positives integers. We say that* J *has a local* (n,β)*-linking near the origin if there exist a neighborhood* U *of* 0 *and subsets* A, S, B *of* U *with* $A\cap S = \emptyset$, $0\notin A$, $A\subset B$ *such that*

1. 0 *is the only critical point of* J *in* $U_0\cap U$, *where* $J_0 = \{u\in X\ ;\ J(u)\le 0\}$,
2. *denoting by* $i_1 : H_{n-1}(A)\to H_{n-1}(U\setminus S)$ *and* $i_2 : H_{n-1}(A)\to H_{q-1}(B)$ *the embeddings of the groups induced by inclusions,*
$$\mathrm{rank} i_1 - \mathrm{rank} i_2 \ge \beta,$$
3. $J\le 0$ *on* B, *and*
4. $J > 0$ *on* $S\setminus\{0\}$.

Let $u\in X$ be an isolated critical point of J with $J(u) = c\in\mathbb{R}$, the group

$$C_k(J,u) = H_k(J^c, J^c\setminus\{u\}),\qquad k = 0,\ 1,\ 2,\ \ldots,$$

is called the k-th critical group of J at u, where $J^c = \{u \in X \ ; \ J(u) \leq c\}$ and $H_k(.,.)$ is the k-th singular relative group with integer coefficients. We say that u is an homological nontrivial critical point of J if at least one of its critical points is nontrivial.

Example 3.1. If u is a strict local minimum of J, then

$$C_k(J,u) = \begin{cases} \mathbb{Z} & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k \neq 0. \end{cases}$$

Theorem 3.1. (Theorem 3.1 [11]) *If F has a local (n,β)-linking near the origin, then*

$$\text{rank } C_n(F,0) \geq \beta.$$

Theorem 3.2. (Theorem 2.1 [9]) *Suppose that F satisfy the (PS) condition and be bounded from below. If J has a critical point which is homological nontrivial and is not a minimizer of J, then J has at least three critical points.*

Some Lemmata. In this subsection we show that the functional Φ satisfies the hypotheses of Theorem 3.2.

Lemma 3.1. *Suppose that either*

(i) $\mu_1(K_\pm) > 1$, *or*

(ii) $\min\{\mu_1(K_\pm)\} = 1$ *and*

$$\lim_{|t|\to\infty} [tg(x,t) - pG(x,t)] = \infty.$$

Then the functional Φ is coercive.

Proof. **(i):** Given $\epsilon > 0$, we have, for a constant $c = c(\epsilon)$,

$$pG(x,t) \leq \begin{cases} (K_+(x) + \epsilon)|t|^p + c & \text{for} \quad t > 0 \\ (K_-(x) + \epsilon)|t|^p + c & \text{for} \quad t < 0 \end{cases}$$

So we can estimate

$$\begin{aligned} \Phi(u) \quad \geq \quad & \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p}\int_\Omega K_+(x)|u^+|^p dx \\ & -\frac{1}{p}\int_\Omega K_-(x)|u^-|^p dx - \frac{\epsilon}{p}\int_\Omega |u|^p dx - c|\Omega| \end{aligned}$$

By the variational characterization of the first eigenvalue we obtain

$$\begin{aligned} \Phi(u) \quad \geq \quad & \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p\mu_1(K_+)}\int_\Omega |\nabla u^+|^p dx \\ & -\frac{1}{p\mu_1(K_-)}\int_\Omega |\nabla u^-|^p dx - \frac{\epsilon}{p\lambda_1}\int_\Omega |\nabla u|^p dx - c|\Omega| \\ \geq \quad & \frac{1}{p}\Big(1 - \frac{1}{\mu_1(K_+)} - \frac{\epsilon}{\lambda_1}\Big)\int_\Omega |\nabla u^+|^p dx \\ & +\frac{1}{p}\Big(1 - \frac{1}{\mu_1(K_-)} - \frac{\epsilon}{\lambda_1}\Big)\int_\Omega |\nabla u^-|^p dx - c|\Omega| \end{aligned}$$

Since $\mu_1(K_\pm) > 1$ we can get $\epsilon > 0$ such that $\min\{(1 - \frac{1}{\mu_1(K_\pm)} - \frac{\epsilon}{\lambda_1})\} > 0$. Therefore Φ is coercive.

(ii): For that matter we introduce the functions $F : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$G(x,t) = \frac{1}{p} K_\pm(x)|t|^p + F(x,t), \quad \text{for } t > 0, \ (t < 0),$$

and

$$g(x,t) = k_\pm(x)|t|^{p-2}t + f(x,t), \quad \text{for } t > 0, \ (t < 0).$$

Then

$$\limsup_{t\to\pm\infty} \frac{pF(x,t)}{|t|^p} = 0 \quad \text{and} \quad \liminf_{t\to\pm\infty} \frac{f(x,t)}{|t|^{p-2}t} = 0.$$

And since $k_\pm(x) \leq K_\pm(x)$ a.e. $x \in \Omega$ (it is clear), we get

$$\lim_{|t|\to\infty} [tf(x,t) - pF(x,t)] = \infty.$$

It follows that for every $M > 0$, there exists $R_M > 0$ such that

$$tf(x,t) - pF(x,t) \geq M, \quad \forall\ |t| \geq R_M, \quad \text{a.e. } x \in \Omega.$$

Now consider $t > 0$

$$\begin{aligned}
\frac{d}{dt}\left[\frac{F(x,t)}{|t|^p}\right] &= \frac{\big(g(x,t) - K_\pm(x)|t|^{p-2}t\big)|t|^p - pF(x,t)|t|^{p-2}t}{|t|^{2p}} \\
&= \frac{tg(x,t) - pF(x,t) - K_\pm(x)|t|^p}{|t|^p t} \\
&= \frac{tf(x,t) - pF(x,t) + \big(k_\pm(x) - K_\pm(x)\big)|t|^p}{|t|^p t} \\
&\geq \frac{tf(x,t) - pF(x,t)}{|t|^p t}.
\end{aligned}$$

It follows that (see the proof of Lemma 3.2 in [9])

$$\lim_{|t|\to\infty} F(x,t) = -\infty \quad \text{a.e } x \in \Omega. \tag{16}$$

Let $\{u_n\} \subset W_0^p$ be such that $||u_n|| \to \infty$. Assume by contradiction that $\Phi(u_n) \leq C$ for some constant C. Taking $v_n = u_n/||u_n||$, we may assume that there is some $v_0 \in W_0^p$ such that $v_n \rightharpoonup v_0$ in W_0^p, $v_n \to v_0$ in L^p, and $v_n(x) \to v_0(x)$ a.e. on Ω. Now

$$\begin{aligned}
\frac{pC}{||u_n||^p} \geq \frac{p\Phi(u_n)}{||u_n||^p} &= \int_\Omega |\nabla v_n|^p dx - \int_\Omega \frac{pG(x,u_n)}{||u_n||^p} dx \\
&= \int_\Omega |\nabla v_n|^p dx - \int_\Omega K_+(v_n^+)^p dx - \int_\Omega K_-(v_n^-)^p dx - \int_\Omega \frac{pF(x,u_n)}{||u_n||^p} dx \\
&\geq \int_\Omega |\nabla v_n|^p dx - \int_\Omega K_+(v_n^+)^p dx - \int_\Omega K_-(v_n^-)^p dx - \frac{C_1}{||u_1||^p}.
\end{aligned}$$

So

$$1=\limsup_{n\to\infty}\int_\Omega|\nabla v_n|\le\int_\Omega K_+(v_0^+)^p dx+\int_\Omega K_-(v_0^-)^p dx. \tag{17}$$

If $\max\{\mu_1(K_\pm)\}>1$, we have

$$\begin{aligned}1\le\int_\Omega K_+(v_0^+)^p dx+\int_\Omega K_-(v_0^-)^p dx &\le \frac{1}{\mu_1(K_+)}\int_\Omega|\nabla v_0^+|^p dx+\frac{1}{\mu_1(K_-)}\int_\Omega|\nabla v_0^-|^p dx\\ &< \int_\Omega|\nabla v_0^+|^p dx+\int_\Omega|\nabla v_0^-|^p dx=\int_\Omega|\nabla v_0|^p dx\\ &\le \lim_{n\to\infty}\int_\Omega|\nabla v_n|^p dx=1,\end{aligned}$$

a contradiction. If $\mu_1(K_\pm)=1$, it follows that

$$\begin{aligned}1\le\int_\Omega K_+(v_0^+)^p dx+\int_\Omega K_-(v_0^-)^p dx &\le \int_\Omega|\nabla v_0^+|^p dx+\int_\Omega|\nabla v_0^-|^p dx\\ &\le \int_\Omega|\nabla v_0|^p dx\le\lim_{n\to\infty}\int_\Omega|\nabla v_n|^p dx=1.\end{aligned}$$

This implies that $||v_0||=1$ and so $v_n\to v_0$ in W_0^p. By (17), we have that

$$\int_\Omega K_+(v_0^+)^p dx+\int_\Omega K_-(v_0^-)^p dx=\int_\Omega|\nabla v_0|^p dx.$$

Hence either $v_0=\varphi_{K_+}$ or $v_0=-\varphi_{K_-}$. Take $v_0=\varphi_{K_+}$, then $u_n(x)\to\infty$ a.e. on Ω. So by (16) we have $F(x,u_n)\to-\infty$ a.e. in Ω. Therefore,

$$C\ge-\int_\Omega F(x,u_n)dx\to\infty \quad\text{as}\quad n\to\infty.$$

This is a contradiction. Hence Φ is coercive on W_0^p. □

Remark 3.1. The coercivity of the functional Φ implies that it satisfies the (PS) condition. Since the (PS) sequences should be bounded and the nonlinearity g is subcritical.

Now we show that the hypotheses (H1) and (H2) imply that the functional Φ has a homological local $(1,1)$-linking at origin.

Let $c(L_+,L_-)$ be defined by (8), and Z defined by

$$Z=\Big\{u\in W_0^{1,p}\ ;\ \int_\Omega|\nabla u|^p dx\ge c(L_+,L_-)\int_\Omega\big(L_+(u^+)^p+L_-(u^-)^p\big)dx\Big\}. \tag{18}$$

Lemma 3.2. *There exists $\rho>0$ such that $\Phi(u)>0$ if $u\in Z$ and $||u||\le\rho$.*

Proof. Given $\epsilon>0$ there exists $\delta>0$ such that

$$pG(x,t)\le\begin{cases}(K_+(x)+\epsilon)|t|^p & \text{for}\quad 0<t<\delta\\ (K_-(x)+\epsilon)|t|^p & \text{for}\quad 0<-t<\delta.\end{cases}$$

And by (2), we have, for $p < q < p^*$,

$$G(x,t) \leq \frac{1}{p}L(x)|t|^p + \frac{\epsilon}{p}|t|^p + C|u|^q, \quad \forall\, t \in \mathbb{R}.$$

Let be $u \in Z$, using the estimate above, we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega G(x,u)dx \\
&\geq \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p}\int_\Omega \big(L_+(x)|u^+|^p + L_-(x)|u^-|^p\big)dx - \frac{\epsilon}{p}\int_\Omega |u|^p - C\int_\Omega |u|^q \\
&\geq \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{pc(L_+,L_-)}\int_\Omega |\nabla u|^p dx - \frac{\epsilon}{p\lambda_1}\int_\Omega |\nabla u|^p - C||u||^q \\
&= \frac{1}{p}\Big(1 - \frac{1}{c(L_+,L_-)} + \frac{\epsilon}{\lambda_1}\Big)||u||^p - C||u||^q
\end{aligned}$$

Since $c(L_+,L_-) > 1$ we can get $\epsilon > 0$ such that $(1 - \frac{1}{c(L_+,L_-)} + \frac{\epsilon}{\lambda_1}) > 0$, so there exists $\rho > 0$ such that $\Phi(u) > 0$ if $u \in Z_L$ and $||u|| \leq \rho$, since $p < q$. □

Now assume that (H1) holds. Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$pG(x,t) \geq l_-(x)|t|^p - \epsilon|t|^p, \quad \text{for} \quad -\delta < t \leq 0.$$

Let $\varphi_{l_\pm} > 0$ be the eigenfunction associated to $\mu_1(l_\pm)$, such that $||\varphi_{l_\pm}|| = 1$. Since $\varphi_{l_\pm} \in L^\infty$, consider $t_- < 0$ so that $-\delta < t\varphi_{l_-} \leq 0$ for all $t_- < t \leq 0$. Then for $t_- < t < 0$, we have

$$\begin{aligned}
\Phi(t\varphi_{l_-}) &\leq \frac{|t|^p}{p}\int_\Omega |\nabla\varphi_{l_-}|^p - \frac{|t|^p}{p}\int_\Omega l_-(x)\varphi_{l_-}^p + \frac{\epsilon|t|^p}{p}\int_\Omega |\nabla\varphi_{l_-}|^p dx \\
&\leq \frac{|t|^p}{p}\int_\Omega |\nabla\varphi_{l_-}|^p - \frac{|t|^p}{p\mu_1(l_-)}\int_\Omega |\nabla\varphi_{l_-}|^p + \frac{\epsilon|t|^p}{p\lambda_1}\int_\Omega |\nabla\varphi_{l_-}|^p \\
&= \frac{|t|^p}{p}\Big(1 - \frac{1}{\mu_1(l_-)} + \frac{\epsilon}{\lambda_1}\Big)||\varphi_{l_-}||^p
\end{aligned}$$

Since $\mu_1(l_-) < 1$ we can get $\epsilon > 0$ such that $(1 - \frac{1}{\mu_1(l_-)} + \frac{\epsilon}{\lambda_1}) < 0$. Therefore $\Phi(t\varphi_{l_-}) < 0$ for $t_- < t < 0$ (and so $u = 0$ is not a minimizer). Analogously , there exists $t_+ > 0$ such that $\Phi(t\varphi_{l_+}) < 0$ for $0 < t < t_+$.

Now let $r > 0$ be defined by $r = \min\{\rho, t_+, -t_-\}$, and consider $U = \overline{B}_r(0)$, $A = \{r\varphi_{l_\pm}\}$, $S = U \cap Z$ and $B = \{t\varphi_{l_+}\ ;\ 0 \leq t \leq r\} \cup \{t\varphi_{l_-}\ ;\ 0 \leq -t \leq r\}$. It is easy to see that U, A, S and B satisfy the Definition 3.1, i.e., Φ has a $(1,1)$-linking near the origin (observe that $\varphi_{l_\pm} \notin Z$) . Thus, by the Theorem 3.1, we have

$$C_1(\Phi, 0) \neq 0. \tag{19}$$

In particular 0 is not a minimizing of Φ.

Assume that (H2) holds, and let $t_+ > 0$ be such that $t\varphi_{l_+} < \eta$ for $0 \leq t < t_+$, then we have

$$pG(x, t\varphi_{l_+}) \geq (t\varphi_{l_+})^p l_+(x), \quad \forall\ 0 \leq t < t_+.$$

Thus for $0 \leq t < t_+$,

$$\begin{aligned}
\Phi(t\varphi_{l_+}) &= \frac{|t|^p}{p}\int_\Omega |\nabla\varphi_{l_+}|^p dx - \int_\Omega G(x, t\varphi_{l_+})dx \\
&= \frac{|t|^p}{p}\int_\Omega l_+(x)\varphi_{l_+}^p dx - \int_\Omega G(x, t\varphi_{l_+})dx \\
&= \int_\Omega \Big(l_+(x)\frac{(t\varphi_{l_+})^p}{p} - G(x, t\varphi_{l_+})\Big)dx \\
&\leq 0.
\end{aligned}$$

Analogously, there exists $t_- < 0$ such that $\Phi(t\varphi_{l_-}) \leq 0$ for $t_- < t \leq 0$. Like in the case (H1), Φ has a $(1,1)$-linking near origin, 0 is not a minimizing of Φ, and we have that

$$C_1(\Phi, 0) \neq 0. \tag{20}$$

Proofs of Theorems 1.2 and 1.3. By Lemma 3.1 the functional Φ is coercive, hence Φ is bounded below and satisfies the (PS) condition (Remark 3.1). Since Φ has a $(1,1)$-linking near the origin, $u = 0$ is homological nontrivial and is not a minimizing (it follows from (19) and (20)). The conclusion follows from Theorem 3.2.

References

[1] A. Ambrosetti & P.H. Rabinowitz, *Dual variational Methods in Critical Point Theory and Applications*, J. Func. Anal. **14** (1973), 349-381.
[2] M. Arias, J. Campos, M. Cuesta & J.-P. Gossez, *Asymmetric Elliptics Problems with Indefinite Weights*, Ann. I.H.P. Analyse Non linéaire **19** (2002), 581-616.
[3] K.C. Chang, *Infinite Dimensional Morse Theory and Multiple Solutions Problems*, Birkhäuser, Boston (1993).
[4] M. Cuesta, *Eigenvalue Problem for the p-Laplacian with Indefinite weights*, Elec. J. Dif. Equ. **2001**, N. 33, (2001), 1-9.
[5] D.G. De Figueiredo, *Positive Solutions of Semilinear Elliptic Problems*, LNM, **957**, Spring-Verlag (1982).
[6] D.G. De Figueiredo & I. Massabò, *Semilinear Elliptic Equations with the Primitive of the Nonlinearity Interacting with the First Eigenvalue*, J. Math. Anal. Appl. **156** (1991), 381-394.
[7] M.F. Furtado & E.A.B. Silva, *Double resonant problems whitc are locally non-quadratic at infinity*, Proceedings of the USA-Chile Workshop on Nonlinear Analysis. Electron. J. Differential Equations. Conf. 06 (2001), 155-171.
[8] S.J. Li & J.Q. Liu, *Some existence theorem on multiple critical points and their aplications*, Kexue Tongbao **17** (1984), 1025-1027.
[9] J. Liu & J. Su, *Remarks on Multiple Solutions for Quasi-Linear Resonant Problems*, J. Math. Anal. Appl. **258** (2001), 209-222.
[10] P. Magrone, *Critical Points Methods for Indefinite Nonlinear Elliptic Equations and Hamiltonian Systems*, PhD Thesis 2001.
[11] K. Perera, *Homological Local Linking*, Abstr. Appl. Anal. **8** (1998), 181-189.

[12] N.S. Trundinger, *On Harnack Type Inequalities and Their Application to Quasilinear Elliptic Equations*, Comm. Pure Appl. Math. **XX** (1967), 721-747.
[13] H.S. Zhou, *An Application of a Mountain Pass Theorem*, Acta Math. Sinica (N.S.) **18** (2002), 27-36.
[14] H.S. Zhou, *title*, Nonlinear Anal. (2003), - .

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