Clifford Valued Differential Forms, Algebraic Spinor Fields, Gravitation, Electromagnetism and "Unified" Theories

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Abstract

In this paper we show how to describe the general theory of a linear metric compatible connection with the theory of Clifford valued differential forms. This is done by realizing that for each spacetime point the algebra of Clifford bivectors is isomorphic to the algebra of $Sl(2,\mathbb{C})$. In that way the pullback of the linear connection under a trivialization of the bundle is represented by a Clifford valued 1-form. That observation makes it possible to realize Einstein's gravitational theory can be formulated in a way which is similar to a $Sl(2,\mathbb{C})$ gauge theory. Some aspects of such approach is discussed. Also, the theory of the covariant spinor derivative of spinor fields is introduced in a novel way, allowing for a physical interpretation of some rules postulated for that covariant spinor derivative in the standard theory of these objects. We use our methods to investigate some polemical issues in gravitational theories and in particular we scrutinize a supposedly "unified" field theory of gravitation and electromagnetism proposed by M. Sachs and recently used in a series of papers. Our results show that Sachs did not attain his objective and that recent papers based on that theory are ill conceived and completely invalid both as Mathematics and Physics.

1 Introduction

In this paper we introduce the concept of Clifford valued differential forms[1], mathematical entities which are sections of $\mathcal{C}\ell(TM) \otimes \bigwedge T^*M$. We show how with the aid of this concept we can produce a very beautiful *description* of the theory of linear connections, where the representative of a given linear connection in a given gauge is represented by a *bivector* valued 1-form. The notion of an exterior covariant differential and exterior covariant derivative of sections of $\mathcal{C}\ell(TM) \otimes \bigwedge T^*M$ is crucial for our program and is thus discussed in details. Our *natural* definitions (to be compared with other approaches on related subjects, as described, e.g., in [17, 18, 48, 50, 69, 70, 92]) parallel in a noticeable way the formalism of the theory of connections in principal bundles and their associated covariant derivative operators acting on associated vector bundles. We identify Cartan curvature 2-forms and *curvature bivectors*. The curvature 2-forms satisfy Cartan's second structure equation and the curvature bivectors satisfy equations in complete analogy with equations of gauge theories. This immediately suggests to write Einstein's theory in that formalism, something that has already been done and extensively studied in the past (see e.g., [21, 23]). Our methodology suggest new ways of taking advantage of such a formulation, but this is postpone for a later paper. Here, our investigation of the $Sl(2,\mathbb{C})$ nonhomogeneous gauge equation for the curvature bivector is restricted to the relationship between that equation and Sachs theory [85, 86, 87] and the shameful problem of the energy-momentum 'conservation' in General Relativity.

We recall also the concept of covariant derivatives of (*algebraic*) spinor fields in our formalism, where these objects are represented as sections of real spinor bundles[2] and study how this theory has as matrix representative the standard two components spinor fields (*dotted* and undotted) already introduced long ago, see, e.g., [21, 71, 72, 73]. What is *new* here is that we identify that in the theory of algebraic spinor fields the *realization* of some rules which used in the standard formulation of the matrix spinor fields, e.g., why the covariant derivative of the Pauli matrices must be null, imply some constraints, with admit a very interesting geometrical interpretation. Indeed, a possible realization of that rules in the Clifford bundle formalism is one where the vector fields defining a global tetrad $\{\mathbf{e_a}\}$ must be such that $D_{\mathbf{e_0}}\mathbf{e_0} = 0$, i.e., $\mathbf{e_0}$ is geodesic reference frame and along each one of its integral lines, say σ, the $\mathbf{e_d}$ ($\mathbf{i} = 1, 2, 3$) are Fermi transported, i.e., they are not rotating relative to the local gyroscope axes. For the best of our knowledge this important fact is here disclosed for the first time.

We use the Clifford bundle formalism and the theory of Clifford valued differential forms to analyze some polemic issues in presentations of gravitational theory and some other theories. In particular, we scrutinized Sachs "unified" theory as described recently in [87] and originally introduced in [85]. We show

[1] Analogous, but non equivalent concepts have been introduced in [29, 96, 95]. In particular [29] is a very complete paper using cliffor ms, i.e., forms with values in a abstract Clifford algebra.

[2] Real Spinor fields have been introduced by Hestenes in [52], but a rigorous theory of that objects in a Lorentzian spacetime has only recently been achieved [64, 79].

that unfortunately there are some *serious* mathematical errors in Sachs theory. To start, he identified erroneously his basic variables q_μ as being quaternion fields over a Lorentzian spacetime. Well, they are *not*. The real mathematical structure of these objects is that they are matrix representations of particular sections of the even Clifford bundle of multivectors $\mathcal{C}\ell(TM)$ (called paravector fields in mathematical literature) as we proved in section 2. Next we show that the identification of a 'new' antisymmetric field in his theory is indeed nothing more than the identification of some combinations of the curvature bivectors[3], an object that appears naturally when we try to formulate Einstein's gravitational theory as a $Sl(2,\mathbb{C})$ gauge theory. In that way, any tentative of identifying such an object with any kind of *electromagnetic field* as did by Sachs in [85, 86, 87] is clearly *wrong*. We note that recently in a series of papers, Evans&AIAS group ([1]-[15],[32]-[36],[26][37]-[41]) uses Sachs theory in order to justify some very *odd* facts, which must be denounced. Indeed, we recall that:

(**i**) On March 26 2002, the United States Patent and Trademark Office (USPTO) in Washington issued US Patent no. 6,362,718 for a Motionless Electromagnetic Generator (MEG). This would be 'remarkable' device has been projected by retired lieutenant colonel Tom E. Bearden of Alabama and collaborators. They claimed MEG produces more output energy than the input energy used for its functioning!

Of course, nobody could think that the officers at the US Patent office do *not* know the law of energy-momentum conservation, which in general prevents all Patent offices to veto all free energy machines, and indeed that energy momentum conservation law has been used since a long time ago as a *golden rule*.

So, affording a patent to that device must have a reason. A possible one is that the patent officers must somehow been convinced that there are theoretical reasons for the functioning of MEG. How, did the patent officers get convinced?

We think that the answer can be identified in a long list of papers published in respectable (?) Physics journals signed by Evan&AIAS group and quoted above[4]. There, they claimed that using Sachs theory there is a 'natural' justification for an entity that they called the $\mathbf{B}_3$ field and that appears (according to them) in their 'new' $O(3)$ electrodynamics and 'unified' field theory. According to them, the $\mathbf{B}_3$ field is to be identified with $\mathbf{F}_{12}$, where $\mathbf{F}_{\mu\nu} = -\mathbf{F}_{\nu\mu}$ (see Eq.(70) below) is a mathematical object that Sachs identified in [85, 86, 87] with an electromagnetic field after 'taking the trace in the spinor indices'. Evans&AIAS group claim to explain the operation of MEG. It simply

[3] The curvature bivectors are physically and mathematically equivalent to the Cartan curvature 2-forms, since they carry the same information. This statement will become obvious from our study in section 4.

[4] Note that Bearden is one of the members of the AIAS group. We mention also that in the AIAS website the following people among others are listed as emeritus fellows of the Foundation: Prof. Alwyn van der Merwe, Univ. of Denver, Colorado, USA, Prof. Mendel Sachs, SUNY, Buffalo, USA, Prof. Jean Pierre Vigier, Institut Henri Poincare, France. Well, van der Merwe is editor of *Foundations of Physics* and *Foundations of Physics Letters*, Sachs is one of the authors we criticize here and Vigier is on the editorial board of *Physics Letters A* for decades and is one of the AIAS authors. This eventually could explain how AIAS got their papers published...

pumps energy from the $\mathbf{F}_{12}$ existing in spacetime. However, the Mathematics and Physics of Evans&AIAS used in their papers are unfortunately only a pot pourri of nonsense as we already demonstrated elsewhere[5] and more below. This, of course invalidate any theoretical justification for the patent.

It would be great if the officers of USPTO would know enough Mathematics and Physics in order to reject immediately the *theoretical* explanations offered by the MEG inventors. But that unfortunately was not the case, because it seems that the knowledge of Mathematics and Physics of that officers was no great than the knowledge of these disciplines by the referees of the Evans&AIAS papers.

Of course, theoretical explanations apart and the authors prejudices it can happen that MEG *works*. However, having followed with interest in the internet[6] the work of supposedly MEG builders, we arrived at the conclusion that MEG did not work until now, and all claims of its inventors and associates are simply due to *wrong* experimental measurements. And, of course, that must also been the case with the USPTO officers, if they did realize any single experiment on the MEG device. And indeed, this may be really the case, for in a recent article [61] we are informed that in August last year the Commissioner of Patents, Nicholas P. Godici informed that it was a planned a re-examination of the MEG patent. We do not know what happened since then.

(**ii**) Now, is energy-momentum conservation a *trustworthy* law of the physical world? To answer that question we discuss in this paper the shameful problem of the energy-momentum 'conservation' in General Relativity.

Yes, in General Relativity there are *no* conservation laws of energy, momentum and angular momentum *in general*, and this fact must be clear once and for ever for all (even for school boys, that are in general fooled in reading science books for laymen).

To show this result in an economic and transparent way a presentation of Einstein's gravitational theory is given in terms of tetrads fields, which has a very elegant description in terms of the calculus in the Clifford bundle $\mathcal{C}\ell(T^*M)$ described in Appendix. Using that toll, we recall also the *correct* wave like equations solved by the tetrad fields[7] $\theta^{\mathbf{a}}$ in General Relativity. This has been done here in order to complete the debunking of recent Evans&AIAS papers ([26],[37]-[41]) claiming to have achieved (yet) another 'unified' field theory. Indeed, we show that, as it is the case with almost all other papers written by those authors, these new ones are again a compendium of very bad Mathematics and Physics.

2 Spacetime, Pauli and Quaternion Algebras

[5] For more details on the absurdities propagated by Evans&AIAS in ISI indexed journals and books see [24, 81]. The second citation is a reply to Evans' paper [37].

[6] See http://groups.yahoo.com/group/free_energy/.

[7] The set $\{\theta^{\mathbf{a}}\}$ is the dual basis of $\{e_{\mathbf{a}}\}$.

In this section we recall very well known facts concerning three special *real* Clifford algebras, namely, the *spacetime* algebra $\mathbb{R}_{1,3}$, the *Pauli* algebra $\mathbb{R}_{3,0}$ and the *quaternion* algebra $\mathbb{R}_{0,2} = \mathbb{H}$ and the relation between them.[8]

2.1 Spacetime Algebra

We define the spacetime algebra $\mathbb{R}_{1,3}$ as being the Clifford algebra associated with Minkowski vector space $\mathbb{R}^{1,3}$, which is a four dimensional real vector space, equipped with a Lorentzian bilinear form

$$\boldsymbol{\eta} : \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \to \mathbb{R}. \tag{1}$$

Let $\{\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be an arbitrary orthonormal basis of $\mathbb{R}^{1,3}$, i.e.,

$$\boldsymbol{\eta}(\mathbf{m}_\mu, \mathbf{m}_\nu) = \eta_{\mu\nu} = \begin{cases} 1 & \text{if} \quad \mu = \nu = 0 \\ -1 & \text{if} \quad \mu = \nu = 1,2,3 \\ 0 & \text{if} \quad \mu \neq \nu \end{cases} \tag{2}$$

As usual we resume Eq.(2) writing $\eta_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)$. We denote by $\{\mathbf{m}^0, \mathbf{m}^1, \mathbf{m}^2, \mathbf{m}^3\}$ the *reciprocal* basis of $\{\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$, i.e., $\boldsymbol{\eta}(\mathbf{m}^\mu, \mathbf{m}_\nu) = \delta^\mu_\nu$. We have in obvious notation $\eta(\mathbf{m}^\mu, \mathbf{m}^\nu) = \eta^{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)$.

The spacetime algebra $\mathbb{R}_{1,3}$ is generate by the following algebraic fundamental relation

$$\mathbf{m}^\mu \mathbf{m}^\nu + \mathbf{m}^\nu \mathbf{m}^\mu = 2\eta^{\mu\nu}. \tag{3}$$

We observe that (as with the conventions fixed in the Appendix) in the above formula and in all the text the Clifford product is denoted by *juxtaposition* of symbols. $\mathbb{R}_{1,3}$ as a vector space over the real field is isomorphic to the exterior algebra $\bigwedge \mathbb{R}^{1,3} = \sum_{j=0}^{4} \bigwedge^j \mathbb{R}^{1,3}$ of $\mathbb{R}^{1,3}$. We code that information writing $\bigwedge \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$. Also, $\bigwedge^0 \mathbb{R}^{1,3} \equiv \mathbb{R}$ and $\bigwedge^1 \mathbb{R}^{1,3} \equiv \mathbb{R}^{1,3}$. We identify the exterior product of vectors by

$$\mathbf{m}^\mu \wedge \mathbf{m}^\nu = \frac{1}{2} \left(\mathbf{m}^\mu \mathbf{m}^\nu - \mathbf{m}^\nu \mathbf{m}^\mu \right), \tag{4}$$

and also, we identify the scalar product by

$$\boldsymbol{\eta}(\mathbf{m}^\mu, \mathbf{m}^\nu) = \frac{1}{2} \left(\mathbf{m}^\mu \mathbf{m}^\nu + \mathbf{m}^\nu \mathbf{m}^\mu \right). \tag{5}$$

Then we can write

$$\mathbf{m}^\mu \mathbf{m}^\nu = \boldsymbol{\eta}(\mathbf{m}^\mu, \mathbf{m}^\nu) + \mathbf{m}^\mu \wedge \mathbf{m}^\nu. \tag{6}$$

[8] This material is treated in details e.g, in the books [53, 57, 74, 75]. See also [43, 44, 45, 46, 65, 66, 67].

From the observations given in the Appendix it follows that an arbitrary element $\mathbf{C} \in \mathbb{R}_{1,3}$ can be written as sum of *nonhomogeneous multivectors*, i.e.,

$$\mathbf{C} = s + c_\mu \mathbf{m}^\mu + \frac{1}{2} c_{\mu\nu} \mathbf{m}^\mu \mathbf{m}^\nu + \frac{1}{3!} c_{\mu\nu\rho} \mathbf{m}^\mu \mathbf{m}^\nu \mathbf{m}^\rho + p\mathbf{m}^5 \tag{7}$$

where $s, c_\mu, c_{\mu\nu}, c_{\mu\nu\rho}, p \in \mathbb{R}$ and $c_{\mu\nu}, c_{\mu\nu\rho}$ are completely antisymmetric in all indices. Also $\mathbf{m}^5 = \mathbf{m}^0\mathbf{m}^1\mathbf{m}^2\mathbf{m}^3$ is the generator of the pseudo scalars. As matrix algebra we have that $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, the algebra of the 2×2 quaternionic matrices.

2.2 Pauli Algebra

Now, the Pauli algebra $\mathbb{R}_{3,0}$ is the Clifford algebra associated with the Euclidean vector space $\mathbb{R}^{3,0}$, equipped as usual, with a positive definite bilinear form. As a matrix algebra we have that $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$, the algebra of 2×2 complex matrices. Moreover, we recall that $\mathbb{R}_{3,0}$ is isomorphic to the even subalgebra of the spacetime algebra, i.e., writing $\mathbb{R}_{1,3} = \mathbb{R}_{1,3}^{(0)} \oplus \mathbb{R}_{1,3}^{(1)}$ we have,

$$\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)}. \tag{8}$$

The isomorphism is easily exhibited by putting $\boldsymbol{\sigma}^i = \mathbf{m}^i\mathbf{m}^0$, $i = 1, 2, 3$. Indeed, with $\delta^{ij} = \operatorname{diag}(1, 1, 1)$, we have

$$\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j + \boldsymbol{\sigma}^j\boldsymbol{\sigma}^j = 2\delta^{ij}, \tag{9}$$

which is the fundamental relation defining the algebra $\mathbb{R}_{3,0}$. Elements of the Pauli algebra will be called Pauli numbers[9]. As vector space we have that $\bigwedge \mathbb{R}^{3,0} \hookrightarrow \mathbb{R}_{3,0} \subset \mathbb{R}_{1,3}$. So, any Pauli number can be written as

$$\mathbf{P} = s + p^i\boldsymbol{\sigma}^i + \frac{1}{2} p^i_{ij}\boldsymbol{\sigma}^i\boldsymbol{\sigma}^j + p\textsc{i}, \tag{10}$$

where $s, p_i, p_{ij}, p \in \mathbb{R}$ and $p_{ij} = -p_{ji}$ and also

$$\textsc{i} = \boldsymbol{\sigma}^1\boldsymbol{\sigma}^2\boldsymbol{\sigma}^3 = \mathbf{m}^5. \tag{11}$$

Note that $\textsc{i}^2 = -1$ and that $\textsc{i}$ commutes with any Pauli number. We can trivially verify that

$$\begin{aligned} \boldsymbol{\sigma}^i\boldsymbol{\sigma}^j &= \textsc{i}\varepsilon_k^{i\,j}\boldsymbol{\sigma}^k, \\ [\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j] \equiv \boldsymbol{\sigma}^i\boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j\boldsymbol{\sigma}^i = 2\boldsymbol{\sigma}^i \wedge \boldsymbol{\sigma}^j &= 2\textsc{i}\varepsilon_k^{i\,j}\boldsymbol{\sigma}^k. \end{aligned} \tag{12}$$

[9] Sometimes they are also called 'complex quaternions'. This last terminology will be obvious in a while.

In that way, writing $\mathbb{R}_{3,0}=\mathbb{R}_{3,0}^{(0)}+\mathbb{R}_{3,0}^{(1)}$, any Pauli number can be written as

$$\mathbf{P}=\mathbf{Q}_1+\textsc{i}\mathbf{Q}_2, \quad \mathbf{Q}_1\in\mathbb{R}_{3,0}^{(0)}, \quad \textsc{i}\mathbf{Q}_2\in\mathbb{R}_{3,0}^{(1)}, \tag{13}$$

with

$$\begin{aligned}\mathbf{Q}_1&=a_0+a_k(\textsc{i}\boldsymbol{\sigma}^k), \quad a_0=s, \quad a_k=\frac{1}{2}\varepsilon_k^{i\,j}p_{ij}, \\ \mathbf{Q}_2&=\textsc{i}\left(b_0+b_k(\textsc{i}\boldsymbol{\sigma}^k)\right), \quad b_0=p, \quad b_k=-p_k.\end{aligned} \tag{14}$$

2.3 Quaternion Algebra

Eqs.(14) show that the quaternion algebra $\mathbb{R}_{0,2}=\mathbb{H}$ can be identified as the even subalgebra of $\mathbb{R}_{3,0}$, i.e.,

$$\mathbb{R}_{0,2}=\mathbb{H}\simeq\mathbb{R}_{3,0}^{(0)}. \tag{15}$$

The statement is obvious once we identify the basis $\{1,\hat{\imath},\hat{\jmath},\hat{k}\}$ of $\mathbb{H}$ with

$$\{\mathbf{1},\textsc{i}\boldsymbol{\sigma}^1,\textsc{i}\boldsymbol{\sigma}^2,\textsc{i}\boldsymbol{\sigma}^3\}, \tag{16}$$

which are the generators of $\mathbb{R}_{3,0}^{(0)}$. We observe moreover that the even subalgebra of the quaternions can be identified (in an obvious way) with the complex field, i.e., $\mathbb{R}_{0,2}^{(0)}\simeq\mathbb{C}$.

Returning to Eq.(10) we see that any $\mathbf{P}\in\mathbb{R}_{3,0}$ can also be written as

$$\mathbf{P}=\mathbf{P}_1+\textsc{i}\mathbf{L}_2, \tag{17}$$

where

$$\begin{aligned}\mathbf{P}_1&=(s+p_k^k\boldsymbol{\sigma})\in\bigwedge^0\mathbb{R}^{3,0}\oplus\bigwedge^1\mathbb{R}^{3,0}\equiv\mathbb{R}\oplus\bigwedge^1\mathbb{R}^{3,0}, \\ \textsc{i}\mathbf{L}_2&=\textsc{i}(p+\textsc{i}l_k^k\boldsymbol{\sigma})\in\bigwedge^2\mathbb{R}^{3,0}\oplus\bigwedge^3\mathbb{R}^{3,0},\end{aligned} \tag{18}$$

with $l_k=-\varepsilon_k^{i\,j}p_{ij}\in\mathbb{R}$. The important fact that we want to recall here is that the subspaces $(\mathbb{R}\oplus\bigwedge^1\mathbb{R}^{3,0})$ and $(\bigwedge^2\mathbb{R}^{3,0}\oplus\bigwedge^3\mathbb{R}^{3,0})$ do not close separately any algebra. In general, if $\mathbf{A},\mathbf{C}\in(\mathbb{R}\oplus\bigwedge^1\mathbb{R}^{3,0})$ then

$$\mathbf{AC}\in\mathbb{R}\oplus\bigwedge^1\mathbb{R}^{3,0}\oplus\bigwedge^2\mathbb{R}^{3,0}. \tag{19}$$

To continue, we introduce

$$\boldsymbol{\sigma}_i=\mathbf{m}_i\mathbf{m}_0=-\boldsymbol{\sigma}^i, \quad i=1,2,3. \tag{20}$$

Then, $\textsc{i}=-\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$ and the basis $\{1,\hat{\imath}\ ,\hat{\jmath},\hat{k}\}$ of $\mathbb{H}$ can be identified with $\{1,-\textsc{i}\boldsymbol{\sigma}_1,-\textsc{i}\boldsymbol{\sigma}_2,-\textsc{i}\boldsymbol{\sigma}_3\}$.

Now, we already said that $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$. This permit us to represent the Pauli numbers by 2×2 complex matrices, in the usual way ($\mathrm{i} = \sqrt{-1}$). We write $\mathbb{R}_{3,0} \ni \mathbf{P} \mapsto P \in \mathbb{C}(2)$, with

$$\begin{aligned} \boldsymbol{\sigma}^1 &\mapsto \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \boldsymbol{\sigma}^2 &\mapsto \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \\ \boldsymbol{\sigma}^3 &\mapsto \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{21}$$

2.4 Minimal left and right ideals in the Pauli Algebra and Spinors

It is not our intention to present the details of algebraic spinor theory here (see, e.g., [47, 76, 57]). However, we will need to recall some facts. The elements $\mathbf{e}_{\pm} = \frac{1}{2}(1 + \boldsymbol{\sigma}_3) = \frac{1}{2}(1 + \mathbf{m}_3\mathbf{m}_0) \in \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{R}_{3,0}$, $\mathbf{e}_{\pm}^2 = \mathbf{e}_{\pm}$ are minimal idempotents. They generate the minimal left and right ideals

$$\mathbf{I}_{\pm} = \mathbb{R}_{1,3}^{(0)}\mathbf{e}_{\pm}, \quad \mathbf{R}_{\pm} = \mathbf{e}_{\pm}\mathbb{R}_{1,3}^{(0)}. \tag{22}$$

From now on we write $\mathbf{e} = \mathbf{e}_+$. It can be easily shown (see below) that, e.g., $\mathbf{I} = \mathbf{I}_+$ has the structure of a 2-dimensional vector space over the complex field [47, 76], i.e., $\mathbf{I} \simeq \mathbb{C}^2$. The elements of the vector space $\mathbf{I}$ are called algebraic *contravariant undotted spinors* and the elements of $\mathbb{C}^2$ are the usual *contravariant undotted spinors* used in physics textbooks. They carry the $D^{(\frac{1}{2},0)}$ representation of $Sl(2,\mathbb{C})$ [60]. If $\boldsymbol{\varphi} \in \mathbf{I}$ we denote by $\varphi \in \mathbb{C}^2$ the usual matrix representative[10] of $\boldsymbol{\varphi}$ is

$$\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \quad \varphi^1, \varphi^2 \in \mathbb{C}. \tag{23}$$

We denote by $\dot{\mathbf{I}} = \mathbf{e}\mathbb{R}_{1,3}^{(0)}$ the space of the algebraic covariant dotted spinors. We have the isomorphism, $\dot{\mathbf{I}} \simeq (\mathbb{C}^2)^\dagger \simeq \mathbb{C}_2$, where $\dagger$ denotes Hermitian conjugation. The elements of $(\mathbb{C}^2)^\dagger$ are the usual contravariant spinor fields used in physics textbooks. They carry the $D^{(0,\frac{1}{2})}$ representation of $Sl(2,\mathbb{C})$ [60]. If $\dot{\boldsymbol{\xi}} \in \dot{\mathbf{I}}$ its matrix representation in $(\mathbb{C}^2)^\dagger$ is a row matrix usually denoted by

$$\dot{\xi} = \begin{pmatrix} \xi_{\dot{1}} & \xi_{\dot{2}} \end{pmatrix}, \quad \xi_{\dot{1}}, \xi_{\dot{2}} \in \mathbb{C}. \tag{24}$$

The following representation of $\dot{\boldsymbol{\xi}} \in \dot{\mathbf{I}}$ in $(\mathbb{C}^2)^\dagger$ is extremely convenient. We say that to a covariant undotted spinor ξ there corresponds a covariant dotted spinor $\dot{\xi}$ given by

$$\dot{\mathbf{I}} \ni \dot{\boldsymbol{\xi}} \mapsto \dot{\xi} = \bar{\xi}\varepsilon \in (\mathbb{C}^2)^\dagger, \quad \bar{\xi}_1, \bar{\xi}_2 \in \mathbb{C}, \tag{25}$$

[10] The matrix representation of elements of ideals are of course, 2×2 complex matrices (see, [47], for details). It happens that both colums of that matrices have the *same* information and the representation by column matrices is enough here for our purposes.

with

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{26}$$

We can easily find a basis for $\mathbf{I}$ and $\dot{\mathbf{I}}$. Indeed, since $\mathbf{I} = \mathbb{R}_{1,3}^{(0)}\mathbf{e}$ we have that any $\boldsymbol{\varphi}\in \mathbf{I}$ can be written as

$$\boldsymbol{\varphi}=\boldsymbol{\varphi}^1\boldsymbol{\vartheta}_1+\boldsymbol{\varphi}^2\boldsymbol{\vartheta}_2$$

where

$$\begin{aligned} &\boldsymbol{\vartheta}_1=\mathbf{e}, \quad \boldsymbol{\vartheta}_2 = \boldsymbol{\sigma}_1\mathbf{e} \\ &\boldsymbol{\varphi}^1 = a+\mathbf{i}b, \quad \boldsymbol{\varphi}^2 = c+\mathbf{i}d, \quad a,b,c,d\in\mathbb{R}. \end{aligned} \tag{27}$$

Analogously we find that any $\dot{\boldsymbol{\xi}} \in \dot{\mathbf{I}}$ can be written as

$$\begin{aligned} &\dot{\boldsymbol{\xi}} = \boldsymbol{\xi}^{\dot{1}}\mathbf{s}^{\dot{1}} + \boldsymbol{\xi}_{\dot{2}}\mathbf{s}^{\dot{2}} \\ &\mathbf{s}^{\dot{1}} = \mathbf{e}, \quad \mathbf{s}^{\dot{2}} = \mathbf{e}\boldsymbol{\sigma}_1. \end{aligned} \tag{28}$$

Defining the mapping

$$\begin{aligned} \boldsymbol{\iota} : \mathbf{I}\otimes\dot{\mathbf{I}} &\rightarrow \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{R}_{3,0}, \\ \boldsymbol{\iota}(\boldsymbol{\varphi}\otimes\dot{\boldsymbol{\xi}}) &= \boldsymbol{\varphi}\dot{\boldsymbol{\xi}} \end{aligned} \tag{29}$$

we have

$$\begin{aligned} 1 \equiv \boldsymbol{\sigma}_0 &= \boldsymbol{\iota}(\mathbf{s}_1\otimes\mathbf{s}^{\dot{1}} + \mathbf{s}_2\otimes\mathbf{s}^{\dot{2}}), \\ \boldsymbol{\sigma}_1 &= -\boldsymbol{\iota}(\mathbf{s}_1\otimes\mathbf{s}^{\dot{2}} + \mathbf{s}_2\otimes\mathbf{s}^{\dot{1}}), \\ \boldsymbol{\sigma}_2 &= \boldsymbol{\iota}[\mathbf{i}(\mathbf{s}_1\otimes\mathbf{s}^{\dot{2}} - \mathbf{s}_2\otimes\mathbf{s}^{\dot{1}})], \\ \boldsymbol{\sigma}_3 &= -\boldsymbol{\iota}(\mathbf{s}_1\otimes\mathbf{s}^{\dot{1}} - \mathbf{s}_2\otimes\mathbf{s}^{\dot{2}}). \end{aligned} \tag{30}$$

From this it follows that we have the identification

$$\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{C}(2) = \mathbf{I}\otimes_{\mathbb{C}}\dot{\mathbf{I}}, \tag{31}$$

from where it follows that each Pauli number can be written as an appropriate Clifford product of sums of algebraic contravariant undotted spinors and algebraic covariant dotted spinors, and of course a representative of a Pauli number in $\mathbb{C}^2$ can be written as an appropriate Kronecker product of a complex column vector by a complex row vector.

Take an arbitrary $\mathbf{P} \in\mathbb{R}_{3,0}$ such that

$$\mathbf{P} = \frac{1}{j!}p_{\mu}^{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j}\ \boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j}, \tag{32}$$

where $p_\mu^{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j} \in \mathbb{R}$ and

$$\boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j} = \boldsymbol{\sigma}_{\mathbf{k}_1}...\boldsymbol{\sigma}_{\mathbf{k}_j}, \quad \text{and } \boldsymbol{\sigma}_0 \equiv 1 \in \mathbb{R}. \tag{33}$$

With the identification $\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)} \simeq \mathbf{I} \otimes_{\mathbb{C}} \dot{\mathbf{I}}$, we can write also

$$\mathbf{P} = \mathbf{P}^A{}_{\dot{B}} \boldsymbol{\iota}(\mathbf{s}_A \otimes \mathbf{s}^{\dot{B}}) = \mathbf{P}^A{}_{\dot{B}} s_B \mathbf{s}^{\dot{B}}, \tag{34}$$

where the $\mathbf{P}^A{}_{\dot{B}} = \mathbf{X}^A{}_{\dot{B}} + \mathbf{i}\mathbf{Y}^A{}_{\dot{B}}$, $\mathbf{X}^A{}_{\dot{B}}, \mathbf{Y}^A{}_{\dot{B}} \in \mathbb{R}$.

Finally, the matrix representative of the Pauli number $\mathbf{P} \in \mathbb{R}_{3,0}$ is $P \in \mathbb{C}(2)$ given by

$$P = P^A{}_{\dot{B}} s_A s^{\dot{B}}, \tag{35}$$

with $P^A{}_{\dot{B}} \in \mathbb{C}$ and

$$\begin{array}{cc} s_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & s_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ s^{\dot{1}} = \begin{pmatrix} 1 & 0 \end{pmatrix} & s^{\dot{2}} = \begin{pmatrix} 0 & 1 \end{pmatrix}. \end{array} \tag{36}$$

It is convenient for our purposes to introduce also covariant undotted spinors and contravariant dotted spinors. Let $\varphi \in \mathbb{C}^2$ be given as in Eq.(23). We define the *covariant* version of undotted spinor $\varphi \in \mathbb{C}^2$ as $\varphi^* \in (\mathbb{C}^2)^t \simeq \mathbb{C}_2$ such that

$$\begin{aligned} &\varphi^* = (\varphi_1, \varphi_2) \equiv \varphi_A s^A, \\ &\varphi_A = \varphi^B \varepsilon_{BA}, \quad \varphi^B = \varepsilon^{BA} \varphi_A, \\ &s^1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad s^2 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \end{aligned} \tag{37}$$

where[11] $\varepsilon_{AB} = \varepsilon^{AB} = \text{adiag}(1,-1)$. We can write due to the above identifications that there exists $\varepsilon \in \mathbb{C}(2)$ given by Eq.(26) which can be written also as

$$\varepsilon = \varepsilon^{AB} s_A \boxtimes s_B = \varepsilon_{AB} s^A \boxtimes s^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma_2 \tag{38}$$

where $\boxtimes$ denote the *Kronecker* product of matrices. We have, e.g.,

$$\begin{aligned} s_1 \boxtimes s_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \boxtimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ s^1 \boxtimes s^1 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \boxtimes \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{39}$$

We now introduce the *contravariant* version of the dotted spinor

$$\dot{\xi} = \begin{pmatrix} \xi_{\dot{1}} & \xi_{\dot{2}} \end{pmatrix} \in \mathbb{C}_2$$

[11] The symbol adiag means the antidiagonal matrix.

as being $\dot{\xi}^* \in \mathbb{C}^2$ such that

$$
\begin{aligned}
\dot{\xi}^* &= \begin{pmatrix} \xi^{\dot{1}} \\ \xi^{\dot{2}} \end{pmatrix} = \xi^{\dot{A}} s_{\dot{A}}, \\
\xi^{\dot{B}} &= \varepsilon^{\dot{B}\dot{A}} \xi_{\dot{A}}, \quad \xi_{\dot{A}} = \varepsilon_{\dot{B}\dot{A}}\, \xi^{\dot{B}}, \\
s_{\dot{1}} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, s_{\dot{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{aligned} \tag{40}
$$

where $\varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \text{adiag}(1,-1)$. We can write due to the above identifications that there exists $\dot{\varepsilon} \in \mathbb{C}(2)$ such that

$$
\dot{\varepsilon} = \varepsilon^{\dot{A}\dot{B}} s_{\dot{A}} \boxtimes s_{\dot{B}} = \varepsilon_{\dot{A}\dot{B}} s^{\dot{A}} \boxtimes \dot{s}^{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon. \tag{41}
$$

Also, recall that even if $\{\mathbf{s}_A\}$,$\{\mathbf{s}_{\dot{A}}\}$ and $\{s^{\dot{A}}\}$,$\{s^{A}\}$ are bases of distinct spaces, we can identify their matrix representations, as it is obvious from the above formulas. So, we have $s_A \equiv s_{\dot{A}}$ and also $s^{\dot{A}} = s^{A}$. This is the reason for the representation of a dotted covariant spinor as in Eq.(25). Moreover, the above identifications permit us to write the *matrix representation* of a Pauli number $\mathbf{P} \in \mathbb{R}_{3,0}$ as, e.g.,

$$
P = P_{AB} s^{A} \boxtimes s^{B} \tag{42}
$$

besides the representation given by Eq.(35).

3 Clifford and Spinor Bundles

3.1 Preliminaries

To characterize in a rigorous mathematical way the *basic* field variables used in Sachs 'unified' field theory [85, 86], we shall need to recall some results of the theory of spinor fields on Lorentzian spacetimes. Here we follow the approach given in [79, 64].[12]

Recall that a Lorentzian manifold is a pair (M, g), where $g \in \sec T^{2,0}M$ is a Lorentzian metric of signature $(1,3)$, i.e., for all $x \in M$, $T_xM \simeq T_x^*M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the vector Minkowski space.

Recall that a Lorentzian spacetime is a pentuple $(M, g, D, \tau_g, \uparrow)$ where $(M, g, \tau_g, \uparrow)$ is an oriented Lorentzian manifold[13] which is also time oriented by an appropriated equivalence relation[14] (denoted $\uparrow$) for the timelike vectors at the tangent space T_xM, $\forall x \in M$. D is a linear connection for M such that $Dg = 0$, $\mathbf{\Theta}(D) = 0$, $\mathcal{R}(D) \neq 0$, where $\mathbf{\Theta}$ and $\mathcal{R}$ are respectively the torsion and curvature tensors of D.

[12] Another important reference on the subject of spinor fields is [56], which however only deals with the case of spinor fields on Riemannian manifolds.

[13] Oriented by the volume element $\tau_g \in \sec \bigwedge^4 T^*M$.

[14] See [88] for details.

Now, Sachs theory uses spinor fields. These objects are sections of so-called *spinor bundles*, which only exist in *spin manifolds*. The ones of interest in Sachs theory are the matrix representation of the bundle of dotted spinor fields, i.e., $S(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{D^{(\frac{1}{2},0)}} \mathbb{C}^2$ and the matrix representation of the bundle of undotted spinor fields (here denoted by) $\bar{S}(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{D^{(0,\frac{1}{2})}} \mathbb{C}_2$. In the previous formula $D^{(\frac{1}{2},0)}$ and $D^{(0,\frac{1)}{2}}$ are the two fundamental non equivalent 2-dimensional representations of $Sl(2,\mathbb{C}) \simeq \mathrm{Spin}^e_{1,3}$, the universal covering group of $\mathrm{SO}^e_{1,3}$, the restrict orthochronous Lorentz group. $P_{\mathrm{Spin}^e_{1,3}}(M)$ is a principal bundle called the spin structure bundle[15]. We recall that it is a classical result (Geroch theorem [49]) that a 4-dimensional Lorentzian manifold is a spin manifold if and only if $P_{\mathrm{SO}^e_{1,3}}(M)$ has a global section[16], i.e., if there exists a set[17] $\{\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ of orthonormal fields defined for all $x \in M$. In other words, in order for spinor fields to exist in a 4-dimensional spacetime the orthonormal frame bundle must be *trivial.*

Now, the so-called tangent (TM) and cotangent (T^*M) bundles, the tensor bundle $(\oplus_{r,s} \otimes^r_s TM)$ and the bundle of differential forms for the spacetime are the bundles denoted by

$$TM = P_{\mathrm{SO}^e_{1,3}}(M) \times_{\rho_{1,3}} \mathbb{R}^{1,3}, \quad T^*M = P_{\mathrm{SO}^e_{1,3}}(M) \times_{\rho^*_{1,3}} \mathbb{R}^{1,3}, \tag{43}$$
$$\oplus_{r,s} \otimes^r_s TM = P_{\mathrm{SO}^e_{1,3}}(M) \times_{\otimes^r_s \rho_{1,3}} \mathbb{R}^{1,3}, \quad \bigwedge T^*M = P_{\mathrm{SO}^e_{1,3}}(M) \times_{\Lambda^k_{\rho^*_{1,3}}} \bigwedge \mathbb{R}^{1,3}.$$

In Eqs.(43)

$$\rho_{1,3} : \mathrm{SO}^e_{1,3} \to \mathrm{SO}^e(\mathbb{R}^{1,3}) \tag{44}$$

is the standard vector representation of $\mathrm{SO}^e_{1,3}$ usually denoted by [18] $D^{(\frac{1}{2},\frac{1}{2})} = D^{(\frac{1}{2},0)} \otimes D^{(0,\frac{1}{2})}$and $\rho^*_{1,3}$ is the dual (vector) representation $\rho^*_{1,3}(l) = \rho_{1,3}(l^{-1})^t$. Also $\otimes^r_s \rho_{1,3}$ and $\Lambda^k_{\rho^*_{1,3}}$ are the induced tensor product and induced exterior power product representations of $\mathrm{SO}^e_{1,3}$. We now briefly recall the definition and some properties of Clifford bundle of multivector fields [79]. We have,

$$\begin{aligned} \mathcal{C}\ell(TM) &= P_{\mathrm{SO}^e_{1,3}}(M) \times_{c\ell_{\rho_{1,3}}} \mathbb{R}_{1,3} \\ &= P_{\mathrm{Spin}^e_{1,3}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}. \end{aligned} \tag{45}$$

Now, recall that [57] $\mathrm{Spin}^e_{1,3} \subset \mathbb{R}^{(0)}_{1,3}$. Consider the 2-1 homomorphism $\mathrm{h} : \mathrm{Spin}^e_{1,3} \to \mathrm{SO}^e_{1,3}, \mathrm{h}(\pm u) = l$. Then $c\ell_{\rho_{1,3}}$ is the following representation of

[15] It is a covering space of $P_{\mathrm{SO}^e_{1,3}}(M)$. See, e.g., [64] for details. Sections of $P_{\mathrm{Spin}^e_{1,3}}(M)$ are the so-called spin frames, i.e., a pair (Σ, u) where for any $x \in M$, $\Sigma(x)$ is an othonormal frame and $u(x)$ belongs to the $\mathrm{Spin}^e_{1,3}$. For details see [64, 79, 82].

[16] In what follows $P_{\mathrm{SO}^e_{1,3}}(M)$ denotes the principal bundle of oriented *Lorentz tetrads*. We presuppose that the reader is acquainted with the structure of $P_{\mathrm{SO}^e_{1,3}}(M)$, whose sections are the time oriented and oriented orthonormal frames, each one associated by a local trivialization to a *unique* element of $SO^e_{1,3}(M)$.

[17] Called vierbein.

[18] See, e.g., [60] if you need details.

$\mathrm{SO}_{1,3}^{e}$,

$$
\begin{aligned}
c\ell_{\rho_{1,3}} &: \mathrm{SO}_{1,3}^{e} \to \mathrm{Aut}(\mathbb{R}_{1,3}),\\
c\ell_{\rho_{1,3}}(L) &= \mathrm{Ad}_u : \mathbb{R}_{1,3} \to \mathbb{R}_{1,3},\\
\mathrm{Ad}_u(\mathbf{m}) &= u\mathbf{m}u^{-1}
\end{aligned} \tag{46}
$$

i.e., it is the standard orthogonal transformation of $\mathbb{R}_{1,3}$ induced by an orthogonal transformation of $\mathbb{R}^{1,3}$. Note that Ad_u act on vectors as the $D^{(\frac{1}{2},\frac{1}{2})}$ representation of $\mathrm{SO}_{1,3}^{e}$ and on multivectors as the induced exterior power representation of that group. Indeed, observe, e.g., that for $\mathbf{v} \in \mathbb{R}^{1,3} \subset \mathbb{R}_{1,3}$ we have in standard notation

$$
L\mathbf{v} = \mathbf{v}^{\nu} L_{\nu}^{\mu} \mathbf{m}_{\mu} = \mathbf{v}^{\nu} u\mathbf{m}_{\nu} u^{-1} = u\mathbf{v}u^{-1}.
$$

The proof of the second line of Eq.(45) is as follows. Consider the representation

$$
\begin{aligned}
&\mathrm{Ad} : \mathrm{Spin}_{1,3}^{e} \to \mathrm{Aut}(\mathbb{R}_{1,3}),\\
&\mathrm{Ad}_u : \mathbb{R}_{1,3} \to \mathbb{R}_{1,3}, \quad \mathrm{Ad}_u(m) = umu^{-1}.
\end{aligned} \tag{47}
$$

Since $\mathrm{Ad}_{-1} = 1 (= \text{identity})$ the representation Ad descends to a representation of $\mathrm{SO}_{1,3}^{e}$. This representation is just $c\ell(\rho_{1,3})$, from where the desired result follows.

Sections of $\mathcal{C}\ell(TM)$ can be called Clifford fields (of multivectors). The sections of the even subbundle $\mathcal{C}\ell^{(0)}(TM) = P_{\mathrm{Spin}_{1,3}^{e}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}^{(0)}$ may be called Pauli fields (of multivectors). Define the real spinor bundles

$$
\mathcal{S}(M) = P_{\mathrm{Spin}_{1,3}^{e}}(M) \times_{l} \mathbf{I}, \quad \dot{\mathcal{S}}(M) = P_{\mathrm{Spin}_{1,3}^{e}}(M) \times_{r} \dot{\mathbf{I}} \tag{48}
$$

where l stands for a left modular representation of $\mathrm{Spin}_{1,3}^{e}$ in $\mathbb{R}_{1,3}$ that mimics the $D^{(\frac{1}{2},0)}$ representation of $Sl(2,\mathbb{C})$ and r stands for a right modular representation of $\mathrm{Spin}_{1,3}^{e}$ in $\mathbb{R}_{1,3}$ that mimics the $D^{(0,\frac{1}{2})}$ representation of $Sl(2,\mathbb{C})$.

Also recall that if $\bar{S}(M)$ is the bundle whose sections are the spinor fields $\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2) = \dot{\varphi}\varepsilon = (\varphi^{\dot{1}}, \varphi^{\dot{2}})$, then it is isomorphic to the space of contravariant dotted spinors. We have,

$$
S(M) \simeq P_{\mathrm{Spin}_{1,3}^{e}}(M) \times_{D^{(\frac{1}{2},0)}} \mathbb{C}^2, \quad \dot{S}(M) \simeq P_{\mathrm{Spin}_{1,3}^{e}}(M) \times_{D^{(0,\frac{1}{2})}} \mathbb{C}_2 \simeq \bar{S}(M), \tag{49}
$$

and from our playing with the Pauli algebra and dotted and undotted spinors in section 2 we have that:

$$
\mathcal{S}(M) \simeq S(M), \quad \dot{\mathcal{S}}(M) \simeq \dot{S}(M) \simeq \bar{S}(M). \tag{50}
$$

Then, we have the obvious isomorphism

$$
\begin{aligned}
\mathcal{C}\ell^{(0)}(TM) &= P_{\mathrm{Spin}_{1,3}^{e}}(M) \times_{\mathrm{Ad}} \mathbb{R}_{1,3}^{(0)}\\
&= P_{\mathrm{Spin}_{1,3}^{e}}(M) \times_{l\otimes r} \mathbf{I} \otimes_{\mathbb{C}} \dot{\mathbf{I}}\\
&= \mathcal{S}(M) \otimes_{\mathbb{C}} \dot{\mathcal{S}}(M).
\end{aligned} \tag{51}
$$

Let us now introduce the following bundle,

$$\mathbb{C}\ell^{(0)}(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{D^{(\frac{1}{2}0)} \otimes D^{(0,\frac{1}{2})}} \mathbb{C}(2). \tag{52}$$

It is clear that

$$\mathbb{C}\ell^{(0)}(M) = S(M) \otimes_{\mathbb{C}} \bar{S}(M) \simeq \mathcal{C}\ell^{(0)}(M). \tag{53}$$

Finally, we consider the bundle

$$\mathcal{C}\ell^{(0)}(TM) \otimes \bigwedge T^*M \simeq \mathbb{C}\ell^{(0)}(M) \otimes \bigwedge T^*M. \tag{54}$$

Sections of $\mathcal{C}\ell^{(0)}(TM) \otimes \bigwedge T^*M$ may be called *Pauli valued differential forms* and sections of $\mathbb{C}\ell^{(0)}(M) \otimes \bigwedge T^*M$ may be called *matrix Pauli valued differential forms*.

Denote by $\mathcal{C}\ell^{(0)}_{(0,2)}(TM)$ the seven dimensional subbundle $\left(\mathbb{R} \oplus \bigwedge^2 TM\right) \subset \bigwedge TM \hookrightarrow \mathcal{C}\ell^{(0)}(TM) \subset \mathcal{C}\ell(TM)$. Now, let $\langle x^\mu \rangle$ be the coordinate functions of a chart of the maximal atlas of M. The fundamental field variable of Sachs theory can be described as

$$\mathbf{Q} = \mathbf{q}_\mu \otimes dx^\mu \equiv q_\mu dx^\mu \in \sec \mathcal{C}\ell^{(0)}_{(0,2)}(TM) \otimes \bigwedge T^*M \subset \sec \mathcal{C}\ell^{(0)}(TM) \otimes \bigwedge T^*M$$

i.e., a Pauli valued 1-form obeying certain conditions to be presented below. If we work (as Sachs did) with $\mathbb{C}\ell^{(0)}(M) \otimes \bigwedge T^*M$, a representative of $\mathbf{Q}$ is $Q \in \sec \mathbb{C}\ell^{(0)}(M) \otimes \bigwedge T^*M$ such that[19]

$$Q = q_\mu(x) dx^\mu = h^{\mathbf{a}}_\mu(x) dx^\mu \sigma_{\mathbf{a}}, \tag{55}$$

where $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and σ_j (j=1, 2, 3) are the Pauli matrices. We observe that the notation anticipates the fact that in Sachs theory the variables $h^{\mathbf{a}}_\mu(x)$ define the set $\{\theta^{\mathbf{a}}\} \equiv \{\theta^{\mathbf{0}}, \theta^{\mathbf{1}}, \theta^{\mathbf{2}}, \theta^{\mathbf{3}}\}$ with

$$\theta^{\mathbf{a}} = h^{\mathbf{a}}_\mu dx^\mu \in \sec \bigwedge T^*M, \tag{56}$$

which is the dual basis of $\{\mathbf{e_a}\} \equiv \{\mathbf{e_0}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, $\mathbf{e_a} \in \sec TM$. We denote by $\{e_\mu\} = \{e_0, e_1, e_2, e_3\}$, a coordinate basis associated with the local chart $\langle x^\mu \rangle$ covering $U \subset M$. We have $e_\mu = h^{\mathbf{a}}_\mu \mathbf{e_a} \in \sec TM$, and the set $\{e_\mu\}$ is the dual basis of $\{dx^\mu\} \equiv \{dx^0, dx^1, dx^2, dx^3\}$. We will also use the *reciprocal basis* to a given basis $\{\mathbf{e_a}\}$, i.e., the set $\{\mathbf{e^a}\} \equiv \{\mathbf{e^0}, \mathbf{e^1}, \mathbf{e^2}, \mathbf{e^3}\}, \mathbf{e^a} \in \sec TM$, with $g(\mathbf{e_a}, e^{\mathbf{b}}) = \delta^{\mathbf{b}}_a$ and the *reciprocal basis* to $\{\theta^{\mathbf{a}}\}$, i.e., the set $\{\theta_{\mathbf{a}}\} = \{\theta_{\mathbf{0}}, \theta_{\mathbf{1}}, \theta_{\mathbf{2}}, \theta_{\mathbf{3}}\}$, with $\theta_{\mathbf{a}}(e^{\mathbf{b}}) = \delta^{\mathbf{b}}_a$. Recall that since $\eta_{\mathbf{ab}} = g(\mathbf{e_a}, \mathbf{e_b})$, we have

[19] Note that a bold index (sub or superscript), say $\mathbf{a}$ take the values 0, 1, 2, 3.

$$g_{\mu\nu} = g\left(e_\mu, e_\nu\right) = h_\mu^{\mathbf{a}} h_\nu^{\mathbf{b}} \eta_{\mathbf{ab}}. \tag{57}$$

To continue, we define the

$$\check{\sigma}_0 = -\sigma_0 \text{ and } \check{\sigma}_{\mathbf{j}} = \sigma_{\mathbf{j}}, \mathbf{j} = 1,2,3 \tag{58}$$

and

$$\check{Q} = \check{q}_\mu(x) dx^\mu = h_\mu^{\mathbf{a}}(x) dx^\mu \check{\sigma}_{\mathbf{a}}. \tag{59}$$

Also, note that

$$\sigma_{\mathbf{a}} \check{\sigma}_{\mathbf{b}} + \sigma_{\mathbf{b}} \check{\sigma}_{\mathbf{a}} = -2\eta_{\mathbf{ab}}. \tag{60}$$

Readers of Sachs books [85, 87] will recall that he said that Q is a representative of a *quaternion*.[20] From our previous discussion we see that this statement is *wrong*.[21] Sachs identification is a dangerous one, because the quaternions are a division algebra, also-called a noncommutative field or skew-field and objects like $\mathbf{Q} = \mathbf{q}_\mu \otimes dx^\mu \in \sec \mathcal{C}\ell^{(0)}_{(0,2)}(TM) \otimes \bigwedge T^*M \subset \sec \mathcal{C}\ell^{(0)}(TM) \otimes \bigwedge T^*M$ are called *paravector* fields. As it is clear from our discussion they did not close a *division* algebra.

Next we introduce a tensor product of sections $\mathbf{A}, \mathbf{B} \in \sec \mathcal{C}\ell^{(0)}(M) \otimes \bigwedge T^*M$. Before we do that we recall that from now on

$$\{1, \boldsymbol{\sigma}_{\mathbf{k}}, \boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2}, \boldsymbol{\sigma}_{\mathbf{123}}\}, \tag{61}$$

refers to a basis of $\mathcal{C}\ell^{(0)}(M)$, i.e., they are fields.[22]

Recalling Eq.(33) we introduce the (obvious) notation

$$\mathbf{A} = \frac{1}{j!} a_\mu^{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j}\ \boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j} dx^\mu, \quad \mathbf{B} = \frac{1}{l!} b_\mu^{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_l} \boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_l} dx^\mu, \tag{62}$$

where the $a_\mu^{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j}, b_\mu^{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j}$ are, in general, *real* scalar functions. Then, we define

$$\mathbf{A} \otimes \mathbf{B} = \frac{1}{j!l!} a_\mu^{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j}\, b_\nu^{\mathbf{p}_1\mathbf{p}_2\ldots\mathbf{p}_l} \boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2\ldots\mathbf{k}_j} \boldsymbol{\sigma}_{\mathbf{p}_1\mathbf{p}_2\ldots\mathbf{p}_l} dx^\mu \otimes dx^\mu. \tag{63}$$

Let us now compute the tensor product of $\mathbf{Q} \otimes \check{\mathbf{Q}}$ where $\mathbf{Q} \in \sec \mathcal{C}\ell^{(0)}_{(0,2)}(M) \otimes \bigwedge T^*M$. We have,

[20] Note that Sachs represented Q by $d\mathsf{S}$, which is a very dangerous notation, which we avoid.

[21] Nevertheless the calculations done by Sachs in [85] are correct because he worked always with the matrix representation of $\mathbf{Q}$. However, his claim of having produce an unified field theory of gravitation and electromagnetism is wrong as we shall prove in what follows.

[22] We hope that in using (for symbol economy) the same notation as in section 2 where the $\{1, \boldsymbol{\sigma}_{\mathbf{k}}, \boldsymbol{\sigma}_{\mathbf{k}_1\mathbf{k}_2}, \boldsymbol{\sigma}_{\mathbf{123}}\}$ is a basis of $\mathbb{R}^{(0)}_{1,3} \simeq \mathbb{R}_{3.0}$ will produce no confusion.

$$
\begin{aligned}
\mathbf{Q}\otimes\check{\mathbf{Q}} &= \mathbf{q}_\mu(x)dx^\mu\otimes\check{\mathbf{q}}_\nu(x)dx^v = \mathbf{q}_\mu(x)\check{\mathbf{q}}_\nu(x)dx^\mu\otimes dx^\nu \\
&= \mathbf{q}_\mu(x)\check{\mathbf{q}}_\nu(x)\frac{1}{2}(dx^\mu\otimes dx^\nu + dx^\nu\otimes dx^\mu) \\
&+ \frac{1}{2}\mathbf{q}_\mu(x)\check{\mathbf{q}}_\nu(x)(dx^\mu\otimes dx^\nu - dx^\nu\otimes dx^\mu) \\
&= \frac{1}{2}(\mathbf{q}_\mu(x)\check{\mathbf{q}}_\nu(x) + \mathbf{q}_\nu(x)\check{\mathbf{q}}_\mu(x))dx^\mu\otimes dx^\nu \\
&+ \frac{1}{2}\mathbf{q}_\mu(x)\check{\mathbf{q}}_\nu(x)dx^\mu\wedge dx^\nu \qquad (64)\\
&= (-g_{\mu\nu}\boldsymbol{\sigma}_0)dx^\mu\otimes dx^\nu \\
&+ \frac{1}{4}(\mathbf{q}_\mu(x)\check{\mathbf{q}}_\nu(x) - \mathbf{q}_\nu(x)\check{\mathbf{q}}_\mu(x))dx^\mu\wedge dx^\nu \\
&= -g_{\mu\nu}dx^\mu\otimes dx^\nu + \frac{1}{2}\mathbf{F}'_{\mu\nu}dx^\mu\wedge dx^\nu.
\end{aligned}
$$

In writing Eq.(64) we have used $dx^\mu\wedge dx^\nu \equiv dx^\mu\otimes dx^\nu - dx^\nu\otimes dx^\mu$. Also, using

$$
\begin{aligned}
g_{\mu\nu} &= \eta_{\mathbf{ab}}h^{\mathbf{a}}_\mu(x)h^{\mathbf{b}}_\nu(x), \quad g = g_{\mu\nu}dx^\mu\otimes dx^v = \eta_{\mathbf{ab}}\theta^{\mathbf{a}}\otimes\theta^{\mathbf{b}} \\
\mathbf{F}'_{\mu\nu} &= \mathbf{F}'^k_{\mu\nu}\mathbf{i}\boldsymbol{\sigma}_k = -\frac{1}{2}(\varepsilon^k_{i\ j}h^i_\mu(x)h^j_\nu(x)\ )\mathbf{i}\boldsymbol{\sigma}_k; \quad i,j,k=1,2,3, \\
\mathbf{F}' &= \frac{1}{2}\mathbf{F}'_{\mu\nu}dx^\mu\wedge dx^v = \frac{1}{2}(\mathbf{F}'^{ij}_{\mu\nu}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j)dx^\mu\wedge dx^v = (\frac{1}{2}\mathbf{F}'^k_{\mu\nu}\mathbf{i}\boldsymbol{\sigma}_k)dx^\mu\wedge dx^v \\
&= -\varepsilon^k_{i\ j}h^i_\mu(x)h^j_\nu(x)\ dx^\mu\wedge dx^v\mathbf{i}\sigma_k \in \sec\bigwedge^2 T^*M\otimes\mathcal{C}\ell^{(0)}_{(2)}(M) \qquad (65)
\end{aligned}
$$

we can write Eq.(64) as

$$
\begin{aligned}
\mathbf{Q}\otimes\check{\mathbf{Q}} &= \mathbf{Q}\overset{s}{\otimes}\check{\mathbf{Q}} + \mathbf{Q}\wedge\check{\mathbf{Q}} \\
&= -g + \mathbf{F}. \qquad (66)
\end{aligned}
$$

We can also write

$$
\mathbf{Q}\otimes\check{\mathbf{Q}} = -\eta_{\mathbf{ab}}\boldsymbol{\sigma}_0\theta^{\mathbf{a}}\otimes\theta^{\mathbf{b}} + \varepsilon^k_{i\ j}\mathbf{i}\boldsymbol{\sigma}_k\theta^i\wedge\theta^j. \qquad (67)
$$

The above formulas show very clearly the mathematical nature of $\mathbf{F}$, it is a 2-form with values on the subspace of multivector Clifford fields, i.e., $\mathbf{F}:\bigwedge^2 TM \hookrightarrow \mathcal{C}\ell^{(0)}_{(2)}(TM)\subset\mathcal{C}\ell^{(0)}(TM)$. Now, we write the formula for $Q\otimes\tilde{Q}$ where $Q\in\mathbb{C}(2)\otimes\bigwedge^1 T^*M$ given by Eq.(55) is the matrix representation of $\mathbf{Q}\in\sec\mathcal{C}\ell^{(0)}_{(0,2)}(M)\otimes\bigwedge^1 T^*M$.

We have,

$$
\begin{aligned}
Q \otimes \check{Q} &= Q \overset{s}{\otimes} \tilde{Q} + Q \wedge \check{Q} \\
&= (-g_{\mu\nu} dx^{\mu} \otimes dx^{v})\sigma_0 + (\varepsilon_{i\ j}^{k} v_{\mu}^{i}(x) v_{\nu}^{j}(x)\ dx^{\mu} \wedge dx^{v})(-\mathrm{i}\sigma_k) \\
&= -g\sigma_0 + \mathbf{F}'^{k} \mathrm{i}\sigma_k, \qquad (68)
\end{aligned}
$$

with

$$
\mathbf{F}'^{k} = \frac{1}{2} \mathbf{F}'^{k}_{\mu\nu} dx^{\mu} \wedge dx^{v} = \varepsilon_{i\ j}^{k} v_{\mu}^{i}(x) v_{\nu}^{j}(x) dx^{\mu} \wedge dx^{\nu}. \qquad (69)
$$

For future reference we also introduce

$$
\mathbf{F}'_{\mu\nu} = \mathbf{F}'^{k}_{\mu\nu} \mathrm{i}\sigma_k. \qquad (70)
$$

3.2 Covariant Derivatives of Spinor Fields

We now briefly recall the concept of covariant spinor derivatives [25, 56, 64, 79]. The idea is the following:

(i) Every connection on the principal bundle of orthonormal frames $P_{\mathrm{SO}^e_{1,3}}(M)$ determines in a canonical way a unique connection on the principal bundle $P_{\mathrm{Spin}^e_{1,3}}(M)$.

(ii) Let D be a covariant derivative operator acting on sections of an associate vector bundle to $P_{\mathrm{SO}^e_{1,3}}(M)$, say, the tensor bundle τM and let D^s be the corresponding covariant spinor derivative acting on sections of associate vector bundles to $P_{\mathrm{Spin}^e_{1,3}}(M)$, say, e.g., the spinor bundles where $\mathcal{P}(M)$ may be called *Pauli spinor bundle*. Of course, $\mathcal{P}(M) \simeq \mathcal{C}\ell^{(0)}(M)$. The matrix representations of the above bundles are:

$$
\begin{aligned}
&S\,(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{D^{(\frac{1}{2}0)}} \mathbb{C}^2, \quad \dot{S}(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{D^{(0,\frac{1}{2})}} \mathbb{C}_2 \\
&P(M) = S\,(M) \otimes \dot{S}(M) = P_{\mathrm{Spin}^e_{1,3}}(M) \times_{D^{(\frac{1}{2}0)} \otimes D^{(0,\frac{1}{2})}} \mathbb{C}^2 \otimes \mathbb{C}_2, \qquad (71)
\end{aligned}
$$

and $P(M)$ may be called *matrix Pauli spinor bundle*. Of course, $P(M) \simeq \mathbb{C}\ell^{(0)}(M)$.

(iv) We have for $\mathbf{T} \in \sec \bigwedge TM \hookrightarrow \mathcal{C}\ell^{(0)}(M)$ and $\boldsymbol{\xi} \in \sec \mathcal{S}(M)$, $\dot{\boldsymbol{\xi}} \in \sec \dot{\mathcal{S}}(M)$, $\mathbf{P} \in \sec \mathcal{P}(M)$and $\mathbf{v} \in \sec TM$. Then,

$$
\begin{aligned}
D^s_{\mathbf{v}}(\mathbf{T} \otimes \boldsymbol{\xi}) &= D_{\mathbf{v}}\mathbf{T} \otimes \boldsymbol{\xi} + \mathbf{T} \otimes D^s_{\mathbf{v}} \boldsymbol{\xi}, \\
D^s_{\mathbf{v}}(\mathbf{T} \otimes \dot{\boldsymbol{\xi}}) &= D_{\mathbf{v}}\mathbf{T} \otimes \dot{\boldsymbol{\xi}} + \mathbf{T} \otimes D^s_{\mathbf{v}} \dot{\boldsymbol{\xi}}, \qquad (72)
\end{aligned}
$$

where

$$
\begin{aligned}
D_{\mathbf{v}}\mathbf{T} &= \partial_{\mathbf{v}}\mathbf{T} + \frac{1}{2}[\boldsymbol{\omega}_{\mathbf{v}}, \mathbf{T}],\\
D^s_{\mathbf{v}}\boldsymbol{\xi} &= \partial_{\mathbf{v}}\boldsymbol{\xi} + \frac{1}{2}\boldsymbol{\omega}_{\mathbf{v}}\boldsymbol{\xi},\\
D^s_{\mathbf{v}}\dot{\boldsymbol{\xi}} &= \partial_{\mathbf{v}}\dot{\boldsymbol{\xi}} - \frac{1}{2}\dot{\boldsymbol{\xi}}\boldsymbol{\omega}_{\mathbf{v}},\\
D_{\mathbf{v}}P &= \partial_{\mathbf{v}}\mathbf{P} + \frac{1}{2}\boldsymbol{\omega}_{\mathbf{v}}\mathbf{P} - \frac{1}{2}\mathbf{P}\ \boldsymbol{\omega}_{\mathbf{v}} = \partial_{\mathbf{v}}\mathbf{P} + \frac{1}{2}[\boldsymbol{\omega}_{\mathbf{v}}, \mathbf{P}]. \qquad (73)
\end{aligned}
$$

(v) For $\mathbf{T} \in \sec \bigwedge TM \hookrightarrow \mathcal{C}\ell^{(0)}(TM)$ and $\xi \in \sec S(M)$, $\bar{\xi} \in \sec \bar{S}(M)$, $P \in \sec\ P(M)$and $\mathbf{v} \in \sec TM$, we have

$$
\begin{aligned}
D^s_{\mathbf{v}}(\mathbf{T}\otimes\xi) &= D_{\mathbf{v}}\mathbf{T}\otimes\xi + \mathbf{T}D^s_{\mathbf{v}}\xi, \qquad (74)\\
D^s_{\mathbf{v}}(\mathbf{T}\otimes\bar{\xi}) &= D_{\mathbf{v}}\mathbf{T}\otimes\bar{\xi} + \mathbf{T}D^s_{\mathbf{v}}\bar{\xi}
\end{aligned}
$$

and

$$
\begin{aligned}
D_{\mathbf{v}}\mathbf{T} &= \partial_{\mathbf{v}}\mathbf{T} + \frac{1}{2}[\boldsymbol{\omega}_{\mathbf{v}}, \mathbf{T}],\\
D^s_{\mathbf{v}}\xi &= \partial_{\mathbf{v}}\xi + \frac{1}{2}\Omega_{\mathbf{v}}\xi,\\
D^s_{\mathbf{v}}\dot{\xi} &= \partial_{\mathbf{v}}\dot{\xi} - \frac{1}{2}\dot{\xi}\Omega_{\mathbf{v}},\\
D_{\mathbf{v}}P &= \partial_{\mathbf{v}}P + \frac{1}{2}\Omega_{\mathbf{v}}P - \frac{1}{2}P\ \Omega_{\mathbf{v}} = \partial_{\mathbf{v}}P + \frac{1}{2}[\Omega_{\mathbf{v}}, P]. \qquad (75)
\end{aligned}
$$

In the above equations $\boldsymbol{\omega}_{\mathbf{v}} \in \sec \mathcal{C}\ell^{(0)}(TM)$ and $\Omega_{\mathbf{v}} \in \sec\ P(M)$. Writing as usual, $\mathbf{v} = v^{\mathbf{a}}\mathbf{e}_{\mathbf{a}}$, $D_{\mathbf{e}_{\mathbf{a}}}e^{\mathbf{b}} = -\omega^{\mathbf{b}}_{\mathbf{ac}}e^{\mathbf{c}}$, $\omega^{\mathbf{b}}_{\mathbf{ac}} = -\omega^{\mathbf{b}}_{\mathbf{ca}}$, $\sigma_{\mathbf{b}} = \mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{0}}$ and[23] $i = -\sigma_{\mathbf{1}}\sigma_{\mathbf{2}}\sigma_{\mathbf{3}}$, we have

$$
\begin{aligned}
\boldsymbol{\omega}_{\mathbf{e}_{\mathbf{a}}} &= \frac{1}{2}\omega^{\mathbf{bc}}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{c}} = \frac{1}{2}\omega^{\mathbf{bc}}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}}\wedge\mathbf{e}_{\mathbf{c}}\\
&= \frac{1}{2}\omega^{\mathbf{bc}}_{\mathbf{a}}\boldsymbol{\sigma}_{\mathbf{b}}\overset{\vee}{\boldsymbol{\sigma}}_{\mathbf{c}}\\
&= \frac{1}{2}(-2\omega^{\mathbf{0i}}_{\mathbf{a}}\sigma_{\mathbf{i}} + \omega^{\mathbf{ji}}_{\mathbf{a}}\sigma_{\mathbf{i}}\sigma_{\mathbf{j}})\\
&= \frac{1}{2}(-2\omega^{\mathbf{0i}}_{\mathbf{a}}\sigma_{\mathbf{i}} - i\ \varepsilon^{\mathbf{k}}_{\mathbf{i}\ \mathbf{j}}\omega^{\mathbf{ji}}_{\mathbf{a}}\sigma_{\mathbf{k}}) = \boldsymbol{\Omega}^{\mathbf{b}}_{\mathbf{a}}\sigma_{\mathbf{b}}. \qquad (76)
\end{aligned}
$$

Note that the $\boldsymbol{\Omega}^{\mathbf{b}}_{\mathbf{a}}$ are ‘formally’ complex numbers. Also, observe that we can write for the ‘formal’ Hermitian conjugate $\boldsymbol{\omega}^{\dagger}_{\mathbf{e}_{\mathbf{a}}}$ of $\boldsymbol{\omega}_{\mathbf{e}_{\mathbf{a}}}$ of

$$
\boldsymbol{\omega}^{\dagger}_{\mathbf{e}_{\mathbf{a}}} = -\mathbf{e}^{\mathbf{0}}\boldsymbol{\omega}_{\mathbf{e}_{\mathbf{a}}}\mathbf{e}^{\mathbf{0}}. \qquad (77)
$$

[23] Have in mind that i is a *Clifford field* here.

Also, write $\Omega_{\mathbf{e_a}}$ for the matrix representation of $\boldsymbol{\omega}_{\mathbf{e_a}}$, i.e.,

$$\Omega_{\mathbf{e_a}} = \Omega_{\mathbf{a}}^{\mathbf{b}} \sigma_{\mathbf{b}},$$

where $\Omega_{\mathbf{a}}^{\mathbf{b}}$ are complex numbers with the same coefficients as the 'formally' complex numbers $\boldsymbol{\Omega}_{\mathbf{a}}^{\mathbf{b}}$. We can easily verify that

$$\Omega_{\mathbf{e_a}} = \varepsilon \Omega_{\mathbf{e_a}}^{\dagger} \varepsilon. \tag{78}$$

We can prove the third line of Eq.(75) as follows. First take the Hermitian conjugation of the second line of Eq.(75), obtaining

$$D_{\mathbf{v}} \bar{\xi} = \partial_{\mathbf{v}} \bar{\xi} + \frac{1}{2} \bar{\xi} \Omega_{\mathbf{v}}^{\dagger}.$$

Next multiply the above equation on the left by ε and recall that $\dot{\xi} = \bar{\xi}\varepsilon$ and Eq.(78). We get

$$\begin{aligned} D_{\mathbf{v}} \dot{\xi} &= \partial_{\mathbf{v}} \dot{\xi} - \frac{1}{2} \dot{\xi} \varepsilon \Omega_{\mathbf{v}}^{\dagger} \varepsilon \\ &= D_{\mathbf{v}} \dot{\xi} = \partial_{\mathbf{v}} \dot{\xi} - \frac{1}{2} \dot{\xi} \Omega_{\mathbf{v}}. \end{aligned}$$

Note that this is compatible with the identification $\mathcal{C}\ell^{(0)}(TM) \simeq \mathcal{S}(M) \otimes_{\mathbb{C}} \dot{\mathcal{S}}(M)$ and $\mathbb{C}\ell^{(0)}(M) \simeq S(M) \otimes_{\mathbb{C}} \dot{S}(M)$.

Note moreover that if $\mathbf{q}_{\mu} = e_{\mu} \mathbf{e_0} = h_{\mu}^{\mathbf{a}} \mathbf{e_a} \mathbf{e_0} = h_{\mu}^{\mathbf{a}} \boldsymbol{\sigma}_{\mathbf{a}} \in \mathcal{C}\ell^{(0)}(TM) \simeq \mathcal{S}(M) \otimes_{\mathbb{C}} \dot{\mathcal{S}}(M)$ we have,

$$D_{\mathbf{v}} \mathbf{q}_{\mu} = \partial_{\mathbf{v}} q_{\mu} + \frac{1}{2} \boldsymbol{\omega}_{\mathbf{v}} \mathbf{q}_{\mu} + \frac{1}{2} \mathbf{q}_{\mu} \boldsymbol{\omega}_{\mathbf{v}}^{\dagger}. \tag{79}$$

For $q_{\mu} = h_{\mu}^{\mathbf{a}} \sigma_{\mathbf{a}} \in \sec \mathbb{C}\ell^{(0)}(M) \simeq S(M) \otimes_{\mathbb{C}} \bar{S}(M)$, the matrix representative of the $\mathbf{q}_{\mu}$ we have for any vector field $\mathbf{v} \in \sec TM$

$$D_{\mathbf{v}} q_{\mu} = \partial_{\mathbf{v}} q_{\mu} + \frac{1}{2} \Omega_{\mathbf{v}} q_{\mu} + \frac{1}{2} q_{\mu} \ \Omega_{\mathbf{v}}^{\dagger} \tag{80}$$

which is the equation used by Sachs for the *spinor* covariant derivative of his 'quaternion' fields. Note that M. Sachs in [85] introduced also a kind of total covariant derivative for his 'quaternion' fields. That 'derivative' denoted in this text by $D_{\mathbf{v}}^{\mathbf{S}}$ will be discussed below.

3.3 Geometrical Meaning of $D_{e_\nu} q_\mu = \Gamma_{\nu\mu}^{\alpha} q_\alpha$

We recall that Sachs wrote [24] that

$$D_{e_\nu} q_\mu = \Gamma_{\nu\mu}^{\alpha} q_\alpha, \tag{81}$$

[24] See Eq.(3.69) in [85].

where $\Gamma^{\alpha}_{\nu\mu}$ are the connection coefficients of the coordinate basis $\{e_\mu\}$, i.e.,

$$D_{e_\nu} e_\mu = \Gamma^{\alpha}_{\nu\mu} e_\alpha \tag{82}$$

How, can Eq.(81) be true? Well, let us calculate $D_{e_\nu}\mathbf{q}_\mu$ in $\mathcal{C}\ell(TM)$. We have,

$$\begin{aligned} D_{e_\nu}\mathbf{q}_\mu &= D_{e_\nu}(e_\mu \mathbf{e_0}) \\ &= (D_{e_\nu} e_\mu)\mathbf{e_0} + e_\mu (D_{e_\nu}\mathbf{e_0}) \\ &= \Gamma^{\alpha}_{\nu\mu}\mathbf{q}_\alpha + e_\mu (D_{e_\nu}\mathbf{e_0}). \end{aligned} \tag{83}$$

So, Eq.(81) follows if, and only if

$$D_{e_\nu}\mathbf{e_0} = 0. \tag{84}$$

To understand the physical meaning of Eq.(84) let us recall the following. In relativity theory reference frames are represented by time like vector fields $Z \in \sec TM$ pointing to the future [80, 88]. If we write the $\alpha_\mathbf{Z} = g(\mathbf{Z},) \in \bigwedge^1 T^*M$ for the physically equivalent 1-form field we have the well known *decomposition*

$$D\alpha_\mathbf{Z} = \mathbf{a_Z} \otimes \alpha_\mathbf{Z} + \varpi_\mathbf{Z} + \sigma_\mathbf{Z} + \frac{1}{3}E_\mathbf{Z}\mathbf{p}, \tag{85}$$

where

$$\mathbf{p} = \mathbf{g} - \alpha_\mathbf{Z} \otimes \alpha_\mathbf{Z} \tag{86}$$

is called the projection tensor (and gives the metric of the rest space of an instantaneous observer [88]), $\mathbf{a_Z} = g(D_\mathbf{Z}\mathbf{Z},)$ is the (form) acceleration of $\mathbf{Z}$, $\varpi_\mathbf{Z}$ is the rotation of $\mathbf{Z}$, $\sigma_\mathbf{Z}$ is the shear of $\mathbf{Z}$ and $E_\mathbf{Z}$ is the expansion ratio of $\mathbf{Z}$. In a coordinate chart (U, x^μ), writing $\mathbf{Z} = Z^\mu \partial/\partial x^\mu$ and $\mathbf{p} = (g_{\mu\nu} - Z_\mu Z_\nu)dx^\mu \otimes dx^\nu$ we have

$$\begin{aligned} \varpi_{\mathbf{Z}\mu\nu} &= Z_{[\alpha;\beta]} p^\alpha_\mu p^\beta_\nu, \\ \sigma_{\mathbf{Z}\alpha\beta} &= [Z_{(\mu;\nu)} - \frac{1}{3}\mathbf{E_Z} h_{\mu\nu}] p^\mu_\alpha p^\nu_\beta, \\ E_\mathbf{Z} &= Z^\mu{}_{;\mu}\,. \end{aligned} \tag{87}$$

Now, in Special Relativity where the space time manifold is $< M = \mathcal{R}^4, \mathbf{g} = \eta, D^\eta, \tau_\eta, \uparrow >$[25] an *inertial reference frame* (*IRF*) $\mathbf{I} \in \sec TM$ is defined by $D^\eta \mathbf{I} = 0$. We can show very easily (see, e.g., [88]) that in General Relativity Theory (*GRT*) where each gravitational field is modelled by a spacetime[26] $< M, \mathbf{g}, D, \tau_g, \uparrow >$ there is *in general* no shear free frame ($\sigma_\mathfrak{Z} = 0$) on any open

[25] η is a constant metric, i.e., there exists a chart $\langle x^\mu \rangle$ of $M = \mathcal{R}^4$ such that $\eta(\partial/\partial x^\mu, \partial/\partial x^\nu) = \eta_{\mu\nu}$, the numbers $\eta_{\mu\nu}$ forming a diagonal matrix with entries $(1, -1, -1, -1)$. Also, D^η is the Levi-Civita connection of η.

[26] More precisely, by a diffeomorphism equivalence class of Lorentzian spacetimes.

neighborhood U of any given spacetime point. The reason is clear in local coordinates $\langle x^\mu \rangle$ covering U. Indeed, $\sigma_{\mathfrak{Q}} = 0$ implies five independent conditions on the components of the frame $\mathfrak{Q}$. Then, we arrive at the conclusion that in a general spacetime model[27] there is no frame $\mathfrak{Q} \in \sec TU \subset \sec TM$ satisfying $D\mathfrak{Q} = 0$, and in general there is no *IRF* in any model of *GRT*.

The following question arises naturally: which characteristics a reference frame on a *GRT* spacetime model must have in order to reflect as much as possible the properties of an *IRF* of *SRT*?

The answer to that question [80] is that there are two kind of frames in *GRT* such that each frame in one of these classes share some important aspects of the *IRFs* of *SRT*. Both concepts are important and it is important to distinguish between them in order to avoid misunderstandings. These frames are the *pseudo inertial reference frame* (*PIRF*) and the and the local Lorentz reference frames ($LLRF\gamma s$), but we don not need to enter the details here.

On the open set $U \subset M$ covered by a coordinate chart $\langle x^\mu \rangle$ of the maximal atlas of M multiplying Eq.(84) by $h^\nu_{\mathbf{a}}$ such that $\mathbf{e_a} = h^\nu_{\mathbf{a}} e_\nu$, we get

$$D_{\mathbf{e_a}} \mathbf{e_0} = 0; \;\; \mathbf{a} = 0,1,2,3. \tag{88}$$

Then, it follows that

$$D_X \mathbf{e_0} = 0, \, \forall X \in \sec TM \tag{89}$$

which characterizes $\mathbf{e_0}$ as an inertial frame. This imposes several restrictions on the spacetime described by the theory. Indeed, if **Ric** is the Ricci tensor of the manifold modeling spacetime, we have[28]

$$\mathbf{Ric}(\mathbf{e_0}, X) = 0, \, \forall X \in \sec TM. \tag{90}$$

In particular, this condition cannot be realized in Einstein-de Sitter spacetime. This fact is completely hidden in the matrix formalism used in Sachs theory, where no restriction on the spacetime manifold (besides the one of being a spin manifold) need to be imposed.

3.4 Geometrical Meaning of $D_{e_\mu} \sigma_{\mathbf{i}} = 0$ in General Relativity

We now discuss what happens in the usual theory of dotted and undotted two component *matrix* spinor fields in general relativity, as described, e.g., in [21, 71, 72]. In that formulation it is postulated that the covariant spinor derivative of Pauli matrices must satisfy

$$D_{e_\mu} \sigma_{\mathbf{i}} = 0, \;\; \mathbf{i} = 1,2,3 \tag{91}$$

[27] We take the opprotunity to correct an statement in [80]. There it is stated that in General Relativity there are no inertial frames. Of, course, the correct statement is that in a general spacetime model there are in general no inertial frames. But, of course, there are spacetime models where there exist frames $\mathfrak{Q} \in \sec TU \subset \sec TM$ satisfying $D\mathfrak{Q} = 0$. See below.

[28] See, exercise 3.2.12 of [88].

Eq.(91) translate in our formalism as

$$D_{e_\mu}\boldsymbol{\sigma}_{\mathbf{i}} = D_{e_\mu}(\mathbf{e_i e_0}) = 0. \tag{92}$$

Differently from the case of Sachs theory, Eq.(92) can be satisfied if

$$D_{e_\mu}\mathbf{e_i} = \mathbf{e_i}(D_{e_\mu}\mathbf{e}_0)\mathbf{e_0} \tag{93}$$

or, writing $D_{e_\mu}\mathbf{e_a} = \omega^{\mathbf{b}}_{\mu\mathbf{a}}\mathbf{e_b}$,

$$\omega^{\mathbf{b}}_{\mu\mathbf{i}} = \mathbf{e^b} \lrcorner (\omega^{\mathbf{a}}_{\mu\mathbf{0}}\mathbf{e_i e_a e_0}). \tag{94}$$

This certainly implies some restrictions on possible spacetime models, but that is the price in order to have spinor fields. At least we do not need to necessarily have $D\mathbf{e_0} = 0$.

We analyze some possibilities of satisfying Eq.(91)

(**i**) Suppose that $\mathbf{e_0}$ satisfy $D_{e_\mu}\mathbf{e_0} = 0$, i.e., $D\mathbf{e_0} = 0$. Then, a necessary and sufficient condition for the validity of Eq.(92) is that

$$D_{e_\mu}\mathbf{e_i} = 0. \tag{95}$$

Multiplying Eq.(95) by $h^{\mu}_{\mathbf{a}}$ we get

$$D_{\mathbf{e_a}}e_{\mathbf{i}} = 0,\ \mathbf{i} = 1,2,3;\ \mathbf{a} = 0,1,2,3 \tag{96}$$

In particular,

$$D_{\mathbf{e}_0}\mathbf{e_i} = 0,\ \mathbf{i} = 1,2,3 \tag{97}$$

Eq.(97) means that the fields $\mathbf{e_i}$ following each integral line of $\mathbf{e_0}$ are Fermi transported[29] [88]. Physicists interpret that equation saying that the $\mathbf{e_i}$ are physically realizable by gyroscopic axes, which gives the local standard of no rotation.

The above conclusion sounds fine. However it follows from Eq.(89) and Eq.(96) that

$$D_{\mathbf{e_a}}\mathbf{e_b} = 0,\ \mathbf{a} = 0,1,2,3;\ \mathbf{b} = 0,1,2,3. \tag{98}$$

Recalling that existence of spinor fields implies that $\{\mathbf{e_a}\}$ is a global tetrad [49], Eq.(98) implies that the connection D must be teleparallel. Then, under the above conditions the curvature tensor of a spacetime admitting spinor fields must be *null*. This, is in particular, the case of special relativity.

(**ii**) Suppose now that $\mathbf{e_0}$ is a geodesic frame, i.e., $D_{\mathbf{e}_0}\mathbf{e_0} = 0$. Then, $h^{\nu}_{\mathbf{0}}D_{e_\nu}\mathbf{e}_0 = 0$ and Eq. (93) implies only that

$$D_{\mathbf{e}_0}\mathbf{e_i} = 0;\ \mathbf{i} = 1,2,3 \tag{99}$$

and we do not have *any* inconsistency. If we take an integral line of $\mathbf{e_0}$, say γ, then the set $\{\mathbf{e_a}|_\gamma\}$ may be called an *inertial moving frame* along γ. The set

[29] An original approach to the Fermi transport using Clifford bundle methods has been given in [78]. There an equivalent spinor equation to the famous Darboux equations of differential geometry is derived.

$\{\mathbf{e_a}|_\gamma\}$ is also *Fermi* transported and since γ is a geodesic worldline they define the standard of *no* rotation along γ.

In conclusion, a consistent definition of spinor fields in general relativity using the Clifford and spin bundle formalism of this paper needs triviality of the frame bundle, i.e., existence of a global tetrad, say $\{\mathbf{e_a}\}$ and validity of Eq.(93). A nice physical interpretation follows moreover if the tetrad satisfies

$$D_{\mathbf{e}_0}\mathbf{e_a} = 0;\ \mathbf{a} = 0,1,2,3. \tag{100}$$

Of course, as it is the case in Sachs theory, the matrix formulation of spinor fields do not impose any constrains in the possible spacetime models, besides the one needed for the existence of a spinor structure. Saying that we have an important comment.

3.5 Covariant Derivative of the Dirac Gamma Matrices

If we use a real spin bundle where we can formulate the Dirac equation, e.g., one where the typical fiber is the ideal of (algebraic) Dirac spinors, i.e., the ideal generated by a idempotent $\frac{1}{2}(1+E_0)$, $E_0 \in \mathbb{R}_{1,3}$, then no restriction is imposed on the global tetrad field $\{\mathbf{e_a}\}$ defining the spinor structure of spacetime (see [79, 64]). In particular, since

$$D_{\mathbf{e}_a}\mathbf{e_b} = \omega^{\mathbf{c}}_{\mathbf{ab}}\mathbf{e_c}, \tag{101}$$

we have,

$$D_{\mathbf{e}_a}\mathbf{e_b} = \frac{1}{2}[\boldsymbol{\omega}_{\mathbf{e_a}}, \mathbf{e_b}] \tag{102}$$

Then,

$$\omega^{\mathbf{c}}_{\mathbf{ab}}\mathbf{e_c} - \frac{1}{2}\boldsymbol{\omega}_{\mathbf{e_a}}\mathbf{e_b} + \frac{1}{2}\mathbf{e_b}\boldsymbol{\omega}_{\mathbf{e_a}} = 0. \tag{103}$$

The matrix representation of the real spinor bundle, of course, sends $\{\mathbf{e_a}\} \mapsto \{\gamma_{\mathbf{a}}\}$, where the $\gamma_{\mathbf{a}}$'s are the standard representation of the Dirac matrices. Then, the matrix translation of Eq.(103) is

$$\omega^{\mathbf{c}}_{\mathbf{ab}}\gamma_{\mathbf{c}} - \frac{1}{2}\boldsymbol{\omega}_{\mathbf{e_a}}\gamma_{\mathbf{b}} + \frac{1}{2}\gamma_{\mathbf{b}}\boldsymbol{\omega}_{\mathbf{e_a}} = 0. \tag{104}$$

For the matrix elements $\gamma^A_{\mathbf{b}B}$ we have

$$\omega^{\mathbf{c}}_{\mathbf{ab}}\gamma^A_{\mathbf{c}B} - \frac{1}{2}\boldsymbol{\omega}^A_{\mathbf{e_a}C}\gamma^C_{\mathbf{b}B} + \frac{1}{2}\gamma^A_{\mathbf{b}C}\boldsymbol{\omega}^C_{\mathbf{e_a}B} = 0. \tag{105}$$

In [25] this last equation is confused with the covariant derivative of $\gamma^A_{\mathbf{c}B}$. Indeed in an exercise in problem 4, Chapter Vbis [25] ask one to prove that

$$\boxed{\nabla_{\mathbf{e_b}}\gamma^A_{\mathbf{c}B} = \omega^{\mathbf{c}}_{\mathbf{ab}}\gamma^A_{\mathbf{c}B} - \tfrac{1}{2}\boldsymbol{\omega}^A_{\mathbf{e_a}C}\gamma^C_{\mathbf{b}B} + \tfrac{1}{2}\gamma^A_{\mathbf{b}C}\boldsymbol{\omega}^C_{\mathbf{e_a}B} = 0.}$$

Of course, the first member of the above equation does not define any covariant derivative operator. Confusions as that one appears over and over again in the literature, and of course, is also present in Sachs theory in a small modified form, as shown in the next subsubsection.

3.6 $D^{\mathbf{S}}_{e_\nu} q_\mu = 0$

Now, taking into account Eq.(80) and Eq.(81) we can write:

$$\partial_\nu \mathbf{q}_\mu + \frac{1}{2}\boldsymbol{\omega}_\nu \mathbf{q}_\mu + \frac{1}{2}\mathbf{q}_\mu \boldsymbol{\omega}_\nu - \Gamma^{\alpha}_{\nu\mu}\mathbf{q}_\alpha = 0. \tag{106}$$

Sachs defined

$$D^{\mathbf{S}}_{e_\nu}\mathbf{q}_\mu = \partial_\nu \mathbf{q}_\mu + \frac{1}{2}\boldsymbol{\omega}_\nu \mathbf{q}_\mu + \frac{1}{2}\mathbf{q}_\mu \boldsymbol{\omega}_\nu - \Gamma^{\alpha}_{\nu\mu}\mathbf{q}_\alpha \tag{107}$$

from where

$$D^{\mathbf{S}}_{e_\nu}\mathbf{q}_\mu = 0. \tag{108}$$

Of course, the matrix representation of the last two equations are:

$$\begin{aligned} D^{\mathbf{S}}_{e_\nu} q_\mu &= \partial_\nu q_\mu + \frac{1}{2}\Omega_\nu q_\mu + \frac{1}{2} q_\mu\ \Omega^{\dagger}_\nu - \Gamma^{\alpha}_{\nu\mu} q_\alpha. \\ D^{\mathbf{S}}_{e_\nu} q_\mu &= 0. \end{aligned} \tag{109}$$

Sachs call [30] $D^{\mathbf{S}}_{e_\nu} q_\mu$ the covariant derivative of a q_μ field. The nomination is an *unfortunate* one, since the equation $D^{\mathbf{S}}_{e_\nu} q_\mu = 0$ is a *trivial* identity and do not introduce any new connection in the game.[31]

After this long exercise we can derive easily all formulas in chapters 3-6 of [85] without using any matrix representation at all. In particular, for future reference we collect some formulas,

$$\begin{aligned} \mathbf{q}^\mu \check{\mathbf{q}}_\mu &= -4, \quad q^\mu \check{q}_\mu = -4\sigma_0 \\ \mathbf{q}^\mu_\rho \boldsymbol{\omega} \check{\mathbf{q}}_\mu &= 0, \quad q^\mu \Omega_\rho \check{q}_\mu = 0, \\ \boldsymbol{\omega}_\rho &= -\frac{1}{2}\check{\mathbf{q}}_\mu(\partial_\rho \mathbf{q}^\mu + \Gamma^{\mu}_{\rho\tau}\mathbf{q}^\tau), \quad \Omega_\rho = -\frac{1}{2}\check{q}_\mu(\partial_\rho q^\mu + \Gamma^{\mu}_{\rho\tau} q^\tau) \end{aligned} \tag{110}$$

Before we proceed, it is important to keep in mind that our 'normalization' of $\boldsymbol{\omega}_\rho$ (and of Ω_ρ) here *differs* from Sachs one by a factor of 1/2. We prefer our normalization, since it is more natural and avoid factors of 2 when we perform contractions.

Before we discuss the equations of Sachs theory we think it is worth, using Clifford algebra methods, to present a formulation of Einstein's gravitational theory which resembles a gauge theory with group $Sl(2,\mathbb{C})$ as the gauge group. This formulation will then be compared with Sachs theory. Our formulation permits to prove that contrary to his claims in [85, 86] he did not produce any unified field theory of gravitation and electromagnetism.

[30] See Eq.(3.69) in [85].

[31] The equation $D^{\mathbf{S}}_{\nu}\mathbf{q}_\mu = 0$ (or its matrix representation) is a reminicescence of an analogous equation for the components of tetrad fields often printed in physics textbooks and confused with the metric compatibility condition of the connection. See,e.g., comments on page 76 of [50].

4 Clifford Valued Differential Forms and the Theory of Linear Connections

4.1 Preliminaries

In the general theory of connections [25, 54] a connection is a 1-form in the cotangent space of a principal bundle, with values in the Lie algebra of a gauge group. In order to develop a theory of a linear connection[32]

$$\overset{\blacktriangle}{\boldsymbol{\omega}} \in \sec T^* P_{\mathrm{SO}^e_{1,3}}(M) \otimes \mathrm{sl}(2,\mathbb{C}), \tag{111}$$

with an *exterior* covariant derivative operator acting on sections of associated vector bundles $P_{\mathrm{SO}^e_{1,3}}(M)$ which reproduces moreover the well known results obtained with the usual covariant derivative of tensor fields in the base manifold, we need to introduce the concept of a *soldering* form

$$\overset{\blacktriangle}{\boldsymbol{\theta}} \in \sec T^* P_{\mathrm{SO}^e_{1,3}}(M) \otimes \mathbb{R}^{1,3}. \tag{112}$$

Let be $U \subset M$ and π_1, π_2 respectively the projections of $T^* P_{\mathrm{SO}^e_{1,3}}(M) \otimes \mathbb{R}^{1,3}$ and $P_{\mathrm{SO}^e_{1,3}}(M)$ to M, naturally associated to the projection π of $P_{\mathrm{SO}^e_{1,3}}(M)$. Let

$$\begin{aligned} \varsigma_1 &: U \to \pi_1^{-1}(U) \subset T^* P_{\mathrm{SO}^e_{1,3}}(M) \otimes \mathbb{R}^{1,3}, \\ \varsigma_2 &: U \to \pi_2^{-1}(U) \subset T^* P_{\mathrm{SO}^e_{1,3}}(M) \otimes \mathrm{sl}(2,\mathbb{C}), \end{aligned} \tag{113}$$

be two cross sections. We are interested in the study of the pullbacks $\boldsymbol{\omega} = \varsigma_1^* \overset{\blacktriangle}{\boldsymbol{\omega}}$ and $\boldsymbol{\theta} = \varsigma_2^* \overset{\blacktriangle}{\boldsymbol{\theta}}$ once we give a local trivialization of the respective bundles. As it is well known we have in a local chart $\langle x^\mu \rangle$ covering U,

$$\boldsymbol{\theta} = e_\mu \otimes dx^\mu \equiv e_\mu dx^\mu \in \sec TM \otimes \bigwedge^1 T^* M. \tag{114}$$

Now, we give the Clifford algebra structure to the tangent bundle, thus generating the Clifford bundle $\mathcal{C}\ell(TM) = \bigcup_x \mathcal{C}\ell_x(M)$, with $\mathcal{C}\ell_x(M) \simeq \mathbb{R}_{1,3}$ introduced previously.

We recall a well known result [57] that for each $x \in U \subset M$ the bivectors of $\mathcal{C}\ell_x(M)$ generate under the product defined by the commutator, the Lie algebra $\mathrm{sl}(2,\mathbb{C})$. We thus are lead to define the representatives in $\mathcal{C}\ell(TM) \otimes \bigwedge T^* M$ for $\boldsymbol{\theta}$ and for the the pullback $\boldsymbol{\omega}$of the connection *in a given gauge* (that we

[32] In words, $\overset{\blacktriangle}{\boldsymbol{\Omega}}$ is a 1-form in the cotangent space of the bundle of ortonornal frames with values in the Lie algebra $\mathrm{so}^e_{1,3} \simeq \mathrm{sl}(2,\mathbb{C})$ of the group $\mathrm{SO}^e_{1,3}(M)$.

represent with the same symbols):

$$
\begin{aligned}
\boldsymbol{\theta} &= e_\mu dx^\mu = \mathbf{e_a}\theta^{\mathbf{a}} \in \sec \bigwedge^1 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M, \\
\boldsymbol{\omega} &= \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{bc}}\mathbf{e_b e_c}\theta^{\mathbf{a}} \\
&= \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{bc}}(\mathbf{e_b} \wedge \mathbf{e_c}) \otimes \theta^{\mathbf{a}} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M.
\end{aligned}
\tag{115}
$$

Before we continue we must recall that whereas $\boldsymbol{\theta}$ is a true tensor, $\boldsymbol{\omega}$ is not a true tensor, since as it is well known, its 'components' do not have the tensor transformation properties. Note that the $\omega_{\mathbf{a}}^{\mathbf{bc}}$ are the 'components' of the connection defined by

$$
D_{\mathbf{e_a}} e^{\mathbf{b}} = -\omega_{\mathbf{ac}}^{\mathbf{b}} e^{\mathbf{c}}, \quad \omega_{\mathbf{ac}}^{\mathbf{b}} = -\omega_{\mathbf{ca}}^{\mathbf{b}}, \tag{116}
$$

where $D_{\mathbf{e_a}}$ is the Levi-Civita covariant derivative operator acting on $\mathcal{C}\ell(TM)$, as defined in Appendix.

4.2 Exterior Covariant Differential

We want now to show how to describe, with our formalism the action of a exterior covariant differential on sections of a Clifford valued differential forms (i.e., sections of $\sec \mathcal{C}\ell(TM) \otimes \bigwedge T^*M$) which *mimics* the action of the pullback of covariant derivative operator acting on sections of a vector bundle associated to the principal bundle $P_{\mathrm{SO}^e_{1,3}}(M)$ once a linear metrical compatible connection is given. We start by recalling the well known definition of the covariant differential $\mathbf{D}$ acting on an arbitrary sections of a vector bundle $E(M)$ associated to $P_{\mathrm{SO}^e_{1,3}}(M)$, having as typical fiber a l-dimensional real vector space. Let $\mathrm{end}E\,(M) = E\,(M) \otimes E^*(M)$ the bundle of endomorphisms of $E\,(M)$ we have

Definition 1 *The covariant exterior differential operator* $\mathbf{D}$ *acting on sections of* $\mathrm{end}E\,(M)$ *is the mapping*

$$
\mathbf{D} : \sec E\,(M) \to \sec E\,(M) \otimes \bigwedge^1 T^*M, \tag{117}
$$

such that for any differentiable function $f : M \to \mathbb{R}$, $A \in \sec E\,(M)$ *and* $F \in \sec(\mathrm{end}E\,(M) \otimes \bigwedge^p T^*M)$, $G \in \sec(\mathrm{end}E\,(M) \otimes \bigwedge^q T^*M)$ *we have:*

$$
\begin{aligned}
\mathbf{D}(fA) &= df \otimes A + f\mathbf{D}A, \\
\mathbf{D}(F \otimes_\wedge A) &= \mathbf{D}F \otimes_\wedge A + (-1)^p F \otimes_\wedge \mathbf{D}A, \\
\mathbf{D}(F \otimes_\wedge G) &= \mathbf{D}F \otimes_\wedge G + (-1)^p F \otimes_\wedge \mathbf{D}G.
\end{aligned}
\tag{118}
$$

In Eq.(118), writing $F=F^{a}\otimes f_{a}^{(p)}$, $G=G^{b}\otimes g_{b}^{(q)}$where F^{a}, $G^{b}\in\sec(\mathrm{end}E\left(M\right))$, $f_{a}^{(p)}\in\sec\bigwedge^{p}T^{\ast}M$ and $g_{b}^{(q)}\in\sec\bigwedge^{q}T^{\ast}M$ we have

$$
\begin{aligned}
F\otimes_{\wedge}A &= \left(F^{a}\otimes f_{a}^{(p)}\right)\otimes_{\wedge}A=\left(F^{a}A\right)\otimes f_{a}^{(p)},\\
F\otimes_{\wedge}G &= \left(F^{a}\otimes f_{a}^{(p)}\right)\otimes_{\wedge}G^{b}\otimes g_{b}^{(q)}=\left(F^{a}G^{b}\right)f_{a}^{(p)}\wedge g_{b}^{(q)},
\end{aligned}
\tag{119}
$$

where[33] $F^{a}A\in\sec E\left(M\right)$ and $F^{a}G^{b}$means the composition of the respective endomorphisms.

Let $U\subset M$ be an open subset of M, $\langle x^{\mu}\rangle$ a coordinate functions of a maximal atlas of M and $\{e_{\mathbf{K}}\}$, $\mathbf{K}=1,2,...l$ a basis for any $\sec E\left(U\right)\subset\sec E\left(M\right)$. Then, a basis for any section of $E\left(M\right)\otimes\bigwedge^{1}T^{\ast}M$ is given by $\{e_{\mathbf{K}}\otimes dx^{\mu}\}$. By definition

$$
\mathbf{D}A\doteq\left(D_{e_{\mu}}A\right)\otimes dx^{\mu},
\tag{120}
$$

where, writing $A=A^{\mathbf{K}}\otimes e_{\mathbf{K}}$ we have

$$
D_{e_{\mu}}A=\partial_{\mu}A^{\mathbf{K}}\otimes e_{\mathbf{K}}+A^{\mathbf{K}}\otimes\mathbf{D}_{e_{\mu}}e_{\mathbf{K}}.
\tag{121}
$$

Now, let first $E\left(M\right)=TM\equiv\bigwedge^{1}(TM)\hookrightarrow\mathcal{C}\ell(TM)$ and as before let $\{e_{\mathbf{j}}\}$, be an orthonormal basis of TM. Then,

$$
\begin{aligned}
\mathbf{D}e_{\mathbf{j}} &= e_{\mathbf{k}}\otimes\boldsymbol{\omega}_{\mathbf{j}}^{\mathbf{k}}\equiv\left(D_{e_{\mathbf{k}}}e_{\mathbf{j}}\right)\otimes\theta^{\mathbf{k}}\\
\boldsymbol{\omega}_{\mathbf{j}}^{\mathbf{k}} &= \boldsymbol{\omega}_{\mathbf{rj}}^{\mathbf{k}}\theta^{\mathbf{r}},
\end{aligned}
\tag{122}
$$

where the $\boldsymbol{\omega}_{\mathbf{j}}^{\mathbf{k}}\in\sec\bigwedge^{1}T^{\ast}M$ are the so-called *connection 1-forms*.

Also, for $\mathbf{v}=v^{\mathbf{i}}\mathbf{e}_{\mathbf{i}}\in\sec TM$, we have

$$
\begin{aligned}
\mathbf{D}\mathbf{v} &= D_{\mathbf{e}_{\mathbf{i}}}\mathbf{v}\otimes\theta^{\mathbf{i}}=\mathbf{e}_{\mathbf{i}}\otimes\mathbf{D}v^{\mathbf{i}},\\
\mathbf{D}v^{\mathbf{i}} &= dv^{\mathbf{i}}+\boldsymbol{\omega}_{\mathbf{k}}^{\mathbf{i}}v^{\mathbf{k}}.
\end{aligned}
\tag{123}
$$

We want now to generalize the concept of covariant exterior differential for the case where $E\left(M\right)=\mathcal{C}\ell(TM)$. In order to do that in an appropriate way for our purposes, we introduced the concept of multivector valued differential forms and their algebra.

4.3 Multivector Valued Differential Forms

Definition: A *homogeneous* multivector valued differential form of type (l,p) is a section of $\bigwedge^{l}TM\otimes\bigwedge^{p}T^{\ast}M\hookrightarrow\mathcal{C}\ell(TM)\otimes\bigwedge T^{\ast}M$, for $0\leq l\leq4$, $0\leq p\leq4$.

[33] We eventully write $\left(F^{a}A\right)\otimes f_{a}^{(p)}\equiv\left(F^{a}A\right)f_{a}^{(p)}$ when there is no possibility of confusion.

We recall, that any $A \in \sec \bigwedge^{l} TM \otimes \bigwedge^{p} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{p} T^{*}M$ can always be written as

$$
\begin{aligned}
A &= m_{(l)} \otimes \psi^{(p)} \equiv m_{(i)}^{\mathbf{i}_1 \ldots \mathbf{i}_l} \mathbf{e}_{\mathbf{i}_1} \ldots \mathbf{e}_{\mathbf{i}_l} \otimes \psi^{(p)} \\
&= \frac{1}{p!} m_{(i)} \otimes \psi^{(p)}_{\mathbf{j}_1 \ldots \mathbf{j}_p} \theta^{\mathbf{j}_1} \wedge \ldots \wedge \theta^{\mathbf{j}_p} \\
&= \frac{1}{p!} m_{(l)}^{\mathbf{i}_1 \ldots \mathbf{i}_l} \mathbf{e}_{\mathbf{i}_1} \ldots \mathbf{e}_{\mathbf{i}_l} \otimes \psi^{(p)}_{\mathbf{j}_1 \ldots . \mathbf{j}_p} \theta^{\mathbf{j}_1} \wedge \ldots \wedge \theta^{\mathbf{j}_p} \qquad (124) \\
&= \frac{1}{p!} A^{\mathbf{i}_1 \ldots \mathbf{i}_l}_{\mathbf{j}_1 \ldots \mathbf{j}_p} \mathbf{e}_{\mathbf{i}_1} \ldots \mathbf{e}_{\mathbf{i}_l} \otimes \theta^{\mathbf{i}_1} \wedge \ldots \wedge \theta^{\mathbf{i}_p}.
\end{aligned}
$$

Definition: The $\otimes_{\wedge}$product of $A = \overset{m}{A} \otimes \psi^{(p)} \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{p} T^{*}M$ and $B = \overset{m}{B} \otimes \chi^{(p)} \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{q} T^{*}M$ is the mapping:

$$
\begin{aligned}
&\otimes_{\wedge} : \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{l} T^{*}M \times \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{p} T^{*}M \\
&\qquad \rightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{l+p} T^{*}M, \\
&A \otimes_{\wedge} B = \overset{m}{A}\overset{m}{B} \otimes \psi^{(p)} \wedge \chi^{(q)}. \qquad (125)
\end{aligned}
$$

Definition: The *commutator* $[A, B]$ of $A \in \sec \bigwedge^{l} TM \otimes \bigwedge^{p} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{p} T^{*}M$ and $B \in \bigwedge^{m} TM \otimes \bigwedge^{q} T^{*}M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{q} T^{*}M$ is the mapping:

$$
\begin{aligned}
&[\quad,\quad] : \sec \bigwedge^{l} TM \otimes \bigwedge^{p} T^{*}M \times \sec \bigwedge^{m} TM \otimes \bigwedge^{q} T^{*}M \\
&\qquad \rightarrow \sec\Big(\Big(\sum_{k=|l-m|}^{|l+m|} \bigwedge^{k} T^{*}M\Big) \otimes \bigwedge^{p+q} T^{*}M\Big) \\
&[A, B] = A \otimes_{\wedge} B - (-1)^{pq} B \otimes_{\wedge} A \qquad (126)
\end{aligned}
$$

Writing $A = A^{\mathbf{j}_1 \ldots \mathbf{j}_l} e_{\mathbf{j}_1} \ldots e_{\mathbf{j}_l} \psi^{(p)}$, $B = B^{\mathbf{i}_1 \ldots \mathbf{i}_m} e_{\mathbf{i}_1} \ldots e_{\mathbf{i}_m} \chi^{(q)}$, with $\psi^{(p)} \in \sec \bigwedge^{p} T^{*}M$ and $\chi^{(q)} \in \sec \bigwedge^{q} T^{*}M$, we have

$$
[A, B] = A^{\mathbf{j}_1 \ldots \mathbf{j}_l} B^{\mathbf{i}_1 \ldots \mathbf{i}_m} \left[e_{\mathbf{j}_1} \ldots e_{\mathbf{j}_l}, e_{\mathbf{i}_1} \ldots e_{\mathbf{i}_m} \right] \psi^{(p)} \wedge \chi^{(q)}, \qquad (127)
$$

The definition of the commutator is extended by linearity to arbitrary sections of $\mathcal{C}\ell(TM) \otimes \bigwedge T^{*}M$.

Now, we have the proposition

Proposition: Let $A \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{p} T^{*}M$, $B \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{q} T^{*}M$, $C \in A \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{r} T^{*}M$. Then,

$$[A,B]=(-1)^{1+pq}[B,A], \tag{128}$$

and

$$(-1)^{pr}\left[[A,B],C\right]+(-1)^{qp}\left[[B,C],A\right]+(-)^{rq}\left[[C,A],B\right]=0. \tag{129}$$

Eq.(129) may be called the *graded Jacobi identity* [19].

Corollary: Let be $A^{(2)}\in\sec\bigwedge^{2}(TM)\otimes\bigwedge^{p}T^{*}M$ and $B\in\sec\bigwedge^{r}(TM)\otimes\bigwedge^{q}T^{*}M$. Then,

$$[A^{(2)},B]=C, \tag{130}$$

where $C\in\sec\bigwedge^{r}(TM)\otimes\bigwedge^{p+q}T^{*}M$.

The proofs of Eq.(128), Eq.(129) and Eq.(130) results from direct calculation and can be obtained without any difficulty.

Definition: The action of the differential operator d acting on

$$A\in\sec\bigwedge^{l}TM\otimes\bigwedge^{p}T^{*}M\hookrightarrow\sec\mathcal{C}\ell(TM)\otimes\bigwedge^{p}T^{*}M,$$

is given by:

$$\begin{aligned}dA &\doteq e_{\mathbf{j}_1}...e_{\mathbf{j}_l}\otimes dA^{\mathbf{j}_1\cdots\mathbf{j}_l}\\ &=e_{\mathbf{j}_1}...e_{\mathbf{j}_l}\otimes d\frac{1}{p!}A^{\mathbf{j}_1\cdots\mathbf{j}_l}_{\mathbf{i}_1\cdots\mathbf{i}_p}\theta^{\mathbf{i}_1}\wedge...\wedge\theta^{\mathbf{i}_p}.\end{aligned} \tag{131}$$

We have the important proposition.

Proposition: Let be $A\in\sec\mathcal{C}\ell(TM)\otimes\bigwedge^{p}T^{*}M$ and let be $B\in\sec\mathcal{C}\ell(TM)\otimes\bigwedge^{q}T^{*}M$. Then,

$$d[A,B]=[dA,B]+(-1)^{p}[A,dB]. \tag{132}$$

The proof of that proposition is a very simple calculation.

Now, note that using the Clifford algebra structure of the space of multivectors we can show very easily that Eq.(122) can be written as:

$$\begin{aligned}\mathbf{D}e_{\mathbf{j}} &=(D_{e_{\mathbf{k}}}e_{\mathbf{j}})\theta^{\mathbf{k}}=\frac{1}{2}[\boldsymbol{\omega},e_{\mathbf{j}}]=-e_{\mathbf{j}}\lrcorner\boldsymbol{\omega}\\ \boldsymbol{\omega} &=\frac{1}{2}\omega_{\mathbf{k}}^{\mathbf{ab}}\mathbf{e_a}\wedge\mathbf{e_b}\otimes\theta^{\mathbf{k}}\\ &\equiv\frac{1}{2}\omega_{\mathbf{k}}^{\mathbf{ab}}\mathbf{e_a}\mathbf{e_b}\otimes\theta^{\mathbf{k}}\in\sec\bigwedge^{2}TM\otimes\bigwedge^{1}T^{*}M\hookrightarrow\sec\mathcal{C}\ell(TM)\otimes\bigwedge^{1}T^{*}M,\end{aligned} \tag{133}$$

where we recall once again that $\boldsymbol{\omega}$ is the representative of the connection in a given gauge.

Recalling the general theory of the exterior covariant derivative acting on a vector bundle $E(M)$, we see that the action of the covariant exterior differential $\mathbf{D}$ on an arbitrary $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(TM)$ is the mapping

$$\mathbf{D} : \sec \bigwedge^l TM \to \sec \bigwedge^l TM \otimes \bigwedge^1 T^*M, \tag{134}$$

such that for any differentiable function $f : M \to \mathbb{R}$, and differentiable $A \in \sec \bigwedge^l TM$ we have

$$\mathbf{D}(fA) = df \otimes A + f\mathbf{D}A. \tag{135}$$

Writing as before and with obvious notation,

$$\mathbf{D}A \doteq (D_{e_\mathbf{i}} A) \otimes \theta^i \tag{136}$$

where $D_{e_\mathbf{i}} A$ is the standard covariant derivative of $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(TM)$, we have the proposition.

Proposition:

$$\mathbf{D}A = dA + \frac{1}{2}[\boldsymbol{\omega}, A]. \tag{137}$$

The proof is a simple calculation.

Eq.(137) can now be extended by linearity for an arbitrary multivector $A \in \sec \mathcal{C}\ell(TM)$.

We are now read to investigate the general case, i.e., we want to give a *definition* of the exterior differential operator acting on an arbitrary section $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$ in such a way as to be compatible with the previous definitions.

Definition: Let $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$, l, $p \geq 1$. The covariant exterior differential of $\mathcal{A}$ is:

$$\mathbf{D}\mathcal{A} = d\mathcal{A} + \frac{p}{2}[\boldsymbol{\omega}, \mathcal{A}]. \tag{138}$$

Writing

$$\mathbf{D}\mathcal{A} = (\mathbf{D}_{e_\mathbf{r}} \mathcal{A}) \otimes_\wedge \theta^\mathbf{r} \equiv (\mathbf{D}_{e_\mathbf{j}} \mathcal{A})\theta^\mathbf{j} \tag{139}$$

we have

$$\begin{aligned} \mathbf{D}_{e_\mathbf{r}} \mathcal{A}(e_{\mathbf{k}_1}, e_{\mathbf{k}_2} ..., e_{\mathbf{k}_p}) &= \partial_{e_\mathbf{r}}(\mathcal{A}(e_{\mathbf{k}_1}, e_{\mathbf{k}_2}, ..., e_{\mathbf{k}_p})) \\ &\quad + \mathcal{A}^{\mathbf{i}_1 ... \mathbf{i}_l}(e_{\mathbf{k}_1}, e_{\mathbf{k}_2}, ..., e_{\mathbf{k}_p}) \frac{p}{2}[\boldsymbol{\omega}_\mathbf{r}, e_{\mathbf{i}_1} e_{\mathbf{i}_2} ... e_{\mathbf{i}_l}]. \end{aligned} \tag{140}$$

Note that

$$\mathbf{D}_{e_\mathbf{r}} \mathcal{A}(e_{\mathbf{k}_1}, e_{\mathbf{k}_2}, ..., e_{\mathbf{k}_p}) = D_{e_\mathbf{r}} \mathcal{A}(e_{\mathbf{k}_1}, e_{\mathbf{k}_2}, ..., e_{\mathbf{k}_p}) + \frac{p-1}{2}[\boldsymbol{\omega}_\mathbf{r}, \mathcal{A}(e_{\mathbf{k}_1}, e_{\mathbf{k}_2}, ..., e_{\mathbf{k}_p})] \tag{141}$$

or

$$\mathbf{D}_{e_\mathbf{r}}\mathcal{A} = D_{e_\mathbf{r}}\mathcal{A} + \frac{p-1}{2}[\boldsymbol{\omega}_\mathbf{r}, \mathcal{A}]. \tag{142}$$

In view of Eq.(141) we call from now on $\mathbf{D}_{e_\mathbf{r}}$ as the *exterior covariant derivative operator*.

Note that only for $\mathcal{A}^{(1)} \in \sec \bigwedge^{l} TM \otimes \bigwedge^{1} T^*M$ we have

$$\mathbf{D}_{e_\mathbf{r}}\mathcal{A}^{(1)} = D_{e_\mathbf{r}}\mathcal{A}^{(1)}. \tag{143}$$

We already know (see Eq. (137)) how to calculate the exterior covariant differential of $A \in \sec \bigwedge^{l} TM \equiv \sec \bigwedge^{l} TM \otimes \bigwedge^{0} T^*M$, and now we write explicitly the above formulas for two special important cases.

4.3.1 Case $p = 1$

When $\mathcal{A} \in \sec \bigwedge^{l} TM \otimes \bigwedge^{1} T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{1} T^*M$, a simple calculation shows that

$$\mathbf{D}_{e_\mathbf{k}}\mathcal{A}(e_\mathbf{i}) = d\mathcal{A}(e_\mathbf{i}) + \frac{1}{2}[\boldsymbol{\omega}_\mathbf{k}, \mathcal{A}(e_\mathbf{i})]$$

or

$$\mathbf{D}\mathcal{A} = d\mathcal{A} + \frac{1}{2}[\boldsymbol{\omega}, \mathcal{A}]. \tag{144}$$

4.3.2 Case $p = 2$

Let be $\mathcal{F} \in \sec \bigwedge^{l} TM \otimes \bigwedge^{2} T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge T^*M$. We have,

$$\begin{aligned} D_{e_\mathbf{r}}\mathcal{F}(e_{\mathbf{k}_1}, e_{\mathbf{k}_2}, ..., e_{\mathbf{k}_p}) &= \partial_{e_\mathbf{r}}(\mathcal{F}(e_{\mathbf{k}_1}, e_{\mathbf{k}_2}, ..., e_{\mathbf{k}_p})) \\ &\quad + \mathcal{F}^{\mathbf{i}_1...\mathbf{i}_l}(e_{\mathbf{k}_1}, e_{\mathbf{k}_2}, ..., e_{\mathbf{k}_p})\frac{1}{2}[\boldsymbol{\omega}_\mathbf{r}, e_{\mathbf{i}_1}e_{\mathbf{i}_2}...e_{\mathbf{i}_l}] \end{aligned} \tag{145}$$

and

$$\mathbf{D}\mathcal{F} = d\mathcal{F} + [\boldsymbol{\omega}, \mathcal{F}]. \tag{146}$$

4.4 Cartan Exterior Differential

Recall that [48] *Cartan* defined the covariant exterior differential of $\mathfrak{C} = e_\mathbf{i} \otimes \mathfrak{C}^\mathbf{i} \in \sec \bigwedge^{1} TM \otimes \bigwedge^{p} T^*M$ as mapping

$$\begin{aligned} &\mathbf{D}^c \colon \bigwedge^{1} TM \otimes \bigwedge^{p} T^*M \longrightarrow \bigwedge^{1} TM \otimes \bigwedge^{p+1} T^*M, \\ &\mathbf{D}^c\mathfrak{C} = \mathbf{D}^c(e_\mathbf{i} \otimes \mathfrak{C}^\mathbf{i}) = e_\mathbf{i} \otimes d\mathfrak{C}^\mathbf{i} + \mathbf{D}^c e_\mathbf{i} \wedge \mathfrak{C}^\mathbf{i}, \\ &\mathbf{D}^c e_\mathbf{i} = \mathbf{D}e_\mathbf{j} = (D_{e_\mathbf{k}}e_\mathbf{j})\theta^\mathbf{k} \end{aligned} \tag{147}$$

which in view of Eq.(131) and Eq.(133) can be written as

$$\mathbf{D}^c\mathfrak{C} = \mathbf{D}^c(e_\mathbf{i} \otimes \mathfrak{C}^\mathbf{i}) = d\mathfrak{C} + \frac{1}{2}[\boldsymbol{\omega}, \mathfrak{C}]. \tag{148}$$

So, we have,

$$\mathbf{D}\mathfrak{C} = \mathbf{D}^c\mathfrak{C} + \frac{p-1}{2}[\boldsymbol{\omega}, \mathfrak{C}]. \tag{149}$$

Note moreover that *only* when $\mathfrak{C}^{(1)} = e_{\mathbf{i}} \otimes \mathfrak{C}^{\mathbf{i}} \in \sec \bigwedge^1 TM \otimes \bigwedge^1 T^*M$ we have

$$\mathbf{D}\mathfrak{C}^{(1)} = \mathbf{D}^c\mathfrak{C}^{(1)}. \tag{150}$$

Finally we have the proposition.

Proposition: If $A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$ and $B \in \bigwedge^m TM \otimes \bigwedge^q T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$, $p, q \geq 2$ then

$$\mathbf{D}(A \otimes_\wedge B) = \mathbf{D}(A) \otimes_\wedge B + (-1)^p A \otimes_\wedge (\mathbf{D}B). \tag{151}$$

The proof follows at once with the use of the above formulas.

We note that Eq.(151) agrees with the third line of Eq.(118), as it may be.

We end this section with two observations:

(i) There are other approaches to the concept of exterior covariant differential acting on sections of a vector bundle $E \otimes \bigwedge^p T^*M$ and also in sections of[34] $\mathrm{end}(E) \otimes \bigwedge^p T^*M$, as e.g., in [17, 18, 48, 50, 69, 70, 92]. Not all are completely equivalent among themselves and to the one presented above. Our definitions have the merit of mimicking coherently the pullback under a local section of the covariant differential acting on sections of vector bundles associated to a given principal bundle as used in gauge theories. Indeed, this consistence will be checked in several situations below.

(ii) Some authors, e.g. [94] find convenient to introduce a *notation* (not a definition) that they call the *exterior covariant derivative* of indexed p-forms, which are objects like the curvature 2-forms (see below) or the connection 1-forms introduced above. This is a very dangerous *notation* according to our view and we avoid its use in this text.

4.5 Torsion and Curvature

The torsion of a connection $\boldsymbol{\omega} \in \sec \bigwedge^2 M \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$ on the *basis manifold* is defined by

$$\boldsymbol{\Theta} = \mathbf{D}\boldsymbol{\theta} \in \sec \bigwedge^1 TM \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^2 T^*M, \tag{152}$$

where $\boldsymbol{\theta} = e_\mu dx^\mu = \mathbf{e_a}\theta^{\mathbf{a}} \in \sec \bigwedge^1 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$ is the soldering form.

The curvature of the connection $\boldsymbol{\omega}$ is defined by

$$\mathcal{R} = \mathbf{D}\boldsymbol{\omega} \in \sec \bigwedge^2 M \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^2 T^*M, \tag{153}$$

[34] $\mathrm{end}(E)$ means the bundle of endomorphisms of the bundle E.

where the connection is $\boldsymbol{\omega} = \frac{1}{2}\left(\omega_{\mathbf{a}}^{\mathbf{bc}}\mathbf{e_b} \wedge \mathbf{e_c}\right) \otimes \theta^{\mathbf{a}} \equiv \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{bc}}\mathbf{e_b}\mathbf{e_c}\theta^{\mathbf{a}} \in \sec \bigwedge^2 M \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$. We now calculate $\boldsymbol{\Theta}$ and $\mathbf{D}\mathcal{R}$. We have,

$$\mathbf{D}\boldsymbol{\theta} = \mathbf{D}(\mathbf{e_a}\theta^{\mathbf{a}}) = \mathbf{e_a}d\theta^{\mathbf{a}} + \frac{1}{2}[\boldsymbol{\omega}_{\mathbf{a}}, \mathbf{e_d}]\theta^{\mathbf{a}} \wedge \theta^{\mathbf{d}} \tag{154}$$

and since $\frac{1}{2}[\boldsymbol{\omega}_{\mathbf{a}}, \mathbf{e_d}] = -\mathbf{e_d} \lrcorner \boldsymbol{\omega}_{\mathbf{a}} = \omega_{\mathbf{ad}}^{\mathbf{c}}\mathbf{e_c}$ we have

$$\mathbf{D}(\mathbf{e_a}\theta^{\mathbf{a}}) = \mathbf{e_a}[d\theta^{\mathbf{a}} + \omega_{\mathbf{bd}}^{\mathbf{a}}\theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}] = \mathbf{e_a}\boldsymbol{\Theta}^{\mathbf{a}}, \tag{155}$$

and we recognize

$$\boldsymbol{\Theta}^{\mathbf{a}} = d\theta^{\mathbf{a}} + \omega_{\mathbf{bd}}^{\mathbf{a}}\theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}, \tag{156}$$

as *Cartan's first structure equation.*

For a torsion free connection, the torsion 2-forms $\boldsymbol{\Theta}^{\mathbf{a}} = 0$, and it follows that $\boldsymbol{\Theta} = 0$. A metrical compatible connection ($Dg = 0$) satisfying $\boldsymbol{\Theta}^{\mathbf{a}} = 0$ is called a Levi-Civita connection. In the remaining of this paper we *restrict* ourself to that case.

Now, according to Eq.(146) we have,

$$\mathbf{D}\mathcal{R} = d\mathcal{R} + [\boldsymbol{\omega}, \mathcal{R}]. \tag{157}$$

Now, taking into account that

$$\mathcal{R} = d\boldsymbol{\omega} + \frac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}], \tag{158}$$

and that from Eqs.(128).(129) and (132) it follows that

$$\begin{aligned} d[\boldsymbol{\omega}, \boldsymbol{\omega})] &= [d\boldsymbol{\omega}, \boldsymbol{\omega}] - [\boldsymbol{\omega}, d\boldsymbol{\omega}], \\ [d\boldsymbol{\omega}, \boldsymbol{\omega}] &= -[\boldsymbol{\omega}, d\boldsymbol{\omega}], \\ [[\boldsymbol{\omega}, \boldsymbol{\omega}], \boldsymbol{\omega}] &= 0, \end{aligned} \tag{159}$$

we have immediately

$$\mathbf{D}\mathcal{R} = d\mathcal{R} + [\boldsymbol{\omega}, \mathcal{R}] = 0. \tag{160}$$

Eq.(160) is known as the *Bianchi identity.*

Note that

$$\begin{aligned} \mathcal{R} &= \frac{1}{4}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e_a} \wedge \mathbf{e_b} \otimes (dx^\mu \wedge dx^\nu) \\ &\equiv \frac{1}{4}\mathcal{R}_{\mathbf{cd}}^{\mathbf{ab}}\mathbf{e_a}\mathbf{e_b} \otimes \theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}} = \frac{1}{4}R_{\rho\sigma}^{\alpha\beta}e_\alpha e_\beta \otimes dx^\rho \wedge dx^\sigma \\ &= \frac{1}{4}R_{\mu\nu\rho\sigma}e^\mu e^\nu \otimes dx^\rho \wedge dx^\sigma, \end{aligned} \tag{161}$$

where $R_{\mu\nu\rho\sigma}$ are the components of the curvature tensor, also known in differential geometry as the Riemann tensor. We recall the well known symmetries

$$
\begin{aligned}
R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma}, \\
R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho}, \\
R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu}.
\end{aligned} \tag{162}
$$

We also write Eq.(161) as

$$
\begin{aligned}
\mathcal{R} &= \frac{1}{4}R^{\mathbf{ab}}_{\mathbf{cd}}\mathbf{e_a}\mathbf{e_b}\otimes(\theta^{\mathbf{c}}\wedge\theta^{\mathbf{d}}) = \frac{1}{2}\mathbf{R}_{\mu\nu}dx^{\mu}\wedge dx^{\nu} \\
&= \frac{1}{2}\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}\ \mathbf{e_a}e^{\mathbf{b}},
\end{aligned} \tag{163}
$$

with

$$
\begin{aligned}
\mathbf{R}_{\mu\nu} &= \frac{1}{2}R^{\mathbf{ab}}_{\mu\nu}\mathbf{e_a}\mathbf{e_b} = \frac{1}{2}R^{\mathbf{ab}}_{\mu\nu}\mathbf{e_a}\wedge\mathbf{e_b}\in\sec\bigwedge\nolimits^{2}TM\hookrightarrow\mathcal{C}\ell(TM), \\
\mathcal{R}^{\mathbf{ab}} &= \frac{1}{2}R^{\mathbf{ab}}_{\mu\nu}dx^{\mu}\wedge dx^{\nu}\in\sec\bigwedge\nolimits^{2}T^{*}M,
\end{aligned} \tag{164}
$$

where $\mathbf{R}_{\mu\nu}$ will be called curvature bivectors and the $\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}$ are called after Cartan the curvature 2-forms. The $\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}$ satisfy *Cartan's second structure equation*

$$
\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} = d\boldsymbol{\omega}^{\mathbf{a}}_{\mathbf{b}} + \boldsymbol{\omega}^{\mathbf{a}}_{\mathbf{c}}\wedge\boldsymbol{\omega}^{\mathbf{c}}_{\mathbf{d}}, \tag{165}
$$

which follows trivially calculating $d\mathcal{R}$ from Eq.(158). Now, we can also write,

$$
\begin{aligned}
\mathbf{D}\mathcal{R} &= d\mathcal{R} + [\boldsymbol{\omega},\mathcal{R}] \\
&= \frac{1}{2}\{d(\frac{1}{2}R^{\mathbf{ab}}_{\mu\nu}\mathbf{e_a}\mathbf{e_b}dx^{\mu}\wedge dx^{\nu}) + \frac{1}{2}[\boldsymbol{\omega}_{\rho},\mathbf{R}_{\mu\nu}]\}dx^{\rho}\wedge dx^{\mu}\wedge dx^{\nu} \\
&= \frac{1}{2}\{\partial_{\rho}\mathbf{R}_{\mu\nu} + [\boldsymbol{\omega}_{\rho},\mathbf{R}_{\mu\nu}]\}dx^{\rho}\wedge dx^{\mu}\wedge dx^{\nu}.
\end{aligned} \tag{166}
$$

In Physics textbooks on gauge theories (see, e.g., [69, 84]), physicists call the operator

$$
\mathbf{D}_{e_{\rho}} \equiv \mathbf{D}_{\rho} = \partial_{\rho} + [\boldsymbol{\omega}_{\rho},], \tag{167}
$$

acting on the curvature bivectors, the *'covariant derivative'*. This is not a very good name, since it is the *exterior covariant derivative* as defined in Eq.(141). The covariant derivative operator acting on sections of $\mathcal{C}\ell(T^{*}M)$ is as we already know another operator. See, Eq.(142).

Note that writing:

$$
\mathbf{D}_{e_{\rho}}\mathbf{R}_{\mu\nu} = \partial_{e_{\rho}}\mathbf{R}_{\mu\nu} + [\boldsymbol{\omega}_{\rho},\mathbf{R}_{\mu\nu}], \tag{168}
$$

we have from Bianchi identity that

$$
\mathbf{D}_{e_{\rho}}\mathbf{R}_{\mu\nu} + \mathbf{D}_{e_{\mu}}\mathbf{R}_{\nu\rho} + \mathbf{D}_{e_{\nu}}\mathbf{R}_{\rho\mu} = 0. \tag{169}
$$

We write now the coordinate expression for the bivector valued 2-forms $\mathbf{R}_{\mu\nu}$. First recall that by definition

$$\mathbf{R}_{\mu\nu} = \mathbf{R}(e_\mu, e_\nu) = -\mathbf{R}(e_\nu, e_\mu) = -\mathbf{R}_{\mu\nu}. \tag{170}$$

Now, observe that using Eqs.(128), (129) and (132) we can easily show that

$$\begin{aligned}[\boldsymbol{\omega}, \boldsymbol{\omega}](e_\mu, e_\nu) &= 2[\boldsymbol{\omega}(e_\mu), \boldsymbol{\omega}(e_\nu)] \\ &= 2[\boldsymbol{\omega}_\mu, \boldsymbol{\omega}_\nu].\end{aligned} \tag{171}$$

Using Eqs. (158), (170) and (171) we get

$$\mathbf{R}_{\mu\nu} = \partial_\mu \boldsymbol{\omega}_v - \partial_v \boldsymbol{\omega}_\mu + [\boldsymbol{\omega}_\mu, \boldsymbol{\omega}_\nu]. \tag{172}$$

4.5.1 Some Useful Formulas

Proposition: Let $A \in \sec \bigwedge^p TM \hookrightarrow \sec \mathcal{C}\ell(TM)$ and $\mathcal{R}$ the curvature of the connection as defined in Eq.(153). Then,

$$\mathbf{D}^2 A = \frac{1}{2}[\mathcal{R}, A]. \tag{173}$$

Proof:

$$\begin{aligned}\mathbf{D}^2 A = \mathbf{D}\mathbf{D}A = \mathbf{D}(dA + \frac{1}{2}[\boldsymbol{\omega}, A] \\ d^2 A + \frac{1}{2}[\boldsymbol{\omega}, dA] + \frac{1}{2}d[\boldsymbol{\omega}, A] + \frac{1}{4}[\boldsymbol{\omega}, [\boldsymbol{\omega}, A]]\end{aligned} \tag{174}$$

Now, as can be easily verified:

$$d[\boldsymbol{\omega}, A] = [d\boldsymbol{\omega}, A] - [\boldsymbol{\omega}, dA], \tag{175}$$

$$[\boldsymbol{\omega}, [\boldsymbol{\omega}, A]] = [[\boldsymbol{\omega}, \boldsymbol{\omega}], A], \tag{176}$$

$$\frac{1}{4}[\boldsymbol{\omega}, [\boldsymbol{\omega}, A]] = \frac{1}{2}[\boldsymbol{\omega} \otimes_\wedge \boldsymbol{\omega}, A]]$$

Using these equations in Eq.(174) we have,

$$\mathbf{D}^2 A = \frac{1}{2}[d\boldsymbol{\omega} + \boldsymbol{\omega} \otimes_\wedge \boldsymbol{\omega}, A] = \frac{1}{2}[\mathcal{R}, A]\blacksquare$$

In particular, when $a \in \sec \bigwedge^1 TM \hookrightarrow \sec \mathcal{C}\ell(TM)$ we have

$$\mathbf{D}^2 a = \mathcal{R} \llcorner a \tag{177}$$

Also, we can show using the previous result that if $\mathcal{A} \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$ it holds

$$\mathbf{D}^2 \mathcal{A} = \frac{1}{2}[\mathcal{R}, \mathcal{A}]. \tag{178}$$

It is a useful text of the consistence of our formalism to derive once again that $\mathbf{D}\mathcal{R}=0$, by calculating $\mathbf{D}^3A$ for $A\in\sec\bigwedge^r TM\hookrightarrow\sec\mathcal{C}\ell(TM)$. We have:

$$\mathbf{D}^3A=\mathbf{D}\left(\mathbf{D}^2A\right)=\mathbf{D}^2(\mathbf{D}A). \tag{179}$$

Now, using the above formulas and recalling Eq.(151), we can write:

$$\begin{aligned}\mathbf{D}^3A&=\mathbf{D}(\mathbf{D}^2A)=\frac{1}{2}\mathbf{D}[\mathcal{R},A]\\&=\frac{1}{2}\mathbf{D}(\mathcal{R}\otimes_\wedge A-A\otimes_\wedge\mathcal{R})\\&=\frac{1}{2}(\mathbf{D}\mathcal{R}\otimes_\wedge A+\mathcal{R}\otimes_\wedge\mathbf{D}A-\mathbf{D}A\otimes_\wedge\mathcal{R}+(-1)^{1+r}A\otimes_\wedge\mathbf{D}\mathcal{R})\end{aligned} \tag{180}$$

and

$$\begin{aligned}\mathbf{D}^3A=\mathbf{D}^2(\mathbf{D}A)&=\frac{1}{2}[\mathcal{R},\mathbf{D}A]\\&=\frac{1}{2}\left(\mathcal{R}\otimes_\wedge\mathbf{D}A-\mathbf{D}A\otimes_\wedge\mathcal{R}\right).\end{aligned} \tag{181}$$

Comparing Eqs.(180) and (181) we get that

$$\mathbf{D}\mathcal{R}\otimes_\wedge A+(-1)^{1+r}A\otimes_\wedge\mathbf{D}\mathcal{R}=[\mathbf{D}\mathcal{R},A]=0, \tag{182}$$

from where it follows that $\mathbf{D}\mathcal{R}=0$, as it may be.

We end this section by collecting some formulas that will be need in the next sections. First recall that[35]

$$\begin{aligned}[][D_{e_\rho},D_{e_\lambda}]e_\mu&=R_{\mu\ \rho\lambda}^{\ \alpha}e_\alpha=-R_{\alpha\mu\rho\lambda}e^\alpha=R_{\mu\alpha\rho\lambda}e^\alpha,\\R_{\mu\ \rho\lambda}^{\ \alpha}&=\mathcal{R}(e_\mu,\theta^\alpha,e_\rho,e_\lambda).\end{aligned} \tag{183}$$

Then a simple calculation shows that

$$[D_{e_\rho},D_{e_\lambda}]e_\mu=e_\mu\lrcorner\mathbf{R}_{\rho\lambda}=-\mathbf{R}_{\rho\lambda}\llcorner e_\mu, \tag{184}$$

$$R_{\mu\alpha\rho\lambda}e^\alpha=\frac{1}{2}(e_\mu\mathbf{R}_{\rho\lambda}-\mathbf{R}_{\rho\lambda}e_\mu). \tag{185}$$

Multiplying Eq.(185) on the left by $\mathbf{e_0}$ we get, recalling that a paravector field is defined as $\mathbf{q}^\alpha=e^\alpha\mathbf{e^0}$ and Eq.(77) we have

$$R_{\mu\alpha\rho\lambda}\mathbf{q}^\alpha=\frac{1}{2}(\mathbf{q}_\mu\mathbf{R}_{\rho\lambda}^\dagger+\mathbf{R}_{\rho\lambda}\mathbf{q}_\mu). \tag{186}$$

[35] In Sachs book he wrote: $[D_{e_\rho},D_{e_\lambda}]e_\mu=R_{\mu\ \rho\lambda}^{\ \alpha}e_\alpha=+R_{\alpha\mu\rho\lambda}e^\alpha$. This produces some changes in signals in relation to our formulas below. Our Eq.(183) agrees with the conventions in [25].

5 General Relativity as a $Sl(2,\mathbb{C})$ Gauge Theory

5.1 The Nonhomogeneous Field Equations

The analogy of the fields $\mathbf{R}_{\mu\nu} = \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e_a e_b} = \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e_a}\wedge\mathbf{e_b} \in \sec\bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(TM)$ with the gauge fields of particle fields is so appealing that it is irresistible to propose some kind of a $Sl(2,\mathbb{C})$ formulation for the gravitational field. And indeed this has already been done, and the interested reader may consult, e.g., [21, 62]. Here, we observe that despite the similarities, the gauge theories of particle physics are in general formulated in flat Minkowski spacetime and the theory here must be for a field on a general Lorentzian spacetime. This introduces additional complications, but it is not our purpose to discuss that issue with all attention it deserves here. Indeed, for our purposes in this paper we will need only to recall some facts.

To start, recall that in gauge theories besides the homogenous field equations given by Bianchi's identities, we also have the nonhomogeneous field equation. This equation, in analogy with the $U(1)$ of the nonhomogeneous equation for the electromagnetic field (see Eq.(258) in Appendix) is written

$$\mathbf{D}\star\mathcal{R} = d\star\mathcal{R} + \frac{1}{2}[\boldsymbol{\omega},\star\mathcal{R}] = -\star\mathbf{T}, \tag{187}$$

where the $\mathbf{T} \in \sec\bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM)\otimes\bigwedge^1 T^*M$ is a current, of course, associated with the energy momentum tensor in Einstein theory. In order to write this equation in components it is very useful to imagine that $\bigwedge T^*M \hookrightarrow \mathcal{C}\ell(T^*M)$, the Clifford bundle of differential forms, for in that case the powerful calculus described in the Appendix can be used. So, we write:

$$\begin{aligned}
&\boldsymbol{\omega} \in \sec\bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM)\otimes\bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM)\otimes\mathcal{C}\ell(T^*M),\\
&\mathcal{R}{=}\mathbf{D}\boldsymbol{\omega} \in \sec\bigwedge^2 TM \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM)\otimes\bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM)\otimes\mathcal{C}\ell(T^*M)\\
&\mathbf{T}{=}\ \mathbf{T}_\nu\theta^\nu \in \sec\bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM)\otimes\mathcal{C}\ell(T^*M).
\end{aligned} \tag{188}$$

Now, using Eq.(247) for the Hodge star operator given in the Appendix and the relation between the operators $d = \boldsymbol{\partial}\wedge$ and $\delta = -\boldsymbol{\partial}\lrcorner$ we can write

$$d\star\mathcal{R} = -\theta^5(-\boldsymbol{\partial}\lrcorner\mathcal{R}) = -\star(\boldsymbol{\partial}\lrcorner\mathcal{R}) = -\star((\partial_\mu\mathbf{R}^\mu_\nu)\theta^\nu). \tag{189}$$

Also,

$$\begin{aligned}
\frac{1}{2}[\boldsymbol{\omega}, \star\mathcal{R}] &= \frac{1}{2}[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \theta^\mu \wedge \star(\theta^\alpha \wedge \theta^\beta) \\
&= -\frac{1}{2}[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \theta^\mu \wedge \theta^5(\theta^\alpha \wedge \theta^\beta) \\
&= -\frac{1}{4}[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu\theta^5(\theta^\alpha \wedge \theta^\beta) + \theta^5(\theta^\alpha \wedge \theta^\beta)\theta^\mu\} \\
&= \frac{\theta^5}{4}[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu(\theta^\alpha \wedge \theta^\beta) - (\theta^\alpha \wedge \theta^\beta)\theta^\mu\} \\
&= \frac{\theta^5}{2}[\boldsymbol{\omega}_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu \lrcorner(\theta^\alpha \wedge \theta^\beta) \\
&= -\star([\boldsymbol{\omega}_\mu, \mathbf{R}^\mu_\beta]\theta^\beta. \qquad (190)
\end{aligned}$$

Using Eqs.(187-190) we get

$$\partial_\mu \mathbf{R}^\mu_\nu + [\boldsymbol{\omega}_\mu, \mathbf{R}^\mu_\nu] = \mathbf{T}_\nu. \qquad (191)$$

Eq.(191) must, of course, be compatible with Einstein's equations, which may be eventually used to determine determines $\mathbf{R}^\mu_\nu, \boldsymbol{\omega}_\mu$ and $\mathbf{T}_\nu$. In order to find $\mathbf{R}^\mu_\nu$ we recall that Einstein's equations can be written in components in an orthonormal basis as

$$R_{\mathbf{ab}} - \frac{1}{2}\eta_{\mathbf{ab}}R = T_{\mathbf{ab}}, \qquad (192)$$

where $R_{\mathbf{ab}} = R_{\mathbf{ba}}$ are the components of the Ricci tensor ($R_{\mathbf{ab}} = R_{\mathbf{a}\ \mathbf{bc}}^{\ \mathbf{c}}$), $T_{\mathbf{ab}}$ are the components of the energy-momentum tensor of matter fields and $R = \eta_{\mathbf{ab}}R^{\mathbf{ab}}$ is the curvature scalar. We next define the *Ricci 1-vectors* and the *energy-momentum 1-vectors* by

$$\mathbf{R}_\mathbf{a} = R_{\mathbf{ab}}e^\mathbf{b}, \quad \mathbf{T}_\mathbf{a} = T_{\mathbf{ab}}e^\mathbf{b}. \qquad (193)$$

We have that

$$\mathbf{R}_\mathbf{a} = -e^\mathbf{b} \lrcorner \mathbf{R}_{\mathbf{ab}}. \qquad (194)$$

Now, multiplying Eq.(192) on the right by $e^\mathbf{b}$ we get

$$\mathbf{R}_\mathbf{a} - \frac{1}{2}\mathbf{R}\mathbf{e}_\mathbf{a} = \mathbf{T}_\mathbf{a}. \qquad (195)$$

Multiplying Eq.(195) first on the right by $\mathbf{e}_\mathbf{b}$ and then on the left by $\mathbf{e}_\mathbf{b}$ and making the difference of the resulting equations we get

$$(-e^\mathbf{c} \lrcorner \mathbf{R}_{\mathbf{ac}})\,\mathbf{e}_\mathbf{b} - \mathbf{e}_\mathbf{b}\,(-e^\mathbf{c} \lrcorner \mathbf{R}_{\mathbf{ac}}) - \frac{1}{2}R(\mathbf{e}_\mathbf{a}\mathbf{e}_\mathbf{b} - \mathbf{e}_\mathbf{b}\mathbf{e}_\mathbf{a}) = (\mathbf{T}_\mathbf{a}\mathbf{e}_\mathbf{b} - \mathbf{e}_\mathbf{b}\mathbf{T}_\mathbf{a}). \qquad (196)$$

5.2 A Set of Maxwell Like Nonhomogeneous Equations

Eq.(196) can be written as

$$\mathcal{F}_{\mathbf{ab}} = (\mathbf{T_a e_b} - \mathbf{e_b T_a}), \tag{197}$$

with

$$\begin{aligned}\mathcal{F}_{\mathbf{ab}} &= (-e^{\mathbf{c}} \lrcorner \mathbf{R_{ac}})\,\mathbf{e_b} - \mathbf{e_b}\,(-e^{\mathbf{c}} \lrcorner \mathbf{R_{ac}}) - \frac{1}{2}R(\mathbf{e_a e_b} - \mathbf{e_b e_a}) \\ &= \frac{1}{2}(\mathbf{R_{ac}} e^{\mathbf{c}} \mathbf{e_b} + \mathbf{e_b} e^{\mathbf{c}} \mathbf{R_{ac}} - e^{\mathbf{c}} \mathbf{R_{ac} e_b} - \mathbf{e_b R_{ac}} e^{\mathbf{c}}) - \frac{1}{2}R(\mathbf{e_a e_b} - \mathbf{e_b e_a}).\end{aligned} \tag{198}$$

It is quite obvious that in a coordinate chart $\langle x^\mu \rangle$ covering an open set $U \subset M$ we can write

$$D_{e_\rho} \mathcal{F}^\rho_\gamma = J_\gamma, \tag{199}$$

with

$$J_\gamma = D_{e_\rho}(\mathbf{T}^\rho e_\gamma - e^\rho \mathbf{T}_\gamma). \tag{200}$$

Remark 2 *Eq.(199) is a set Maxwell like nonhomogeneous equations. It looks like Maxwell equations when that equations are written in components, but Eq.(199) is nothing more than a trivial consequence of the equation of the nonhomogeneous field equations in the* $Sl(2,\mathbb{C})$ *like gauge theory version of Einstein's theory, discussed in the previous section. In particular, keep in mind that any one of the* $\mathcal{F}^\rho_\gamma \in \sec \bigwedge^2 TM$. *Or, in words, each* $\mathcal{F}^\rho_\gamma$ *it is a bivector field, not a set of scalars (which are components of a 2-form) as in Maxwell theory.*

We immediately see that in *vacuum* $\mathcal{F}_{\mathbf{ab}} = 0$, from where we get the identity (valid *only* in vacuum)

$$(e^{\mathbf{c}} \lrcorner \mathbf{R_{ac}})\,\mathbf{e_b} = (e^{\mathbf{c}} \lrcorner \mathbf{R_{bc}})\,\mathbf{e_a}. \tag{201}$$

It is very important to realize that this equation does *not* imply that the curvature bivector $\mathbf{R_{ab}}$ is null in vacuum. Indeed, recalling its definition (Eq.(164)) we have

$$\mathbf{R_{ab}} = R_{\mathbf{abcd}} e^{\mathbf{c}} e^{\mathbf{d}}, \tag{202}$$

and we see that it is zero only if the Riemann tensor is null which is not the case in any non trivial general relativistic model.

The important fact that we want to emphasize here is that Eq.(198) although eventually interesting has nothing new in it, i.e., all information given by that equation is already contained in the original Einstein's equation, for indeed it has been obtained from it by simple algebraic manipulations. The terms

$$\begin{aligned}\mathcal{F}_{\mathbf{ab}} &= \frac{1}{2}(\mathbf{R_{ac}} e^{\mathbf{c}} \mathbf{e_b} + \mathbf{e_b} e^{\mathbf{c}} \mathbf{R_{ac}} - e^{\mathbf{c}} \mathbf{R_{ac} e_b} - \mathbf{e_b R_{ac}} e^{\mathbf{c}}) - \frac{1}{2}R(\mathbf{e_a e_b} - \mathbf{e_b e_a}) \\ \mathfrak{R}_{\mathbf{ab}} &= (\mathbf{T_a e_b} - \mathbf{e_b T_a}) - \frac{1}{2}R(\mathbf{e_a e_b} - \mathbf{e_b e_a}), \\ \mathbf{F_{ab}} &= \frac{1}{2}R(\mathbf{e_a e_b} - \mathbf{e_b e_a}),\end{aligned} \tag{203}$$

are pure gravitational objects, despite being antisymmetric in indices $\mathbf{a}, \mathbf{b}$. Note that $\mathbf{F_{ab}}$ differs from a factor, namely R from the $\mathbf{F'_{ab}}$ give by Eq.(70).

5.3 $Sl\,(2,\mathbb{C})$ Gauge Theory and Sachs Antisymmetric Equation

We discuss in this subsection yet another algebraic exercise. First recall that in section 2 we define the paravector fields,

$$\mathbf{q_a} = \mathbf{e_a e_0} = \boldsymbol{\sigma}_\mathbf{a}, \quad \check{\mathbf{q}}_\mathbf{a} = (-\boldsymbol{\sigma}_\mathbf{0}, \boldsymbol{\sigma}_\mathbf{i}), \quad \boldsymbol{\sigma}_\mathbf{0} = 1.$$

To derive Sachs[36] Eq.(6.50a) all we need to do is to multiply Eq.(195) on the right by $\mathbf{e^0}$ and perform some algebraic manipulations. We then get (with *our* normalization) for the equivalent of Einstein's equations using paravector fields and a coordinate chart $\langle x^\mu \rangle$ covering an open set $U \subset M$, the following equation

$$\mathbf{R}_{\rho\lambda}\mathbf{q}^\lambda + \mathbf{q}^\lambda \mathbf{R}^\dagger_{\rho\lambda} + R\mathbf{q}_\rho = 2\mathbf{T}_\rho. \tag{204}$$

For the Hermitian conjugate we have

$$-\mathbf{R}^\dagger_{\rho\lambda}\check{\mathbf{q}}^\lambda - \check{\mathbf{q}}^\lambda \mathbf{R}_{\rho\lambda} + R\check{\mathbf{q}}_\rho = 2\check{\mathbf{T}}_\rho. \tag{205}$$

where as above $\mathbf{R}_{\rho\lambda}$ are the the curvature bivectors given by Eq.(172) and

$$\mathbf{T}_\rho = T^\mu_\rho \mathbf{q}_\mu. \tag{206}$$

After that, we multiply Eq.(204) on the right by $\check{\mathbf{q}}_\gamma$ and Eq.(205) on the left by $\mathbf{q}_\gamma$ ending with two new equations. If we sum them, we get a 'symmetric' equation[37] completely equivalent to Einstein's equation (from where we started). If we make the difference of the equations we get an antisymmetric equation. The antisymmetric equation can be written, introducing

$$\begin{aligned} F_{\rho\gamma} &= \frac{1}{2}(\mathbf{R}_{\rho\lambda}\mathbf{q}^\lambda\check{\mathbf{q}}_\gamma + \mathbf{q}_\gamma\check{\mathbf{q}}^\lambda\mathbf{R}_{\rho\lambda} + \mathbf{q}^\lambda\mathbf{R}^\dagger_{\rho\lambda}\check{\mathbf{q}}_\gamma + \mathbf{q}_\gamma\mathbf{R}^\dagger_{\rho\lambda}\check{\mathbf{q}}^\lambda) \\ &+ \frac{1}{2}R(\mathbf{q}_\rho\check{\mathbf{q}}_\gamma - \mathbf{q}_\gamma\check{\mathbf{q}}_\rho) \end{aligned} \tag{207}$$

and

$$J_\gamma = D_{e_\rho}(\mathbf{T}^\rho\check{\mathbf{q}}_\gamma - \mathbf{q}_\gamma\check{\mathbf{T}}^\rho), \tag{208}$$

as

$$D_{e_\rho}F^\rho_\gamma = J_\gamma. \tag{209}$$

This equation, of course, is completely equivalent to our Eq.(199). Its matrix translation in $\mathbb{C}\ell^{(0)}(M) \simeq S(M) \otimes_\mathbb{C} \bar{S}(M)$ gives Sachs equation (6.52-) in [85] if

[36] Numeration is from Sachs' book [85].
[37] Eq.(6.52) in Sachs' book [85].

we take into account his different 'normalization' of Ω and the *ad hoc* factor with dimension of electric charge that he introduced. It has no new information.[38]

Using Eqs.(88), (108) and (169) we may verify that

$$D^{\mathbf{S}}_{e_\rho}F_{\mu\nu} + D^{\mathbf{S}}_{e_\mu}F_{\nu\rho} + D^{\mathbf{S}}_{e_\nu}F_{\rho\mu} = 0, \tag{210}$$

where $D^{\mathbf{S}}_{e_\rho}$ is Sachs 'covariant' derivative that we mention in Eq.(108). In [86] Sachs concludes that the last equation implies that there are no magnetic monopoles in nature. Of course, this is equation is only a reflection of Bianchi's identity valid for the curvature bivectors. It has *nothing* to do with the magnetic monopoles.

We thus conclude this section stating that Sachs claims in [85, 86] of having produced an unified field theory of electricity and electromagnetism are wrong.

6 Energy-Momentum "Conservation" in General Relativity

6.1 Einstein's Equations in terms of Superpotentials $\star S^{\mathbf{a}}$

From Eq.(187) it follows that

$$d(\star\mathbf{T}-\frac{1}{2}[\boldsymbol{\omega},\star\mathcal{R}]) = 0, \tag{211}$$

and we could think that we identified a *conservation law* for the energy momentum of matter plus the gravitational field, with $\frac{1}{2}[\boldsymbol{\omega},\star\mathcal{R}]$ describing the energy momentum of the gravitational field. However, this is not the case, because this term (due to the presence of $\boldsymbol{\omega}$) is gauge dependent.

Now, suppose that somehow Eq.(196) can be solved for $\mathbf{R_{ab}}$. Then, the corresponding equation can be used to determine the current term in Eq.(191). We are nothing going to attempt this exercise here, because there is a more easy way to find *appropriate* currents for Einstein's theory that we discuss below after recalling [77, 93] yet another formulation of Einstein's equation where the gravitational field is described by a set of 2-forms $\star S^{\mathbf{a}}$, $\mathbf{a} = 0, 1, 2, 3$ called superpotentials. The calculations that follows are done in the Clifford algebra of multiforms fields $\mathcal{C}\ell\,(T^*M)$.

We start again with Einstein's equations given by Eq.(192), but this time we multiply on the left by $\theta^{\mathbf{b}}$ getting an equation relating the *Ricci 1-forms* $\mathcal{R}^{\mathbf{a}} = R^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}}$ with the *energy-momentum 1-forms* $\mathcal{T}^{\mathbf{a}} = T^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}}$, i.e.,

$$\mathcal{G}^{\mathbf{a}} = \mathcal{R}^{\mathbf{a}} - \frac{1}{2}Re^{\mathbf{a}} = \mathcal{T}^{\mathbf{a}}. \tag{212}$$

[38] It is amusing to read Carmeli's review([22]) of Sachs book, for he did not realize that Sachs theory was simply a description in Pauli bundle of a $Sl(2,\mathbb{C})$ gauge formulation of Einstein's theory as described in his book [21] and as such, it could not be an unified field theory of gravitation and electromagnetism.

We take the dual of this equation,

$$\star\mathcal{G}^{\mathbf{a}} = \star\mathcal{T}^{\mathbf{a}}. \tag{213}$$

Now, we observe that [77, 93] we can write

$$\star\mathcal{G}^{\mathbf{a}} = -d\star S^{\mathbf{a}} - \star t^{\mathbf{a}}, \tag{214}$$

where

$$\begin{aligned} S^{\mathbf{c}} &= -\frac{1}{2}\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}}), \\ \star t^{\mathbf{c}} &= \frac{1}{2}\boldsymbol{\omega}_{\mathbf{ab}} \wedge [\boldsymbol{\omega}_{\mathbf{d}}^{\mathbf{c}} \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) + \boldsymbol{\omega}_{\mathbf{d}}^{\mathbf{b}} \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{d}} \wedge \theta^{\mathbf{c}})]. \end{aligned} \tag{215}$$

The proof of Eq.(215) follows at once from the fact that

$$\star\mathcal{G}^{\mathbf{d}} = \frac{1}{2}\mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}). \tag{216}$$

Indeed, recalling the identities in Eq.(248) we can write

$$\begin{aligned} \frac{1}{2}\mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) &= -\frac{1}{2} \star [\mathcal{R}_{\mathbf{ab}} \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] \\ &= -\frac{1}{2} R_{\mathbf{abcd}} \star [(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] \\ &= - \star (\mathcal{R}^{\mathbf{d}} - \frac{1}{2} R\theta^{\mathbf{d}}). \end{aligned} \tag{217}$$

On the other hand we have,

$$\begin{aligned} & 2\star\mathcal{G}^{\mathbf{d}} = d\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) + \boldsymbol{\omega}_{\mathbf{ac}} \wedge \boldsymbol{\omega}_{\mathbf{b}}^{\mathbf{c}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\ & = d[\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] - \boldsymbol{\omega}_{\mathbf{ab}} \wedge d\star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\ & + \boldsymbol{\omega}_{\mathbf{ac}} \wedge \boldsymbol{\omega}_{\mathbf{b}}^{\mathbf{c}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\ & = d[\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] - \boldsymbol{\omega}_{\mathbf{ab}} \wedge \boldsymbol{\omega}_{\mathbf{p}}^{\mathbf{a}} \star (\theta^{\mathbf{p}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\ & - \boldsymbol{\omega}_{\mathbf{ab}} \wedge \boldsymbol{\omega}_{\mathbf{p}}^{\mathbf{b}} \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{p}} \wedge \theta^{\mathbf{d}}) - \boldsymbol{\omega}_{\mathbf{ab}} \wedge \boldsymbol{\omega}_{\mathbf{p}}^{\mathbf{d}} \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{p}})] \\ & + \boldsymbol{\omega}_{\mathbf{ac}} \wedge \boldsymbol{\omega}_{\mathbf{b}}^{\mathbf{c}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\ & = d[\boldsymbol{\omega}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] - \boldsymbol{\omega}_{\mathbf{ab}} \wedge [\boldsymbol{\omega}_{\mathbf{p}}^{\mathbf{d}} \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{p}}) + \boldsymbol{\omega}_{\mathbf{p}}^{\mathbf{b}} \star (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{p}} \wedge \theta^{\mathbf{d}})] \\ & = -2(d\star S^{\mathbf{d}} + \star t^{\mathbf{d}}). \end{aligned} \tag{218}$$

Now, we can then write Einstein's equation in a very interesting, but *dangerous* form, i.e.:

$$-d\star S^{\mathbf{a}} = \star T^{\mathbf{a}} + \star t^{\mathbf{a}}. \tag{219}$$

In writing Einstein's equations in that way, we have associated to the gravitational field a set of 2-form fields $\star S^{\mathbf{a}}$ called *superpotentials* that have as sources the currents $(\star T^{\mathbf{a}} + \star t^{\mathbf{a}})$. However, superpotentials are not uniquely defined since, e.g., superpotentials $(\star S^{\mathbf{a}} + \star\alpha^{\mathbf{a}})$, with $\star\alpha^{\mathbf{a}}$ closed, i.e., $d\star\alpha^{\mathbf{a}} = 0$ give the same second member for Eq.(219).

6.2 Is There Any Energy-Momentum Conservation Law in GR?

We say that Eq.(219) is a dangerous one. The reason is that (as in the case of Eq.(211)) we can be led to think that we have discovered a conservation law for the energy momentum of matter plus gravitational field, since

$$d(\star\mathcal{T}^{\mathbf{a}} + \star t^{\mathbf{a}}) = 0. \tag{220}$$

This thought however is only an example of wishful thinking, because $\star t^{\mathbf{a}}$ depends on the connection (see Eq.(215)) and thus are gauge dependent. They do not have the same tensor transformation law as the $\star\mathcal{T}^{\mathbf{a}}$. So, Stokes theorem cannot be used to derive from Eq.(220) conserved quantities that are independent of the gauge and the local coordinate chart used to perform calculations. In fact, the currents $\star t^{\mathbf{a}}$ are nothing more than the old pseudo energy momentum tensor of Einstein in a new dress. Non recognition of this fact can lead to many misunderstandings. We present some of them in what follows.

First, it is easy to see that from Eq.(213) it follows that [63]

$$\mathbf{D}\star\mathcal{G} = \mathbf{D}\star\mathcal{T} = 0, \tag{221}$$

where $\star\mathcal{G} = \mathbf{e_a}\otimes\mathcal{G}^{\mathbf{a}}$ and $\star\mathcal{T} = \mathbf{e_a}\otimes\star\mathcal{T}^{\mathbf{a}}$. Now, in [63] it is written a 'Stokes theorem'

$$\boxed{\int_{\text{4-cube}} \mathbf{D}\star\mathcal{T} = \int_{\substack{\text{3 boundary}\\ \text{of this 4-cube}}} \star\mathcal{T}}$$

This equation which appears also in many other texts and scientific papers, as e.g., in [28, 97] is completely misleading and indeed it is a non sense, since we cannot sum tensors at different spacetime points.

In Einstein theory possible superpotentias are, of course, the $\star S^{\mathbf{a}}$ that we found above (Eq.(215)), with

$$\star S_{\mathbf{c}} = [-\frac{1}{2}\omega_{\mathbf{ab}}\lrcorner(\theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}\wedge\theta_{\mathbf{c}})]\theta^{\mathbf{5}}. \tag{222}$$

Then, if we integrate Eq.(219) over a 'certain finite 3-dimensional volume', say a ball B, and use Stokes theorem we have

$$P^{\mathbf{a}} = \int_B \star(T^{\mathbf{a}} + t^{\mathbf{a}}) = -\int_{\partial B} \star S^{\mathbf{a}}. \tag{223}$$

In particular the energy or (inertial mass) of the gravitational field plus matter generating the field is defined by

$$P^{\mathbf{0}} = E = \lim_{R\to\infty}\int_{\partial B} \star S^{\mathbf{0}} \tag{224}$$

Now, a frequent misunderstanding is the following. Suppose that in a *given* gravitational theory there exists an energy-momentum conservation law for matter plus the gravitational field expressed in the form of Eq.(220), where $T^{\mathbf{a}}$ are the energy-momentum 1-forms of matter and $t^{\mathbf{a}}$ are *true*[39] energy-momentum 1-forms of the gravitational field. This means that the 3-forms $(\star T^{\mathbf{a}} + t^{\mathbf{a}})$ are closed, i.e., they satisfy Eq.(220). Is this enough to warrant that the energy of a closed universe is zero? Well, that would be the case if starting from Eq.(220) we could jump to Eq.(219) and then to Eq.(224) (as done in [94]). But that inference is not correct, for indeed, it is not the case that closed three forms are always exact. Take a closed universe with topology, say $\mathbb{R}\times S^3$. In this case $B = S^3$ and we have $\partial B = \partial S^3 = \varnothing$. Now, as it is well known (see, e.g., [68]), the third de Rham cohomology group of $\mathbb{R}\times S^3$ is $H^3\left(\mathbb{R}\times S^3\right) = H^3\left(S^3\right) = \mathbb{R}$. Since this group is non trivial it follows that from Eq.(220) it did not follow the validity of Eq.(219). So, in that case an equation like Eq.(223) cannot even be written. However, in Einstein's theory the energy of a closed universe[40] if it is given by Eq.(224) is indeed zero, since in that theory the 3-forms $(\star T^{\mathbf{a}} + t^{\mathbf{a}})$ are indeed exact (see Eq.(219)).

But, is the above formalism consistent? Given a coordinate chart $\langle x^\mu \rangle$ of the maximal atlas of M , with some algebra we can show that for a gravitational model represented by a diagonal asymptotic flat metric[41], the inertial mass $E = m_i$ is given by

$$m_i = \lim_{R\to\infty} \frac{-1}{16\pi} \int\limits_{\partial B} \frac{\partial}{\partial x^\beta}(g_{11}g_{22}g_{33}g^{\alpha\beta})d\sigma_\alpha, \tag{225}$$

where $\partial B = S^2(R)$ is a 2-sphere of radius R, $(-n_\alpha)$ is the outward unit normal and $d\sigma_\alpha = -R^2 n_\alpha dA$. If we apply Eq.(225) to calculate, e.g., the energy of the Schwarzschild space time[42] generate by a gravitational mass m, we expect to have one unique and unambiguous result, namely $m_i = m$.

However, as showed in details, e.g., in [20] the calculation of E depends on the spatial coordinate system naturally adapted to the reference frame $Z = \frac{1}{\sqrt{\left(1-\frac{2m}{r}\right)}}\frac{\partial}{\partial t}$, even if these coordinates produce asymptotically flat metrics. Then, even if in one given chart we may obtain $m_i = m$ there are others where $m_i \neq m$! This according to our view shows that all discourse concerning pseudo-energy momentum tensors is nonsense.

The fact is: there are no conservation laws of energy-momentum in General Relativity in general. And, at this point it is better to quote page 98 of Sachs&Wu [88]:

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[39] This means that the $t^{\mathbf{a}}$ are not pseudo 1-forms, as in Einstein's theory.

[40] Note that if we suppose that the universe contains spinor fields, then it must be a spin manifold, i.e., it is parallezible according to Geroch's theorem [49].

[41] A metric is said to be asymptotically flat in given coordinates, if $g_{\mu\nu} = n_{\mu\nu}(1 + \mathrm{O}\left(r^{-k}\right))$, with $k = 2$ or $k = 1$ depending on the author. See, eg., [89, 90, 98].

[42] For a Scharzschild spacetime we have $g = \left(1 - \frac{2m}{r}\right) dt \otimes dt - \left(1 - \frac{2m}{r}\right)^{-1} dr \otimes dr - r^2(d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi)$.

" As mentioned in section 3.8, conservation laws have a great predictive power. It is a shame to lose the special relativistic total energy conservation law (Section 3.10.2) in general relativity. Many of the attempts to resurrect it are quite interesting; many are simply garbage."

In General Relativity, every gravitational field is modelled (module diffeomorphisms) by a Lorentzian spacetime. In the particular case, when this spacetime admits a *timelike* Killing vector, we can formulated a law of energy conservation. If the spacetime admits three linearly independent *spacelike* Killing vectors, we have a law of conservation of momentum. The crucial fact to have in mind here is that a general Lorentzian spacetime, does not admits such Killing vectors in general. As one example, we quote that the popular Friedmann-Robertson-Walker expanding universes models do not admit timelike Killing vectors, in general.

The only possibility of resurrecting a *trustworthy* conservation law of energy-momentum valid in all circumstances in a theory of the gravitational field that *resembles* general relativity (in the sense of keeping Einstein's equation) is to reinterpret that theory as a field theory in flat Minkowski spacetime. Such a theory has been extensively studied by Logunov and collaborators [58, 59]. Another presentation of the theory is one where the gravitational field is represented by a distortion field in Minkowski spacetime. A first attempt to such a theory using Clifford bundles has been given in [77]. Another presentation has been developed in [55], but that work contains some very wrong statements that make (in our opinion) the theory invalid. This has been discussed with details in [42].

We quote here the problem associated with the energy momentum tensor in general relativity for two reasons. The first is that recently people think to have solved this problem in the so-called *teleparallel* version of general relativity [30]. However, such a hope unfortunately has not been realized, as we show in a sequel paper [83], which discuss conservation laws in a general Riemann-Cartan spacetime, using Clifford bundle methods.

The second reason was to leave the reader aware of the *shameful* fact of non energy-momentum conservation in General Relativity when we comment in the next section some papers by Evans&AIAS where they try to explain the functioning of MEG, a 'motionless electric generator' that according to those authors pumps energy from the vacuum.

6.3 "Explanation" of MEG according to AIAS

Our comments on AIAS papers dealing with MEG are the following:

(i) AIAS claim[43] that the $\mathbf{B}_3$ electromagnetic field of their new "$O(3)$ electrodynamics" is to be identified with $\mathbf{F_{12}}$ (giving by Eq.(203)).

Well, this is a nonsequitur because we already showed above that $\mathbf{F_{12}}$ has nothing to do with electromagnetic fields, it is only a combination of the curvature bivectors, which is a pure gravitational object.

[43] See the list of their papers related to the subject in the bibliography.

(**ii**) With that identification AIAS claims that it is the energy of the "electromagnetic" field $\mathbf{F_{12}}$ that makes MEG to work. In that way MEG must be understood as motionless electromagnetic generator that (according to AIAS) pumps energy from the 'vacuum' defined by the $\mathbf{B}_3$ field.

Well, Eq.(203) shows that $\mathbf{F_{12}} = 0$ on the vacuum. It follows that if MEG really works, then it is pumping energy from another source, or it is violating the law of energy-momentum conservation. So, it is unbelievable how Physics journals have published AIAS papers on MEG using arguments as the one just discussed, that are completely wrong.

(**iii)** We would like to leave it clear here that it is our my opinion that MEG does not work, even if the USPTO granted a patent for that invention, what we considered a very sad and dangerous fact. We already elaborated on this point in the introduction and more discussion on the subject of MEG can be found[44] at http://groups.yahoo.com/group/free_energy/.

(**iv**) And what to say about the new electrodynamics of the AIAS group and its $\mathbf{B}_3$ field?

Well, in [24, 81] we analyzed in deep all known presentations of the "new $O(3)$ electrodynamics" of the AIAS group. It has been proved beyond any doubt that almost all AIAS papers are simply a pot pourri of non sequitur Mathematics and Physics. That is not only our opinion, and the reader is invited (if he become interested on that issue) to read a review of [24] in [16].

Recently ([37]-[40]) Evans is claiming to have produced an unified theory and succeeded in publishing his odd ideas in ISI indexed Physical journals. In the next section we discuss his 'unified' theory, showing that it is again, as it is the case of the old Evans&AIAS papers, simply a compendium of nonsense Mathematics and Physics.

(**v**) And if we are wrong concerning our opinion that MEG does not work?

Well, in that (improbable) case that MEG works, someone can claim that its functioning vindicates the General Theory of Relativity, since as proved in the last section in that theory there is no trustworthy law of energy-momentum conservation. That would be really amazing...

7 Field Equations for the Tetrad Fields $\theta^{\mathbf{a}}$

In the previous section we gave a Clifford bundle formulation of the field equations of general relativity in a form that resembles a $Sl(2,\mathbb{C})$ gauge theory and also a formulation in terms of a set of 2-form fields $\star S^{\mathbf{a}}$. We are not going to discuss in this paper if the $Sl\,(2,\mathbb{C})$ nonhomogeneous field equation can be of some utility. This will be done in another paper. Here we want to recall how to write field equations directly for the tetrad fields $\theta^{\mathbf{a}}$ in such a way that the obtained equations are equivalent to Einstein's field equations. Of course, we

[44]The reader must be aware that there are many nonsequitur posts in this yahoo group, but there are also many serious papers written by serious and competent people.

could write analogous (and equivalent) equations for the dual tetrads $\mathbf{e_a}$.[45]

As shown in details in papers [77, 91] the correct wave like equations satisfied by the $\theta^{\mathbf{a}}$ are[46]:

$$-(\boldsymbol{\partial}\cdot\boldsymbol{\partial})\theta^{\mathbf{a}}+\boldsymbol{\partial}\wedge(\boldsymbol{\partial}\cdot\theta^{\mathbf{a}})+\boldsymbol{\partial}\lrcorner(\boldsymbol{\partial}\wedge\theta^{\mathbf{a}})=T^{\mathbf{a}}-\frac{1}{2}T\theta^{\mathbf{a}}. \tag{226}$$

When $\theta^{\mathbf{a}}$ is an exact differential, and in this case we write $\theta^{\mathbf{a}}\mapsto\theta^{\mu}=dx^{\mu}$ and if the coordinate functions are harmonic, i.e., $\delta\theta^{\mu}=-\boldsymbol{\partial}\theta^{\mu}=0$, Eq.(226) becomes

$$\square\theta^{\mu}+\frac{1}{2}R\theta^{\mu}=-T^{\mu}, \tag{227}$$

where we have written as defined in the Appendix

$$(\boldsymbol{\partial}\cdot\boldsymbol{\partial})=g^{\mu\nu}D_{\mathbf{e}_{\mu}}D_{\mathbf{e}_{\nu}}=\square \tag{228}$$

i.e., $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ is the (covariant) D' Alembertian operator.

In Eq.(226) $\boldsymbol{\partial}=\theta^{\mathbf{a}}D_{\mathbf{e}_{a}}=\boldsymbol{\partial}\wedge+\boldsymbol{\partial}\lrcorner=d-\delta$ is the Dirac (like) operator acting on sections of the Clifford bundle $\mathcal{C}\ell(T^{*}M)$ defined in the Appendix.

With these formulas we can write

$$\begin{aligned}\boldsymbol{\partial}^{2}&=\boldsymbol{\partial}\cdot\boldsymbol{\partial}+\boldsymbol{\partial}\wedge\boldsymbol{\partial},\\ \boldsymbol{\partial}\wedge\boldsymbol{\partial}&=-\boldsymbol{\partial}\cdot\boldsymbol{\partial}+\boldsymbol{\partial}\wedge\boldsymbol{\partial}\lrcorner+\boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\wedge,\end{aligned} \tag{229}$$

with

$$\begin{aligned}\boldsymbol{\partial}\cdot\boldsymbol{\partial}&=\eta^{\mathbf{ab}}(D_{\mathbf{e_a}}D_{\mathbf{e_b}}-\omega_{\mathbf{ab}}^{\mathbf{c}}D_{\mathbf{e_c}}),\\ \boldsymbol{\partial}\wedge\boldsymbol{\partial}&=\theta^{\mathbf{a}}\wedge\theta^{\mathbf{b}}(D_{\mathbf{e_a}}D_{\mathbf{e_b}}-\omega_{\mathbf{ab}}^{\mathbf{c}}D_{\mathbf{e_c}}).\end{aligned} \tag{230}$$

Note that $D_{e_{a}}\theta^{\mathbf{b}}=-\omega_{\mathbf{ac}}^{\mathbf{b}}\theta^{\mathbf{c}}$ and a somewhat long, but simple calculation [47] shows that

$$(\boldsymbol{\partial}\wedge\boldsymbol{\partial})\theta^{\mathbf{a}}=\mathcal{R}^{\mathbf{a}}, \tag{231}$$

where, as already defined, $\mathcal{R}^{\mathbf{a}}=R_{\mathbf{b}}^{\mathbf{a}}\theta^{\mathbf{b}}$ are the Ricci 1-forms. Also $\mathcal{T}^{\mathbf{a}}=T_{\mathbf{b}}^{\mathbf{a}}\theta^{\mathbf{b}}$ are the energy momentum 1-forms and $R=R_{\mathbf{a}}^{\mathbf{a}}=-T=T_{\mathbf{a}}^{\mathbf{a}}$. We also observe (that for the best of our knowledge) $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ that has been named the Ricci operator in [91] has no analogue in classical differential geometry.

Note that Eq. (226) can be written after some algebra as

$$\mathcal{R}^{\mu}-\frac{1}{2}T\theta^{\mu}=\mathcal{T}^{\mu}, \tag{232}$$

with $\mathcal{R}^{\mu}=R_{\nu}^{\mu}dx^{\nu}$ and $\mathcal{T}^{\mu}=T_{\nu}^{\mu}dx^{\nu}$, $\theta^{\mu}=dx^{\mu}$ in a coordinate chart of the maximal atlas of M covering an open set $U\subset M$.

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[45] Incidentally, our exercise will show that all recent Evans papers ([26, 37, 38, 39, 40]) describing his new 'unified' theory are sheer nonsense.

[46] Of course, there are analogous equations for the $\mathbf{e_a}$ [51], where in that case, the Dirac operator must be defined (in an obvious way) as acting on sections of the Clifford bundle of multivectors, that has been introduced in section 3.

[47] The calculation is done in detail in [77, 91].

8 A Short Comment on Recent Evans&AIAS Papers

We are now prepared to make some crucial comments concerning some recent papers by Evans&AIAS ([26],[37]-[41]).

(**i**) Evans wrote that the $\mathbf{e_a}$, $\mathbf{a} = 0, 1, 2, 3$ satisfy the equations

$$\boxed{(\Box + T)\mathbf{e_a} = 0.}$$

He thought to have produced a valid derivation for that equations. I will not comment on his derivation here. Enough is to say that if that equation was true it would imply that $(\Box + T)\theta^{\mathbf{a}} = 0$, which is not the case, since the true equation satisfied by any one of the $\theta^{\mathbf{a}}$ is Eq.(226).

(**ii**) We note that Eq.(232) looks like an equation written several times by Evans in [38, 39, 40, 41], but Evans equation is a non sequitur because in place of the coframe 1-forms he uses scalar functions !

(**iii**) We quote that Evans explicitly wrote several times in [38, 39, 40] that the "electromagnetic potential"[48] $\mathbf{A}$ of his theory (a 1-form with values in a vector space) satisfies the following wave equation,

$$\boxed{(\Box + T)\mathbf{A} = 0.}$$

Now, this equation is incorrect even for the usual $U(1)$ gauge potential of classical electrodynamics $A \in \sec \bigwedge^1 T^*M \subset \sec \mathcal{C}\ell(T^*M)$. Indeed, in vacuum Maxwell equation reads,

$$\boldsymbol{\partial} F = 0, \tag{233}$$

where $F = \boldsymbol{\partial} A = \boldsymbol{\partial} \wedge A = dA$, if we work in the Lorenz gauge $\boldsymbol{\partial} \cdot A = \boldsymbol{\partial} \lrcorner A = -\delta A = 0$. Now, since we can also write

$$\boldsymbol{\partial}^2 = -(d\delta + \delta d) \tag{234}$$

and we have that

$$\boldsymbol{\partial}^2 A = 0. \tag{235}$$

Now, a simple calculation shows that in the coordinate basis introduced above we have,

$$(\boldsymbol{\partial}^2 A)_\alpha = g^{\mu\nu} D_\mu D_\nu A_\alpha + R^\nu_\alpha A_\nu \tag{236}$$

and we see that Eq.(235) reads in components

$$\Box A_\mu - R^\nu_\mu A_\nu = 0. \tag{237}$$

[48] What Evans did was to identify his "electromagnetic potential" with the bivector valued connection 1-form $\boldsymbol{\omega}$ that we introduced in section above. As we explained with details this is a nonsequitur because that quantity is related to gravitation, not electromagnetism.

Eq.(237) can be found, e.g., in Eddington's book [31] on page 175.

Finally we make a single comment on reference [26]. There we can read at the beginning of section 1.1:

"The antisymmetrized form of special relativity [1] has spacetime metric given by the enlarged structure

$$\eta^{\mu\nu} = \frac{1}{2}\left(\sigma^{\mu}\sigma^{\nu*} + \sigma^{\nu}\sigma^{\mu*}\right), \tag{1.1.}$$

where σ^{μ} are the Pauli matrices satisfying a clifford (sic) algebra

$$\{\sigma^{\mu}, \sigma^{\nu}\} = 2\delta^{\mu\nu},$$

which are represented by

$$\sigma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^{2} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

The $*$ operatorion denotes quaternion conjugation, which translates to a spatial parity transformation."

Well, the $*$ is not really defined anywhere in [26]. If it refers to a spatial parity operation, we infer that $\sigma^{0*} = \sigma^{0}$and $\sigma^{i*} = -\sigma^{i}$. Also, $\eta^{\mu\nu}$ is not defined, but Eq.(3.5) of [26] make us to infers that $\eta^{\mu\nu}$ =diag$(1, -1, -1, 1)$. In that case Eq.(1.1) above is true but the equation $\{\sigma^{\mu}, \sigma^{\nu}\} = 2\delta^{\mu\nu}$ is false. Enough is to see that $\{\sigma^{0}, \sigma^{i}\} = 2\sigma^{i} \neq 2\delta^{0i}$. Any school boy would detect immediately this error, and any competent mathematical physicist would immediately recognize the statement that that the equation $\{\sigma^{\mu}, \sigma^{\nu}\} = 2\delta^{\mu\nu}$ defines a Clifford algebra as *false*.What were the referees doing when they read that paper and the others by Evans&AIAS?

9 Conclusions

In this paper we introduced the concept of Clifford valued differential forms, which are sections of $\mathcal{C}\ell(TM)\otimes\bigwedge T^{*}M$. We showed how this theory can be used to produce a very elegant description the theory of linear connections, where a given linear connection is represented by a bivector valued 1-form. Crucial to the program was the introduction the notion of the exterior covariant derivative of sections of $\mathcal{C}\ell(TM) \otimes \bigwedge T^{*}M$. Our *natural* definitions parallel in a noticeable way the formalism of the theory of connections in a principal bundle and the covariant derivative operators acting on associate bundles to that principal bundle. We identified Cartan curvature 2-forms and *curvature bivectors*. The curvature 2-forms satisfy Cartan's second structure equation and the curvature bivectors satisfy equations in analogy with equations of gauge theories. This immediately suggest to write Einstein's theory in that formalism, something that has already been done and extensively studied in the past. However, we did not enter into the details of that theory in this paper. We only discussed the relation between the nonhomogeneous $Sl(2,\mathbb{C})$ gauge equation satisfied by the curvature bivector and the *shameful* problem of the energy-momentum 'conservation' in General Relativity, and also between that theory and M. Sachs 'unified' field theory as described in [85, 86].

We also recalled the concept of covariant derivatives of spinor fields, when these objects are represented as sections of real spinor bundles ([56, 64, 79]) and study how this theory has as matrix representative the standard spinor fields (dotted and undotted) already introduced long ago, see, e.g., [21, 71, 72, 73]. What was new in our approach is that we identify a possible profound physical meaning concerning some of the rules used in the standard formulation of the (matrix) formulation of spinor fields, e.g., why the covariant derivative of the Pauli matrices must be null. Those rules implies in constraints for the geometry of the spacetime manifold. A possible realization of that constraints is one where the fields defining a global tetrad must be such that $\mathbf{e_0}$ is a geodesic field and the $\mathbf{e_i}$ are Fermi transported (i.e., are not rotating relative to the "fixed stars") along each integral line of $\mathbf{e_0}$. For the best of our knowledge this important fact is here disclosed for the first time.

We use our formalism to discuss several issues in presentations of gravitational theory and other theories. In particular, we scrutinized Sachs "unified" the theory as discussed recently in [86, 87] and as originally introduced in [85]. It is really difficult to believe that after that more than 40 years Sachs succeeded in publishing his wrong results without anyone denouncing his errors. The case is worth to have in mind when we realized that Sachs has more than 900 citations in the Science citation Index. Some one may say how cares? Well, I cared, for reasons mainly described in the introduction, and here we showed that there are some crucial mathematical errors in that theory. To start, [85, 86, 87] identified erroneously his basic variables q_μ as being (matrix representations) of quaternion fields. Well, they are not. The real mathematical structure of these objects is that they are matrix representations of particular sections of the even Clifford bundle of multivectors $\mathcal{C}\ell(TM)$ as we proved in section 2. Next we show that the identification of a 'new' antisymmetric field $\mathcal{F}_{\alpha\beta}$ in his theory is indeed nothing more than the identification of some combinations of the curvature bivectors[49], an object that appears naturally when we try to formulate Einstein's gravitational theory as a $Sl(2,\mathbb{C})$ gauge theory. In that way, any tentative of identifying $\mathcal{F}_{\alpha\beta}$ with any kind of electromagnetic field as did by Sachs in [85, 86] is clearly wrong. We also present the wave like equations solved by the (co)tetrad fields[50] $\theta^{\mathbf{a}}$. Equipped with the correct mathematical formulation of some sophisticated notions of modern Physics theories we identified *fatal mathematical flaws* in several papers by Evans&AIAS[51] that use Sachs 'unified' theory. In a series of papers, quoted in the bibliography Evans&AIAS claims that MEG works with the energy of the $\mathbf{B}_3$ field that they identified with the field $\mathbf{F_{12}}$ (given by Eq.(203)) that appears in Sachs theory. They thought,

[49] The curvature bivectors are physically and mathematically equivalent to the Cartan curvature 2-forms, since they carry the same information. This statement is obvious from our study in section 4.

[50] The set $\{\theta^{\mathbf{a}}\}$ is the dual basis of $\{e_{\mathbf{a}}\}$.

[51] Recall that Evans is as quoted as Sachs, according to the Sceience Citation Index...

following Sachs, that that field represents an electromagnetic field. It is amazing how referees of that papers could accept that argument, for in vacuum $\mathbf{F_{12}} = 0$ (see Eq.(203)). Also, as already said $\mathbf{F_{12}}$ is not an electromagnetic field. However, since there are no conservation laws of energy-momentum in general relativity, if MEG works[52], maybe it is only demonstrating this aspect of General Relativity, that may authors on the subject try (hard) to hide under the carpet.

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A Clifford Bundles $\mathcal{C}\ell(T^*M)$ and $\mathcal{C}\ell(TM)$

Let $\mathcal{L} = (M, g, D)$ be a Lorentzian spacetime. This means that (M, g) is a four dimensional time oriented and space oriented Lorentzian manifold, with $M \simeq \mathbb{R}^4$ and $g \in \sec(T^*M \times T^*M)$ being a Lorentzian metric of signature (1,3). T^*M $[TM]$ is the cotangent [tangent] bundle. $T^*M = \cup_{x\in M} T^*_x M$, $TM = \cup_{x\in M} T_x M$, and $T_x M \simeq T^*_x M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space [88]. D is the Levi-Civita connection of g, *i.e.*, $Dg = 0$, $\mathcal{R}(D) = 0$. Also $\Theta(D) = 0$, $\mathcal{R}$ and Θ being respectively the torsion and curvature tensors. Now, the Clifford bundle of differential forms $\mathcal{C}\ell(T^*M)$ is the bundle of algebras[53] $\mathcal{C}\ell(T^*M) = \cup_{x\in M} \mathcal{C}\ell(T^*_x M)$, where $\forall x \in M, \mathcal{C}\ell(T^*_x M) = \mathbb{R}_{1,3}$, the so-called *spacetime algebra* [57]. Locally as a linear space over the real field R, $\mathcal{C}\ell(T^*_x M)$ is isomorphic to the Cartan algebra $\bigwedge(T^*_x M)$ of the cotangent space and $\bigwedge T^*_x M = \sum_{k=0}^4 \bigwedge^k T^*_x M$, where $\bigwedge^k T^*_x M$ is the $\binom{4}{k}$-dimensional space of k-forms. The Cartan bundle $\bigwedge T^*M = \cup_{x\in M} \bigwedge T^*_x M$ can then be thought [56] as "imbedded" in $\mathcal{C}\ell(T^*M)$. In this way sections of $\mathcal{C}\ell(T^*M)$ can be represented as a sum of nonhomogeneous differential forms. Let $\{\mathbf{e_a}\} \in \sec TM, (\mathbf{a} = 0, 1, 2, 3)$ be an orthonormal basis $g(\mathbf{e_a}, \mathbf{e_b}) = \eta_{\mathbf{ab}} = \mathrm{diag}(1, -1, -1, -1)$ and let $\{\theta^{\mathbf{a}}\} \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$ be the dual basis. Moreover, we denote by g^{-1} the metric in the cotangent bundle.

An analogous construction can be done for the tangent space. The corresponding Clifford bundle is denoted $\mathcal{C}\ell(TM)$ and their sections are called multivector fields. All formulas presented below for $\mathcal{C}\ell(T^*M)$ have a corresponding in $\mathcal{C}\ell(TM)$ and this fact has been used in the text.

[52] We stated above our opinion that despite MEG is a patented device it does not work.

[53] We can show using the definitions of section 5 that $\mathcal{C}\ell(T^*M)$ is a vector bundle associated with the *orthonormal frame bundle*, i.e., $\mathcal{C}\ell(M) = P_{SO_{+(1,3)}} \times_{ad} Cl_{1,3}$. Details about this construction can be found, e.g., in [64].

A.1 Clifford product, scalar contraction and exterior products

The fundamental *Clifford product* (in what follows to be denoted by juxtaposition of symbols) is generated by $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ and if $\mathcal{C} \in \sec \mathcal{C}\ell(T^*M)$ we have

$$C = s + v_{\mathbf{a}}\theta^{\mathbf{a}} + \frac{1}{2!}b_{\mathbf{cd}}\theta^{\mathbf{c}}\theta^{\mathbf{d}} + \frac{1}{3!}a_{\mathbf{abc}}\theta^{\mathbf{a}}\theta^{\mathbf{b}}\theta^{\mathbf{c}} + p\theta^{\mathbf{5}} \,, \tag{238}$$

where $\theta^{\mathbf{5}} = \theta^0\theta^{\mathbf{1}}\theta^{\mathbf{2}}\theta^{\mathbf{3}}$ is the volume element and s, $v_{\mathbf{a}}$, $b_{\mathbf{cd}}$, $a_{\mathbf{abc}}$, $p \in \sec \bigwedge^0 T^*M \subset \sec \mathcal{C}\ell(T^*M)$.

Let $A_r, \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$, $B_s \in \sec \bigwedge^s T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$. For $r = s = 1$, we define the *scalar product* as follows:

For $a, b \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$,

$$a \cdot b = \frac{1}{2}(ab + ba) = g^{-1}(a, b). \tag{239}$$

We also define the *exterior product* ($\forall r, s = 0, 1, 2, 3$) by

$$\begin{aligned} A_r \wedge B_s &= \langle A_r B_s \rangle_{r+s}, \\ A_r \wedge B_s &= (-1)^{rs} B_s \wedge A_r \end{aligned} \tag{240}$$

where $\langle\rangle_k$ is the component in the subspace $\bigwedge^k T^*M$ of the Clifford field. The exterior product is extended by linearity to all sections of $\mathcal{C}\ell(T^*M)$.

For $A_r = a_1 \wedge ... \wedge a_r, B_r = b_1 \wedge ... \wedge b_r$, the *scalar product* is defined as

$$\begin{aligned} A_r \cdot B_r &= (a_1 \wedge ... \wedge a_r) \cdot (b_1 \wedge ... \wedge b_r) \\ &= \det \begin{bmatrix} a_1 \cdot b_1 & \dots & a_1 \cdot b_k \\ \dots & \dots & \dots \\ a_k \cdot b_1 & \dots & a_k \cdot b_k \end{bmatrix}. \end{aligned} \tag{241}$$

We agree that if $r = s = 0$, the scalar product is simple the ordinary product in the real field.

Also, if $r, s \neq 0$ and $A_r \cdot B_s = 0$ if r or s is zero.

For $r \leq s, A_r = a_1 \wedge ... \wedge a_r, B_s = b_1 \wedge ... \wedge b_s$ we define the *left contraction* by

$$\lrcorner : (A_r, B_s) \mapsto A_r \lrcorner B_s = \sum_{i_1 < ... < i_r} \epsilon^{i_1 .... . i_s}_{1 ...... s} (a_1 \wedge ... \wedge a_r) \cdot (b_{i_r} \wedge ... \wedge b_{i_1})^{\sim} b_{i_r+1} \wedge ... \wedge b_{i_s}, \tag{242}$$

where $\sim$ denotes the reverse mapping (*reversion*)

$$\sim : \sec \bigwedge^p T^*M \ni a_1 \wedge ... \wedge a_p \mapsto a_p \wedge ... \wedge a_1, \tag{243}$$

and extended by linearity to all sections of $\mathcal{C}\ell(T^*M)$. We agree that for $\alpha, \beta \in \sec \bigwedge^0 T^*M$ the contraction is the ordinary (pointwise) product in the real field and that if $\alpha \in \sec \bigwedge^0 T^*M$, $A_r, \in \sec \bigwedge^r T^*M, B_s \in \sec \bigwedge^s T^*M$ then $(\alpha A_r) \lrcorner B_s = A_r \lrcorner (\alpha B_s)$. Left contraction is extended by linearity to all pairs of elements of sections of $\mathcal{C}\ell(T^*M)$, i.e., for $A, B \in \sec \mathcal{C}\ell(T^*M)$

$$A \lrcorner B = \sum_{r,s} \langle A \rangle_r \lrcorner \langle B \rangle_s, r \leq s. \tag{244}$$

It is also necessary to introduce in $\mathcal{C}\ell(T^*M)$ the operator of *right contraction* denoted by $\llcorner$. The definition is obtained from the one presenting the left contraction with the imposition that $r \geq s$ and taking into account that now if $A_r, \in \sec \bigwedge^r T^*M, B_s \in \sec \bigwedge^s T^*M$ then $A_r \llcorner (\alpha B_s) = (\alpha A_r) \llcorner B_s$.

A.2 Some useful formulas

The main formulas used in the Clifford calculus in the main text can be obtained from the following ones, where $a \in \sec \bigwedge^1 T^*M$ and $A_r, \in \sec \bigwedge^r T^*M, B_s \in \sec \bigwedge^s T^*M$:

$$\begin{aligned}
aB_s &= a \lrcorner B_s + a \wedge B_s, B_s a = B_s \llcorner a + B_s \wedge a, \qquad (245)\\
a \lrcorner B_s &= \frac{1}{2}(aB_s - (-)^s B_s a),\\
A_r \lrcorner B_s &= (-)^{r(s-1)} B_s \llcorner A_r,\\
a \wedge B_s &= \frac{1}{2}(aB_s + (-)^s B_s a),\\
A_r B_s &= \langle A_r B_s \rangle_{|r-s|} + \langle A_r \lrcorner B_s \rangle_{|r-s-2|} + \ldots + \langle A_r B_s \rangle_{|r+s|}\\
&= \sum_{k=0}^{m} \langle A_r B_s \rangle_{|r-s|+2k}, \; m = \frac{1}{2}(r + s - |r - s|). \qquad (246)
\end{aligned}$$

A.3 Hodge star operator

Let $\star$ be the usual Hodge star operator $\star : \bigwedge^k T^*M \to \bigwedge^{4-k} T^*M$. If $B \in \sec \bigwedge^k T^*M$, $A \in \sec \bigwedge^{4-k} T^*M$ and $\tau \in \sec \bigwedge^4 T^*M$ is the volume form, then $\star B$ is defined by

$$A \wedge \star B = (A \cdot B)\tau.$$

Then we can show that if $A_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(T * M)$ we have

$$\star A_p = \widetilde{A_p} \theta^{\mathbf{5}}. \tag{247}$$

This equation is enough to prove very easily the following identities (which are used in the main text):

$$\begin{aligned} A_r \wedge \star B_s &= B_s \wedge \star A_r; \quad r = s, \\ A_r \lrcorner \star B_s &= B_s \lrcorner \star A_r; \quad r + s = 4, \\ A_r \wedge \star B_s &= (-1)^{r(s-1)} \star (\tilde{A}_r \lrcorner B_s); \quad r \leq s, \\ A_r \lrcorner \star B_s &= (-1)^{rs} \star (\tilde{A}_r \wedge B_s); \quad r + s \leq 4 \end{aligned} \tag{248}$$

Let d and δ be respectively the differential and Hodge codifferential operators acting on sections of $\bigwedge T^*M$. If $\omega_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$, then $\delta\omega_p = (-)^p \star^{-1} d \star \omega_p$, with $\star^{-1}\star =$ identity, where when applied to a p-form

$$\star^{-1} = (-1)^{p(4-p)+1} \star \quad .$$

A.4 Action of $D_{\mathbf{e_a}}$ on Sections of $\mathcal{C}\ell(TM)$ and $\mathcal{C}\ell(T^*M)$

Let $D_{\mathbf{e_a}}$ be the Levi-Civita covariant derivative operator acting on sections of the tensor bundle. It can be easily shown (see, e.g., [27]) that $D_{\mathbf{e_a}}$ is also a covariant derivative operator on the Clifford bundles $\mathcal{C}\ell(TM)$ and $\mathcal{C}\ell(T^*M)$.

Now, if $A_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M)$ we can show, very easily by explicitly performing the calculations[54] that

$$D_{\mathbf{e_a}} A_p = \partial_{\mathbf{e_a}} A_p + \frac{1}{2}[\omega_{\mathbf{e_a}}, A_p], \tag{249}$$

where the $\omega_{\mathbf{e_a}} \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M)$ may be called *Clifford connection 2-forms.* They are given by:

$$\omega_{\mathbf{e_a}} = \omega_{\mathbf{a}}^{\mathbf{bc}} \theta_{\mathbf{b}} \theta_{\mathbf{c}} = \frac{1}{2} \omega_{\mathbf{a}}^{\mathbf{bc}} \theta_{\mathbf{b}} \wedge \theta_{\mathbf{c}}, \tag{250}$$

where (in standard notation)

$$D_{\mathbf{e_a}} \theta_{\mathbf{b}} = \omega_{\mathbf{ab}}^{\mathbf{c}} \theta_{\mathbf{c}}, \quad D_{\mathbf{e_a}} \theta^{\mathbf{b}} = -\omega_{\mathbf{ac}}^{\mathbf{b}} \theta^{\mathbf{c}}, \;\; \omega_{\mathbf{a}}^{\mathbf{bc}} = -\omega_{\mathbf{a}}^{\mathbf{cb}} \tag{251}$$

An analogous formula to Eq.(249) is valid for the covariant derivative of sections of $\mathcal{C}\ell(TM)$ and they are used in several places in the main text.

A.5 Dirac Operator, Differential and Codifferential

The Dirac operator acting on sections of $\mathcal{C}\ell(T^*M)$ is the invariant first order differential operator

$$\boldsymbol{\partial} = \theta^{\mathbf{a}} D_{\mathbf{e_a}}, \tag{252}$$

and we can show(see, e.g., [77]) the very important result:

$$\boldsymbol{\partial} = \boldsymbol{\partial} \wedge + \boldsymbol{\partial} \lrcorner = d - \delta. \tag{253}$$

[54] A derivation of this formula from the genral theory of connections can be found in [64].

The square of the Dirac operator $\boldsymbol{\partial}^2$ is called the *Hodge Laplacian.* It is not to be confused with the standard Laplacian which is given by $\square = \boldsymbol{\partial} \cdot \boldsymbol{\partial}$. The following identities are used in the text

$$\begin{aligned} dd &= \delta\delta = 0, \\ d\boldsymbol{\partial}^2 &= \boldsymbol{\partial}^2 d; \quad \delta\boldsymbol{\partial}^2 = \boldsymbol{\partial}^2\delta, \\ \delta\star &= (-1)^{p+1}\star d; \quad \star\delta = (-1)^p\star d, \\ d\delta\star &= \star d\delta; \quad \star d\delta = \delta d\star; \quad \star\boldsymbol{\partial}^2 = \boldsymbol{\partial}^2\star \end{aligned} \tag{254}$$

A.6 Maxwell Equation

Maxwell equations in the Clifford bundle of differential forms resume in one single equation. Indeed, if $F \in \sec \bigwedge^2 T^*M \subset \sec \mathcal{C}\ell(T^*M)$ is the electromagnetic field and $J_e \in \sec \bigwedge^1 T^*M \subset \sec \mathcal{C}\ell(T^*M)$ is the electromagnetic current, we have Maxwell equation[55]:

$$\boldsymbol{\partial} F = J_e. \tag{255}$$

Eq.(255) is equivalent to the pair of equations

$$dF = 0, \tag{256}$$

$$\delta F = -J_e. \tag{257}$$

Eq.(256) is called the homogenous equation and Eq.(257) is called the nonhomogeneous equation. Note that it can be written also as:

$$d \star F = - \star J_e. \tag{258}$$

References

[1] Anastasovski P. K., Bearden T. E., Ciubotariu, C., et al, Derivation of O(3) Electrodynamics from the Einstein-Sachs Theory of General Relativity, *Found Phys. Lett.* **16** (3), 275-280 (2003).

[2] Anastasovski P. K., Bearden, T. E., Ciubotariu, C., et al., Inverse Faraday Effect From The First Principles of General Relativity, *Found. Phys. Lett.* **16** (2), 195-198 (2003).

[3] Anastasovski P. K., Bearden, T. E., Ciubotariu, C., et al., Aharonov-Bohm Effect As The Basis of Electromagnetic Energy Inherent In The Vacuum, *Found. Phys. Lett.* **15** (6), 561-568 (2002).

[4] Anastasovski P. K., Bearden, T. E, Ciubotariu, C., et al., Derivation of O(3) Electrodynamics from the Irreducible Representations of The Einstein Group, *Found. Phys. Lett.* **15** (2), 179-187 (2002).

[55] Then, there is no misprint in the title of this section.

[5] Anastasovski P. K., Bearden, T. E., Ciubotariu, C., et al., Derivation of The B-(3) Field and Concomitant Vacuum Energy Density from The Sachs Theory of Electrodynamics, *Found. Phys. Let.t* **14** (6): 589-593 (2001).

[6] Anastasovski P. K., Bearden, T. E., Ciubotariu, C., et al., Development of The Sachs Theory of Electrodynamics, *Found. Phys. Lett.* **14** (6), 595-600 (2001).

[7] Anastasovski P. K., Bearden, T.E., Ciubotariu C, et al., Anti-Gravity Effects in The Sachs Theory of Electrodynamics, *Found. Phys. Lett.* **14** (6), 601-605 (2001).

[8] Anastasovski P. K., Bearden, T. E., Ciubotariu, C., et al., Explanation of The Motionless Electromagnetic Generator By Sachs's Theory Of Electrodynamics, *Found. Phys. Lett.* **14** (4), 387-393 (2001).

[9] Anastasovksi P. K., Bearden T. E., Ciubotariu, C., et al., Explanation of The Motionless Electromagnetic Generator with O(3) Electrodynamics, *Found. Phys. Lett.* **14** (1), 87-94 (2001).

[10] Anastasovski P. K., Bearden T.E., Ciubotariu, C., et al., Runaway Solutions Of The Lehnert Equations: The Possibility Of Extracting Energy From The Vacuum, *Optik* **111** (9), 407-409 (2000).

[11] Anastasovski P. K., Bearden, T. E., Ciubotariu, C., et al., Effect of Vacuum Energy on The Atomic Spectra, *Found. Phys. Lett.* **13** (3), 289-296 (2000).

[12] Anastasovski P. K., Bearden, T. E., Ciubotariu, C., et al., Classical Electrodynamics Without The Lorentz Condition: Extracting Energy From The Vacuum, *Phys. Scripta* **61** (5), 513-517 (2000).

[13] Anastasovski P. K., Bearden, T. E., Ciubotariu, C, et al., On The Nature of The $\mathbf{B}_3$ Field, *Phys Scripta* **61** (3), 287-291 (2000).

[14] Anastasovski P. K., Bearden T. E., Ciubotariu, C., et al., Equations of The Yang-Mills Theory of Classical Electrodynamics, *Optik* **111** (2), 53-56 (2000).

[15] Anastasovski P. K., Bearden, T. E., Ciubotariu, C., et al., A General Theory of Non-Abelian Electrodynamics, *Found. Phys. Lett.* **12** (3), 251-265 (1999).

[16] Anglès, P., Review of the paper: "Carvalho, A. L. T., and Rodrigues W. A. Jr., The non Sequitur Mathematics and Physics of the 'New Electrodynamics' Proposed by the AIAS Group, *Random Oper. Stoch. Equs.* **9** , 161-206 (2001)", *Math. Rev.* 2002d:78002.

[17] Baez, J. and Muniain, J. P., Gauge Fields, *Knots and Gravity*, World Scientific Publ. Co., Singapore, 1994.

[18] Been, I. M., and Tucker, R. W., *An Introduction to Spinors and Geometry with Applications in Physics*, Adam Hilger, Bristol and New York, 1987.

[19] Bleecker, D., *Gauge Theory and Variational Principles*, Addison-Wesley Publ. Co., Inc, Reading, MA, 1981.

[20] Bohzkov, Y., and Rodrigues, W. A. Jr., Mass and Energy in General Relativity, *Gen. Rel. and Grav.* **27**, 813- 819 (1995).

[21] Carmeli, M., *Group Theory and Relativity*, McGraw-Hill Int. Publ. Co., New York,1977.

[22] Carmeli, M., Review of M. Sachs book: *General Relativity and Matter*, D. Reidel Publ. Co., Dordrecht, 1982, *Math. Rev.* **84g:83001** (1984).

[23] Carmeli, M., Leibowitz, E., and Nissani, N., *Gravitation*: $Sl(2,\mathbb{C})$ *Gauge Theory and Conservation Laws*, World Scientific Publ. Co., Singapore, 1990.

[24] Carvalho, A. L. T., and Rodrigues W. A. Jr., The non Sequitur Mathematics and Physics of the 'New Electrodynamics' Proposed by the AIAS Group (with), *Random Oper. Stoch. Equs.* **9** , 161-206 (2001), physics/0302016

[25] Choquet-Bruhat, Y., DeWitt-Morete, C., and Dillard-Bleick, M., *Analysis, Manifolds and Physics*, (revised edition), North Holland Publ. Co., Amsterdam, 1982.

[26] Clements, D. J., and Evans, M. .W., The $\mathbf{B}^{(3)}$ Field from Curved Spacetime Embedded in the Irreducible Representation of the Einstein Group, *Found. Phys. Lett.***16**, 465-479 (2003).

[27] Crumeyrolle, A., *Orthogonal and Sympletic Clifford Algebras*, Kluwer Acad. Publ., Dordrecht, 1990.

[28] Dalton, K., Energy and Momentum in General Relativity , *Gen. Rel. Grav.* **21**, 533-544 (1989).

[29] Dimakis, A. and Müller-Hoissen, F., Clifform Calculus with Applications to Classical Field Theories, *Class. Quanum. Grav.* **8**, 2093-2132 (1991).

[30] de Andrade, V. C., Guillen, L. C. T., and Pereira, J. G., Gravitational Energy-Momentum Density in Teleparallel Gravity, *Phys. Rev. Lett.* **84**, 4533-4536 (2000).

[31] Eddington, A. S., *The Mathematical Theory of Relativity* (third unaltered edition), Chelsea Publ. Co., New York, 1975.

[32] Evans M. W., Jeffers S., The Present Status of The Quantum Theory of Light, Part 3, *Adv. Chem. Phys.* **119**, 1-196 (2001).

[33] Evans M. W., O(3) Electrodynamics, Part 1, *Adv. Chem. Phys.* **119**, 79-267 (2001).

[34] Evans M. W., The Link Between The Sachs And O(3) Theories of Electrodynamics, Part 2, *Adv. Chem. Phys.* **119**, 469-494 (2001).

[35] Evans M. W., The Link Between The Topological Theory of Ranada and Trueba, The Sachs Theory, and O(3) Electrodynamics, Part 2, *Adv. Chem. Phys.* **119**, 495-499 (2001).

[36] Evans M. W., Bearden, T. E., Labounsky A., The Most General Form of The Vector Potential In Electrodynamics, *Found. Phys. Lett.* **15** (3), 245-261 (2002).

[37] Evans, M. W., *Comments on A.L. Trovon de Carvalho and Rodrigues "Random Optics(sic) and Stochastic Equations 9(2), 161-206 (2001)"*, preprint www.aias.us (July 2003).

[38] Evans, M. W., A Generally Covariant Field Equation for Gravitation and Electromagnetism, *Found. Phys. Lett.* **16,** 369-377 (2003).

[39] Evans, M. W., A Generally Covariant Wave Equation for Grand Unified Field, *Found. Phys. Lett.* **16**, 513-547 (2003).

[40] Evans, M. W., The Equations of Grand Unified Field in terms of the Maurer-Cartan Structure Relations of Differential Geometry, *Found. Phys. Lett.* **17**, 25-47 (2004).

[41] Evans, M. W., Derivation of Dirac´s Equation form the Evans Wave Equation, *Found. Phys. Lett.* **17**, 149-166 (2004).

[42] Fernández, V. V, Moya, A. M., and Rodrigues, W. A. Jr., Covariant Derivatives on Minkowski Manifolds, in R. Ablamowicz and B. Fauser (eds.), Clifford Algebras and their Applications in Mathematical Physics (Ixtapa-Zihuatanejo, Mexico 1999), vol.1, Algebra and Physics, *Progress in Physics* **18**, pp 373-398, Birkhäuser, Boston, Basel and Berlin, 2000.

[43] Fernández,V. V., Moya, A. M. and Rodrigues, W. A. Jr., Euclidean Clifford Algebra, *Advances in Applied Clifford Algebras* **11**(S3), 1-21 (2001). math-ph/0212043.

[44] Fernández,V. V., Moya, A. M. and Rodrigues, W. A. Jr., Extensors, *Advances in Applied Clifford Algebras* **11**(S3), 23-40 (2001). math-ph/0212046.

[45] Fernández,V. V., Moya, A. M. and Rodrigues, W. A. Jr., Metric Tensor Vs. Metric Extensor, *Advances in Applied Clifford Algebras* **11**(S3), 41-48 (2001). math-ph/0212048.

[46] Fernández,V. V, Moya, A. M., and Rodrigues, W. A. Jr., Metric Clifford Algebra, *Advances in Applied Clifford Algebras* **11**(S3) 49-68, (2001). math-ph/0212049.

[47] Figueiredo, V. L. and Capelas de Oliveira, E., and Rodrigues, W. A. Jr., Covariant, Algebraic and Operator Spinors, *Int. J. Theor. Phys.* **29**, 371 396 (1990).

[48] Frankel, T., *The Geometry of Physics*, Cambridge University Press, Cambridge, 1997.

[49] Geroch, R. Spinor Structure of Space-Times in General Relativity. I, *J. Math. Phys.* **9**, 1739-1744 (1988).

[50] Göckeler, M., and Schücker, *Differential Geometry, Gauge Theories and Gravity*, Cambridge University Press, Cambridge, 1987.

[51] Hestenes, D., *Spacetime Algebra*, Gordon and Breach, New York, 1966.

[52] Hestenes, D., Real Spinor Fields, *J. Math. Phys.*, 798-808, 1967.

[53] Hestenes, D. and Sobczyk, G., *Clifford Algebra to Geometrical Calculus*, Reidel Publ. Co., Dordrecht, 1984.

[54] Kobayashi , S., and Nomizu, K., *Foundations of Differential Geometry*, vol. 1, Intersience Publishers, New York, 1963.

[55] Lasenby, Doran, C. and Gull, S., Gravity, Gauge Theories and Geometric Algebras, *Phil. Trans. R. Soc.* **356**, 487-582 (1998).

[56] Lawson, H. B. Jr., and Michelson, M. L., *Spin Geometry*, Princeton University Press, Princeton, 1989.

[57] Lounesto, P., *Clifford Algebras and Spinors*, London Math. Soc., Lecture Notes Series **239**, Cambridge University Press, Cambridge, 1997.

[58] Logunov, A. A., and Mestvirishvili, M. A., *The Relativistic Theory of Gravitation*, Mir Publishers, Moscow, 1989.

[59] Logunov, A. A., *Relativistic Theory of Gravity*, Nova Science Publ., New York, 1999.

[60] Miller, W. Jr., *Symmetry Groups and their Applications*, Academic Press, New York, 1972.

[61] McMillan, I., The MEG-One Year One, *Newsletter of the JREF*, August 29, 2003 at http://www.randi.org/jr/082903.html.

[62] Mielke, E. W., *Geometrodynamics of Gauge Fields*, Physical Research vol. 3, Akademie-Verlag, Berlin, 1987.

[63] Misner, C. W., Thorne, K.S., and Wheeler, J. A., *Gravitation*, W. H. Freemanand Company, San Francisco, 1973.

[64] Mosna, R. A., and Rodrigues, W. A. Jr., The Bundles of Algebraic and Dirac-Hestenes Spinor Fields, in publication in *J. Math. Phys.* (2004) math-ph/0212033

[65] Moya, A. M., Fernández,V. V., and Rodrigues, W. A. Jr., Multivector Functions of Real Variable, *Advances in Applied Clifford Algebras* **11**(S3), 69-77, (2001). math.GM/0212222.

[66] Moya, A. M., Fernández,V. V., and Rodrigues, W. A. Jr., Multivector Functions of Mutivector Variable, *Advances in Applied Clifford Algebras* **11** (S3), 79-91 (2001). math.GM/0212223.

[67] Moya, A. M., Fernández,V. V., and Rodrigues, W. A. Jr., Multivector Functionals, *Advances in Applied Clifford Algebras* **11** (S3), 93-103 (2001). math.GM/0212224

[68] Nakahara, M., Geometry, *Topology and Physics*, IOP Publishing Ltd, Bristol, 1990.

[69] Nash, C., and Sen, S., *Topology and Geometry for Physicists*, Academic Press, London, New York, 1983.

[70] Palais, R. S., The Geometrization of Physics, Tsing-Hua Univ. Press, Hsinchu, Taiwan, 1981.

[71] Penrose, R., A Spinor Approach to General Relativity, *Ann. Phys.*(N.Y.) **10**, 171-201 (1960).

[72] Penrose, R., and Rindler, W., *Spinors and Space-Time,* volumes 1 and 2, Cambridge University Pres, Cambridge, 1984.

[73] Pirani, F..A. E., Introduction to Gravitational Radiation Theory, in A. Trautman, F. A. E. Pirani and H. Bondi, *Lectures in General Relativity*, vol.1, Brandeis Summer Institute in Theoretical Physics 1964, Prentice-Hall, Int. Inc., London 1965.

[74] Porteous, I. R., *Clifford Algebras and the Classical Groups*, Cambridge Studies in Advanced Mathematics **50**, Cambridge University Press, Cambridge, 1995.

[75] Porteous, I. R., *Topological Geometry*, Van Nostrand Reinhold, London, 1969; 2nd edition, Cambridge University Press, Cambridge, 1981.

[76] Rodrigues, W. A. Jr., and Capelas de Oliveira, E., Dirac and Maxwell Equations in the Clifford and Spin-Clifford Bundles, *Int. J. Theor. Phys.* **29**, 397-412 (1990).

[77] Rodrigues, W. A. Jr., and Souza, Q. A. G., The Clifford Bundle and the Nature of the Gravitational Field, *Found. Phys.* **23**, 1465-1490 (1993).

[78] Rodrigues, W. A. Jr., Vaz, J. Jr., and Pavsic, M., The Clifford Bundle and the Dynamics of the Superparticle, *Banach Center Publications. Polish Acad. Sci.* **37**, 295-314 (1996).

[79] Rodrigues, W. A. Jr., Algebraic and Dirac-Hestenes Spinors and Spinor Fields, in publication in *J. Math. Phys.* (2004) math-ph/0212030

[80] Rodrigues, W. A. Jr., and Sharif, M., Equivalence Principle and the Principle of Local Lorentz Invariance, *Found. Phys.* **31**, 1785-1806 (2001).

[81] Rodrigues, W. A. Jr., *W.A.R. versus AIAS*, RP 28/03 IMECC-UNICAMP, http://www.ime.unicamp.br/rel_pesq/2003/rp28-03.html.

[82] Rodrigues, W. A. Jr., Maxwell-Dirac Equivalence of the First and Second Kinds and the Seiberg-Witten Equations on Minkowski Spacetime, *Int. Journal of Math. and Math. Sci.* **2003**(43), 2707-2734 (2003), math-phys/0212034.

[83] Rodrigues, W. A. Jr. and Souza, Q. A. G., *Conservation Laws of Energy-Momentum on Riemann-Cartan Spacetimes* (in preparation).

[84] Ryder, L. H., *Quantum Field Theory* (second edition), Cambridge University Press, Cambridge, 1996.

[85] Sachs, M., *General Relativity and Matter*, D. Reidel Publ. Co., Dordrecht, 1982.

[86] Sachs, M., On Unification of Gravity and Electromagnetism and the Absence of Magnetic Monopoles, *N.Cimento B* **114**, 123-126 (1999).

[87] Sachs, M., Quantum Mechanics and Gravity, The Frontiers Science IV, Springer-Verlag, Berlin, 2004.

[88] Sachs, R. K., and Wu, H., *General Relativity for Mathematicians*, Springer-Verlag, New York, 1977.

[89] Schoen, R., and Yau, S.-T., Proof of the Positive Mass Conjecture in General Relativity, *Commun. Math. Phys.* **65**, 45-76 (1979).

[90] Schoen, R., and Yau, S.-T., Proof of the Positive Mass Theorem 2, *Commun. Math. Phys.* **79**, 231-260 (1981).

[91] Souza, Q. A. G., and Rodrigues, W. A. Jr., The Dirac Operator and the Structure of Riemann Cartan Weyl Spaces, in P. Letelier and W. A. Rodrigues Jr. (eds.), *Gravitation. The Spacetime Structure*. Proc. SILARG VIII, pp. 177 210, World Scientific Publ. Co., 1994.

[92] Sternberg, S., *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, N. J., 1964.

[93] Thirring, W., *A Course in Mathematical Physics,* vol. 2, Springer-Verlag, New York, 1979.

[94] Thirring, W. and Wallner, R., The Use of Exterior Forms in Einstein's Gravitation Theory, *Brazilian J. Phys.* **8,** 686-723 (1978).

[95] Tucker, R. W., A Clifford Calculus for Physical Theories, in J. S. R. Chisholm and A. K. Common (eds.), *Clifford Algebras and Their Applications in Mathematical Physics*, pp177-200, D. Reidel Publ. Co., Dordrecht, 1985.

[96] Vargas, J. H. G., and Torr, D. G., A Geometric Language for Dirac Equation, in R. Ablamowicz and B. Fauser (eds.), Clifford Algebras and their Applications in Mathematical Physics (Ixtapa-Zihuatanejo, Mexico 1999), vol.1, Algebra and Physics, *Progress in Physics* **18**, pp 135-154, Birkhäuser, Boston, Basel and Berlin, 2000.

[97] Vargas, J. G., and Torr, D. G., Conservation of Vector-Valued Forms and the Question of the Existence of Gravitatioanl Energy-Momentum in General Relativity, *Gen. Rel. Grav.* **23**, 713-732 (1991).

[98] Wald, R., *General Relativity*, Univ. Chicago Press, Chicago, 1984.