Resolvent estimates for plane Couette flow

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Abstract

We discuss the problem of deriving estimates for the resolvent of the linear operator associated with three dimensional perturbations of plane Couette flow, and determining its dependence on the Reynolds number R. Depending on the values of the parameters involved, we derive estimates analytically. For the remaining values of the parameters, we prove that deriving estimates for the resolvent can be reduced to estimating the solutions of a 4th order linear homogeneous ordinary differential equation with non-homogeneous boundary conditions. We study these boundary value problems numerically. Our results indicate the L_2 norm of the resolvent to be proportional to R^2.

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1 Introduction

It is well known that plane Couette flow is stable for infinitesimal perturbations for all values of the Reynolds number R [9]. In laboratory experiments though, transition to turbulence is observed for Reynolds numbers as low as 350 approximately [4, 13]. This discrepancy may be caused by a small domain of attraction of the Couette flow. Therefore, it is of great interest to understand how this domain of attraction scales with the Reynolds number R. Recent works use the resolvent technique to derive a threshold amplitude for perturbations of the base flow, that is, to give a lower bound on the size of perturbations that can lead to turbulence [6, 5, 2].

Successful application of the resolvent method requires estimates for the resolvent $(s\mathcal{I} - \mathcal{L}_R)^{-1}$ of the linear operator $\mathcal{L}_R$ associated with perturbations of the base flow, for the parameter s belonging to the unstable half plane $\mathrm{Re}(s) \geq 0$. For large enough values of $|s|$, depending on the Reynolds number, analytical estimates for the L_2 norm of the resolvent have already been proved [1, 7]. To derive an estimate valid for the whole unstable half plane, direct numerical computations have been used indicating the L_2 norm of the resolvent to be

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proportional to R^2 [5, 14]. In [7], R-dependent weighted norms are used. Direct numerical computations indicate that in one of the norms considered, the resolvent is proportional to R.

We study the 3 dimensional case, with periodic boundary conditions in two of the directions. Our results indicate the L_2 norm of the resolvent to be proportional to R^2, agreeing with the computations in [5, 14]. Our main result is a theorem showing that the problem of proving the resolvent estimates can be reduced to estimating the solutions of a 4th order homogeneous linear ordinary differential equation with non-homogenous boundary conditions. Numerical computations, which are simple and reliable in this case, are used only to study the norms of the solutions of those boundary value problems. The analysis carried out here has other advantages. First of all, it clarifies the reasons for the R^2 growth of the L_2 norm of the resolvent, since it shows exactly where the extra factor of R comes into the game. It also gives some physical insight about the problem, showing that different components of perturbations of the base flow have different scales with respect to R. We also discuss the reasons for the better dependence of the resolvent on R when the weighted norm from [7] is used.

2 The problem

We first give some notations that will be used throughout this work.

In general, elements of $\mathbb{R}^3$ will be represented by bold face letters. The same letter may be used for one of the coordinates of the vector. For example, when convenient, we write $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$. We denote by Ω the set

$$\Omega := [0, 2\pi] \times [0, 2\pi] \times [0, 1].$$

The euclidian inner product in $\mathbb{R}^3$ is denoted by $\cdot$, that is, for $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, we have

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{3} x_i y_i.$$

The L_2 inner product and norm over Ω are denoted respectively by

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \int_\Omega \overline{\mathbf{u}}_1 \cdot \mathbf{u}_2 \, d\mathbf{x} \quad , \quad \|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle.$$

In our choice of coordinates, the Couette flow is the vector field $\mathbf{U} = (0, z, 0)$, which is a steady solution of

$$\begin{aligned}
&\mathbf{U}_t + (\mathbf{U} \cdot \nabla)\mathbf{U} + \nabla P = \frac{1}{R}\Delta \mathbf{U} \\
&\nabla \cdot \mathbf{U} = 0 \\
&\mathbf{U}(x, y, 0, t) = (0, 0, 0) \\
&\mathbf{U}(x, y, 1, t) = (0, 1, 0) \\
&\mathbf{U}(x, y, z, t) = \mathbf{U}(x + 2\pi, y, z, t) \\
&\mathbf{U}(x, y, z, t) = \mathbf{U}(x, y + 2\pi, z, t),
\end{aligned} \tag{2.1}$$

for P a constant. The positive parameter R is the Reynolds number. We consider $R \geq 1$, since this is the physically interesting case. We also note that there are no technical reasons for this assumption, only a slight simplification of the presentation. Problem (2.1) describes the flow of an incompressible fluid between the two parallel planes $z = 0$ and $z = 1$, the plane $z = 0$ at rest and the plane $z = 1$ moving in the y direction with constant velocity 1.

We want to analyze the resolvent of the linear operator associated with perturbations of the Couette flow $\mathbf{U}$. Therefore, we consider the initial boundary value problem

$$\begin{aligned}
&\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{u} + \nabla p = \frac{1}{R}\Delta\mathbf{u} + \mathbf{F}\\
&\nabla\cdot\mathbf{u} = 0\\
&\mathbf{u}(x,y,0,t) = \mathbf{u}(x,y,1,t) = (0,0,0)\\
&\mathbf{u}(x,y,z,t) = \mathbf{u}(x+2\pi,y,z,t)\\
&\mathbf{u}(x,y,z,t) = \mathbf{u}(x,y+2\pi,z,t)\\
&\mathbf{u}(x,y,z,0) = (0,0,0),
\end{aligned} \tag{2.2}$$

which is the linearization of the equations governing 3 dimensional perturbations $\mathbf{u}(\mathbf{x},t) = (u(\mathbf{x},t), v(\mathbf{x},t), w(\mathbf{x},t))$ of $\mathbf{U}$. The forcing $\mathbf{F}(\mathbf{x},t) = (F(\mathbf{x},t), G(\mathbf{x},t), H(\mathbf{x},t))$ is a given C^∞ function, satisfying

$$\int_0^\infty \|\mathbf{F}(\cdot,t)\|^2\,dt < \infty \quad , \quad \nabla\cdot\mathbf{F} = 0. \tag{2.3}$$

The pressure term $p(x,y,z,t)$ in (2.2) is determined up to a constant in terms of $\mathbf{u}$ by the linear elliptic problem

$$\begin{aligned}
\Delta p = -\nabla\cdot((\mathbf{u}\cdot\nabla)\mathbf{U}) - \nabla\cdot((\mathbf{U}\cdot\nabla)\mathbf{u}) &= -2w_y\\
p_z(x,y,0,t) &= \frac{1}{R}w_{zz}(x,y,0,t)\\
p_z(x,y,1,t) &= \frac{1}{R}w_{zz}(x,y,1,t).
\end{aligned} \tag{2.4}$$

Moreover, if p is given by the problem above, the solution $\mathbf{u}$ of (2.2) remains divergence free. Therefore, we drop the continuity equation and write (2.2) as the linear evolution equation

$$\begin{aligned}
&\mathbf{u}_t = \mathcal{L}_R\mathbf{u} + \mathbf{F}\\
&\mathbf{u}(\mathbf{x},0) = (0,0,0),
\end{aligned} \tag{2.5}$$

where the linear operator $\mathcal{L}_R$ is defined by

$$\mathcal{L}_R\mathbf{u} := \frac{1}{R}\Delta\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{U} - (\mathbf{U}\cdot\nabla)\mathbf{u} - \nabla p, \tag{2.6}$$

with p given in terms of $\mathbf{u}$ by (2.4).

It was proven in [9] that all the eigenvalues of $\mathcal{L}_R$ have negative real part for all values of R, and that the eigenvalue with greatest real part is at least at a distance proportional to $\frac{1}{R}$ from the imaginary axis. Our aim is to estimate the L_2 norm of the resolvent $(s\mathcal{I} - \mathcal{L}_R)^{-1}$ of $\mathcal{L}_R$ on the unstable half plane $\mathrm{Re}(s) \geq 0$, and to determine its dependence on R. Our results indicate the resolvent constant $\sup\limits_{\mathrm{Re}(s)\geq 0} \|(s\mathcal{I} - \mathcal{L}_R)^{-1}\|$ to be proportional to R^2, which agrees with the direct numerical computations of [5, 14]. Our analysis clarifies the role played by each component of the function $\mathbf{u}$, and it allows to determine the origin of the R^2 growth of the resolvent constant.

3 Estimates for the resolvent

For large $|s|$, estimates were already proved [1, 7]. We state Theorem 1 from [1]:

Theorem 1 *If $|s| \geq 2\sqrt{2}(1+\sqrt{R})$, then*

$$\| (s\mathcal{I} - \mathcal{L}_R)^{-1} \|^2 \leq \frac{8}{|s|^2}\left(1+\sqrt{R}\right)^2 \leq 1.$$

Using these estimates and the maximum modulus theorem for holomorphic mappings in Banach spaces [3], one can prove (see [1]) that

$$\sup_{\mathrm{Re}(s)\geq 0} \|(s\mathcal{I} - \mathcal{L}_R)^{-1}\| = \sup_{\xi\in\mathbb{R}} \|(i\xi\mathcal{I} - \mathcal{L}_R)^{-1}\|. \tag{3.1}$$

Therefore, to our purposes, it is sufficient to consider $s = i\xi$ purely imaginary. Using this result one can prove (see [1]):

Theorem 2 *Let $s = i\xi$, $\xi \in \mathbb{R}$. If $|\xi| \geq 2(1+\sqrt{R})$, then*

$$\| (s\mathcal{I} - \mathcal{L}_R)^{-1} \|^2 \leq \frac{8}{|s|^2}\left(1+\sqrt{R}\right)^2 \leq 1.$$

Hence, our aim is to estimate the resolvent $(s\mathcal{I} - \mathcal{L}_R)^{-1}$ for $s = i\xi$, $0 \leq |\xi| < 2(1+\sqrt{R})$. We write the problem (2.2) componentwise:

$$\begin{gathered} u_t + zu_y + p_x = \frac{1}{R}\Delta u + F \\ v_t + zv_y + w + p_y = \frac{1}{R}\Delta v + G \\ w_t + zw_y + p_z = \frac{1}{R}\Delta w + H \\ u_x + v_y + w_z = 0 \\ u(\mathbf{x},0) = v(\mathbf{x},0) = w(\mathbf{x},0) = 0, \end{gathered} \tag{3.2}$$

with u, v, w vanishing at $z = 0$, $z = 1$ and 2π periodic in both x and y directions. Taking the Laplace transform with respect to t of the equation in (2.5), we get the resolvent equation

$$s\widetilde{\mathbf{u}} = \mathcal{L}_R\widetilde{\mathbf{u}} + \widetilde{\mathbf{F}}. \tag{3.3}$$

Componentwise, the transformed problem is

$$\begin{aligned} s\widetilde{u} + z\widetilde{u}_y + \widetilde{p}_x &= \frac{1}{R}\Delta\widetilde{u} + \widetilde{F} \\ s\widetilde{v} + z\widetilde{v}_y + \widetilde{w} + \widetilde{p}_y &= \frac{1}{R}\Delta\widetilde{v} + \widetilde{G} \\ s\widetilde{w} + z\widetilde{w}_y + \widetilde{p}_z &= \frac{1}{R}\Delta\widetilde{w} + \widetilde{H} \\ \widetilde{u}_x + \widetilde{v}_y + \widetilde{w}_z &= 0. \end{aligned} \tag{3.4}$$

Our aim is to get an estimate of the form

$$\|\widetilde{\mathbf{u}}\|^2 \le CR^\gamma \|\widetilde{\mathbf{F}}\|^2, \tag{3.5}$$

where C is an absolute constant. Since the most important part of the argument is to determine the exponent γ, we keep the notation simple by representing by C any absolute constant appearing in different parts of this work, possibly with different numerical values. We obtain $\gamma = 4$, which implies the norm of the resolvent to be proportional to R^2. Actually, our analysis show that different components of the velocity have different dependence on R. We get

$$\begin{aligned} \|\widetilde{u}\|^2 &\le CR^4 \|\widetilde{\mathbf{F}}\|^2 \\ \|\widetilde{v}\|^2 &\le CR^4 \|\widetilde{\mathbf{F}}\|^2 \\ \|\widetilde{w}\|^2 &\le CR^2 \|\widetilde{\mathbf{F}}\|^2. \end{aligned} \tag{3.6}$$

The inequalities above provide some physical insight about the problem. For a given forcing, components of the perturbations which are parallel to the planes may grow as R^2, while the worst growth for the normal component is R.

To derive the estimates, we use the well known equivalent formulation of the problem in terms of the normal velocity and the normal vorticity [10, 7]. The vorticity is defined by

$$\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3) := \mathbf{curl}\,\mathbf{u}. \tag{3.7}$$

The transformed normal component of the velocity $\widetilde{w}$ is the solution of

$$\begin{aligned} \left(s + z\frac{\partial}{\partial y}\right)\Delta\widetilde{w} &= \frac{1}{R}\Delta^2\widetilde{w} + \Delta\widetilde{H} \\ \widetilde{w}(x,y,0,s) &= \widetilde{w}(x,y,1,s) = 0 \\ \widetilde{w}_z(x,y,0,s) &= \widetilde{w}_z(x,y,1,s) = 0. \end{aligned} \tag{3.8}$$

The transformed normal component of the vorticity $\widetilde{\eta}_3$ satisfies

$$\begin{aligned} \left(s + z\frac{\partial}{\partial y}\right)\widetilde{\eta}_3 + \widetilde{w}_x &= \frac{1}{R}\Delta\widetilde{\eta}_3 + \widetilde{G}_x - \widetilde{F}_y \\ \widetilde{\eta}_3(x,y,0,s) &= \widetilde{\eta}_3(x,y,1,s) = 0. \end{aligned} \tag{3.9}$$

Expand in a Fourier series in the x and y directions. We represent by k_1 and k_2 the respective parameters. Let $k^2 := k_1^2 + k_2^2$. The transformed functions $\widehat{w}$, $\widehat{\eta}_3$ are the solutions of the system

$$\begin{gathered}\frac{1}{R}\widehat{w}'''' - \Big(s + \frac{2k^2}{R} + ik_2 z\Big)\widehat{w}'' + \Big(sk^2 + \frac{k^4}{R} + ik_2k^2 z\Big)\widehat{w} = k^2\widehat{H} - \widehat{H}'' \\ \widehat{w}(k_1,k_2,0,s) = \widehat{w}(k_1,k_2,1,s) = \widehat{w}'(k_1,k_2,0,s) = \widehat{w}'(k_1,k_2,1,s) = 0,\end{gathered} \tag{3.10}$$

and

$$\begin{gathered}\frac{1}{R}\widehat{\eta}_3'' - \Big(s + \frac{k^2}{R} + ik_2 z\Big)\widehat{\eta}_3 = ik_1\widehat{w} + ik_2\widehat{F} - ik_1\widehat{G} \\ \widehat{\eta}_3(k_1,k_2,0,s) = \widehat{\eta}_3(k_1,k_2,1,s) = 0.\end{gathered} \tag{3.11}$$

In the problems above, $'$ denotes the derivative with respect to z. The equations in (3.10) and (3.11) are respectively the classical Orr-Sommerfeld and Squire equations [10, 8, 11, 12]. The transformed normal velocity $\widehat{w}$, solution of (3.10), acts as a forcing term in the equation of the transformed normal vorticity (3.11). To simplify the notation, we define the differential operators T, T_0 by

$$\begin{gathered}T := \frac{1}{R}\mathcal{D}^2 - \left(s + \frac{k^2}{R} + ik_2 z\right) \\ T_0 := \mathcal{D}^2 - k^2,\end{gathered} \tag{3.12}$$

where $\mathcal{D}$ denotes the derivative with respect to z. Then, the differential equation in (3.10) is written as

$$TT_0\widehat{w} = I := k^2\widehat{H} - \widehat{H}''. \tag{3.13}$$

The equation for the transformed normal vorticity is

$$T\widehat{\eta}_3 = ik_1\widehat{w} + ik_2\widehat{F} - ik_1\widehat{G}. \tag{3.14}$$

The following Lemma follows directly from Parseval's identity:

Lemma 3 *If*

$$\|\widehat{\mathbf{u}}(k_1,k_2,\cdot,s)\|^2 \le CR^\gamma \|\widehat{\mathbf{F}}(k_1,k_2,\cdot,s)\|^2 \tag{3.15}$$

for all $(k_1,k_2) \in \mathbb{Z}\times\mathbb{Z}$ *and for all* $s \in \mathbb{C}$, $\mathrm{Re}(s) \ge 0$*, then*

$$\|(s\mathcal{I} - \mathcal{L}_R)^{-1}\|^2 \le CR^\gamma \ , \ \forall\, s \in \mathbb{C} \ , \ \mathrm{Re}(s) \ge 0. \tag{3.16}$$

Therefore, we aim for an estimate of the form (3.15). We begin by estimating the normal velocity.

3.1 Estimates for the normal velocity

We separate the analysis into three cases: $k^2 \ge \dfrac{R}{\sqrt{2}}$, $k = 0$ and $0 < k^2 < \dfrac{R}{\sqrt{2}}$.

Case $k^2 \geq \dfrac{R}{\sqrt{2}}$ The transformed normal velocity is the solution of problem (3.10). By Theorem 2, we need only to consider $s = i\xi$, $\xi \in \mathbb{R}$, $0 \leq |\xi| < 2(1+\sqrt{R})$. Therefore, (3.10) reads

$$\begin{gathered} \frac{1}{R}\widehat{w}'''' - \Big(i\xi + \frac{2k^2}{R} + ik_2 z\Big)\widehat{w}'' + \Big(i\xi k^2 + \frac{k^4}{R} + ik_2 k^2 z\Big)\widehat{w} = I \\ \widehat{w}(k_1,k_2,0,\xi) = \widehat{w}(k_1,k_2,1,\xi) = \widehat{w}'(k_1,k_2,0,\xi) = \widehat{w}'(k_1,k_2,1,\xi) = 0, \end{gathered} \tag{3.17}$$

where $I = k^2\widehat{H} - \widehat{H}''$. We prove the following:

Theorem 4 *If* $k^2 \geq \dfrac{R}{\sqrt{2}}$, *then*

$$\|\widehat{w}''(k_1,k_2,\cdot,s)\|^2 + (k^2+k_1^2)\|\widehat{w}'(k_1,k_2,\cdot,s)\|^2 + k^4\|\widehat{w}(k_1,k_2,\cdot,s)\|^2 \leq \\ \leq CR^2\|\widehat{H}(k_1,k_2,\cdot,s)\|^2.$$

Proof. Taking the inner product of the differential equation in (3.17) with $\widehat{w}$ and integrating by parts, one obtains

$$\begin{aligned} \frac{1}{R}\|\widehat{w}''\|^2 + \left(\frac{2k^2}{R} + i\xi\right)\|\widehat{w}'\|^2 + \left(i\xi k^2 + \frac{k^4}{R}\right)\|\widehat{w}\|^2 & \\ + ik_2\langle\widehat{w},\widehat{w}'\rangle + ik_2\langle\widehat{w}',z\widehat{w}'\rangle + ik_2k^2\langle\widehat{w},z\widehat{w}\rangle \quad &= \quad \langle\widehat{w},I\rangle. \end{aligned} \tag{3.18}$$

As can be easily checked through integration by parts, $\langle\widehat{w},\widehat{w}'\rangle$ is purely imaginary, and $\langle\widehat{w},z\widehat{w}\rangle$, $\langle\widehat{w}',z\widehat{w}'\rangle$ are both real. Hence, taking the real part of (3.18) and using the triangle inequality, one gets

$$\frac{1}{R}\|\widehat{w}''\|^2 + \frac{2k^2}{R}\|\widehat{w}'\|^2 + \frac{k^4}{R}\|\widehat{w}\|^2 - |k_2||\langle\widehat{w},\widehat{w}'\rangle| \leq |\langle\widehat{w},I\rangle|. \tag{3.19}$$

We note that (3.19) is valid for all values of the parameters.
If $k_2 \neq 0$, use the inequality

$$|\langle\widehat{w},\widehat{w}'\rangle| \leq \frac{R}{4|k_2|}\|\widehat{w}\|^2 + \frac{|k_2|}{R}\|\widehat{w}'\|^2$$

to get

$$\frac{1}{R}\|\widehat{w}''\|^2 + \left(\frac{2k^2}{R} - \frac{k_2^2}{R}\right)\|\widehat{w}'\|^2 + \left(\frac{k^4}{R} - \frac{R}{4}\right)\|\widehat{w}\|^2 \leq |\langle\widehat{w},I\rangle|. \tag{3.20}$$

Since $k^2 \geq \dfrac{R}{\sqrt{2}}$ implies $\dfrac{k^4}{R} - \dfrac{R}{4} \geq \dfrac{k^4}{2R}$, inequality (3.20) gives

$$\frac{1}{R}\|\widehat{w}''\|^2 + \frac{k^2+k_1^2}{R}\|\widehat{w}'\|^2 + \frac{k^4}{2R}\|\widehat{w}\|^2 \leq |\langle\widehat{w},I\rangle|. \tag{3.21}$$

The desired estimates follow from this inequality. To derive them, we first note that the differential equation in (3.17) is linear. Therefore, if $\widehat{w}_1$, $\widehat{w}_2$ are the solutions of

$$TT_0\widehat{w}_1 = k^2\widehat{H}$$
$$TT_0\widehat{w}_2 = -\widehat{H}'',$$

both satisfying the same boundary conditions as $\widehat{w}$, then $\widehat{w} = \widehat{w}_1 + \widehat{w}_2$. We prove estimates for $\widehat{w}_1$ and $\widehat{w}_2$.

Using inequality (3.21) for $\widehat{w}_1$, and the Cauchy-Schwarz inequality, one gets

$$\frac{1}{R}\|\widehat{w}_1''\|^2 + \frac{k^2+k_1^2}{R}\|\widehat{w}_1'\|^2 + \frac{k^4}{2R}\|\widehat{w}_1\|^2 \le k^2\|\widehat{w}_1\|\|\widehat{H}\|. \tag{3.22}$$

This inequality implies

$$\|\widehat{w}_1''\|^2 + (k^2+k_1^2)\|\widehat{w}_1'\|^2 + k^4\|\widehat{w}_1\|^2 \le CR^2\|\widehat{H}\|^2. \tag{3.23}$$

For $\widehat{w}_2$, first note that

$$\langle \widehat{w}_2, \widehat{H}''\rangle = \langle \widehat{w}_2'', \widehat{H}\rangle, \tag{3.24}$$

since the boundary conditions satisfied by $\widehat{w}_2$ imply that the boundary terms after integration by parts vanish. Therefore, using (3.21), and the Cauchy-Schwarz inequality, we get

$$\frac{1}{R}\|\widehat{w}_2''\|^2 + \frac{k^2+k_1^2}{R}\|\widehat{w}_2'\|^2 + \frac{k^4}{2R}\|\widehat{w}_2\|^2 \le \|\widehat{w}_2''\|\|\widehat{H}\|. \tag{3.25}$$

This inequality implies

$$\|\widehat{w}_2''\|^2 + (k^2+k_1^2)\|\widehat{w}_2'\|^2 + k^4\|\widehat{w}_2\|^2 \le CR^2\|\widehat{H}\|^2. \tag{3.26}$$

Since $\widehat{w} = \widehat{w}_1 + \widehat{w}_2$, inequalities (3.23) and (3.26) imply

$$\|\widehat{w}''\|^2 + (k^2+k_1^2)\|\widehat{w}'\|^2 + k^4\|\widehat{w}\|^2 \le CR^2\|\widehat{H}\|^2. \tag{3.27}$$

If $k_2 = 0$, then $k_1 \neq 0$ and inequality (3.19) is

$$\frac{1}{R}\|\widehat{w}''\|^2 + \frac{2k_1^2}{R}\|\widehat{w}'\|^2 + \frac{k_1^4}{R}\|\widehat{w}\|^2 \le |\langle \widehat{w}, I\rangle|.$$

¿From this inequality, estimates follow by the same argument as above, with no restriction on k_1. □

Case $k = 0$ In this case, we prove

Theorem 5 *If $k = 0$, we have*

$$\|\widehat{w}''(0,0,\cdot,s)\|^2 + \|\widehat{w}'(0,0,\cdot,s)\|^2 + \|\widehat{w}(0,0,\cdot,s)\|^2 \le CR^2\|\widehat{H}(0,0,\cdot,s)\|^2. \tag{3.28}$$

Proof. For this case, problem (3.17) is

$$\begin{aligned}
&\frac{1}{R}\widehat{w}'''' - i\xi\widehat{w}'' + = -\widehat{H}'' \\
&\widehat{w}(0,0,0,s) = \widehat{w}(0,0,1,s) = \widehat{w}'(0,0,0,s) = \widehat{w}'(0,0,1,s) = 0,
\end{aligned} \tag{3.29}$$

where $s = i\xi$. Taking the inner product of the equation with $\widehat{w}$ and integrating by parts, one gets

$$\frac{1}{R}\|\widehat{w}''\|^2 + i\xi\|\widehat{w}'\|^2 = -\langle \widehat{w}'', \widehat{H}\rangle. \tag{3.30}$$

Taking the real part of this equation, and using the Cauchy-Schwarz inequality on its right hand side, we obtain

$$\|\widehat{w}''\|^2 \leq R^2\|\widehat{H}\|^2. \tag{3.31}$$

Application of the Poincaré's inequality twice gives us the estimate

$$\|\widehat{w}''\|^2 + \|\widehat{w}'\|^2 + \|\widehat{w}\|^2 \leq CR^2\|\widehat{H}\|^2, \tag{3.32}$$

finishing the proof.

Case $0 < k^2 < \dfrac{R}{\sqrt{2}}$ For this case, we show that the problem can be reduced to estimating the solutions of linear homogeneous ordinary differential equations with non-homogeneous boundary conditions. The method used here is similar to the approach in [1] to estimate the stream function for the case of two space dimensions.

Theorem 6 *If for all $R \geq 1$, The solutions $\phi_1(k_1, k_2, z, s)$ and $\phi_2(k_1, k_2, z, s)$ of*

$$\begin{array}{cc}
TT_0\phi_1 = 0 & TT_0\phi_2 = 0 \\
\phi_1(k_1,k_2,0,s) = 0 & \phi_2(k_1,k_2,0,s) = 0 \\
\phi_1(k_1,k_2,1,s) = 0 & \phi_2(k_1,k_2,1,s) = 0 \\
\phi_1'(k_1,k_2,0,s) = 1 & \phi_2'(k_1,k_2,0,s) = 0 \\
\phi_1'(k_1,k_2,1,s) = 0 & \phi_2'(k_1,k_2,1,s) = 1
\end{array} \tag{3.33}$$

satisfy

$$\begin{array}{cc}
|k|\|\phi_1(k_1,k_2,\cdot,s)\|^2 \leq C & |k|\|\phi_2(k_1,k_2,\cdot,s)\|^2 \leq C \\
\|\phi_1'(k_1,k_2,\cdot,s)\|^2 \leq C & \|\phi_2'(k_1,k_2,\cdot,s)\|^2 \leq C,
\end{array} \tag{3.34}$$

for some absolute constant $C > 0$ and for all $0 < k^2 < \dfrac{R}{\sqrt{2}}$, $s = i\xi$, $\xi \in \mathbb{R}$, $0 \leq |\xi| < 2(1+\sqrt{R})$, then

$$|k|\|\widehat{w}'(k_1,k_2,\cdot,s)\|^2 + k^2\|\widehat{w}(k_1,k_2,\cdot,s)\|^2 \leq CR^2\|\widehat{\mathbf{F}}\|^2, \tag{3.35}$$

for all $R \geq 1$, $0 < k^2 < \dfrac{R}{\sqrt{2}}$, $s = i\xi$, $\xi \in \mathbb{R}$, $0 \leq |\xi| < 2(1+\sqrt{R})$.

Proof. The transformed normal velocity $\widehat{w}$ is the solution of

$$\begin{aligned} TT_0\widehat{w} &= I \\ \widehat{w}(0) = \widehat{w}(1) &= \widehat{w}'(0) = \widehat{w}'(1) = 0, \end{aligned} \tag{3.36}$$

where $I = k^2\widehat{H} - \widehat{H}''$. To simplify the notation, we do not write explicitly the dependence of $\widehat{w}$ on all the parameters.

Let g and h be solution of the system

$$\begin{aligned} Th = \left(\frac{1}{R}\mathcal{D}^2 - \left(i\xi + \frac{k^2}{R} + ik_2 z\right)\right)h = I, \quad h(0) = h(1) = 0 \\ T_0 g = (\mathcal{D}^2 - k^2)g = h, \quad g(0) = g(1) = 0. \end{aligned} \tag{3.37}$$

Taking the inner product of the first equation with h, and integrating by parts, one gets

$$-\frac{1}{R}\|h'\|^2 - \frac{k^2}{R}\|h\|^2 - i\xi\|h\|^2 - ik_2\langle h, zh\rangle = \langle h, I\rangle. \tag{3.38}$$

Taking the real part of the equation above, and noting that $\langle h, zh\rangle \in \mathbb{R}$, we get

$$\frac{1}{R}\|h'\|^2 + \frac{k^2}{R}\|h\|^2 \leq |\langle h, I\rangle|. \tag{3.39}$$

As done before, since the equation satisfied by h in (3.37) is linear, we study separately h_1, h_2, the solutions of

$$\begin{aligned} Th_1 &= k^2\widehat{H}, \quad h_1(0) = h_1(1) = 0 \\ Th_2 &= -\widehat{H}'', \quad h_2(0) = h_2(1) = 0. \end{aligned} \tag{3.40}$$

For h_1, inequality (3.39) is

$$\frac{1}{R}\|h_1'\|^2 + \frac{k^2}{R}\|h_1\|^2 \leq |\langle h_1, k^2\widehat{H}\rangle| \leq k^2\|h_1\|\|\widehat{H}\|, \tag{3.41}$$

which implies

$$\|h_1\|^2 \leq R^2\|\widehat{H}\|^2. \tag{3.42}$$

For h_2, using inequality (3.39) and integrating by parts once, we have

$$\frac{1}{R}\|h_2'\|^2 + \frac{k^2}{R}\|h_2\|^2 \leq |\langle h_2, -\widehat{H}''\rangle| = |\langle h_2', -\widehat{H}'\rangle| \leq \|h_2'\|\|\widehat{H}'\|. \tag{3.43}$$

Therefore,

$$\|h_2'\|^2 + k^2\|h_2\|^2 \leq CR^2\|\widehat{H}'\|^2. \tag{3.44}$$

¿From (2.3), we have

$$\widehat{H}' = -ik_1\widehat{F} - ik_2\widehat{G}. \tag{3.45}$$

Therefore, (3.44) gives

$$\|h_2'\|^2 + k^2\|h_2\|^2 \leq CR^2\big(k_1^2\|\widehat{F}\|^2 + k_2^2\|\widehat{G}\|^2\big) \leq Ck^2R^2\big(\|\widehat{F}\|^2 + \|\widehat{G}\|^2\big), \tag{3.46}$$

which implies

$$\|h_2\|^2 \le CR^2\big(\|\widehat{F}\|^2 + \|\widehat{G}\|^2\big). \tag{3.47}$$

Using (3.42) and (3.47), we conclude that $h = h_1 + h_2$ satisfies

$$\|h\|^2 \le CR^2\big(\|\widehat{F}\|^2 + \|\widehat{G}\|^2 + \|\widehat{H}\|^2\big) = CR^2\|\widehat{\mathbf{F}}\|^2. \tag{3.48}$$

For g, estimates follow in a similar and simpler way. Taking the inner product of the second equation in (3.37) with g and integrating by parts, one can prove that

$$k^2\|g'\|^2 + k^4\|g\|^2 \le C\|h\|^2. \tag{3.49}$$

Using the differential equation for g in (3.37), one can also bound g''. Therefore, one gets

$$\|g''\|^2 + k^2\|g'\|^2 + k^4\|g\|^2 \le C\|h\|^2. \tag{3.50}$$

Using (3.48) and (3.50), we conclude that

$$\|g''\|^2 + k^2\|g'\|^2 + k^4\|g\|^2 \le CR^2\|\widehat{\mathbf{F}}\|^2. \tag{3.51}$$

It follows from the definition of g that it satisfies

$$\begin{aligned} TT_0 g &= k^2\widehat{H} - \widehat{H}'' \\ g(0) &= g(1) = 0. \end{aligned} \tag{3.52}$$

Therefore, g is the satisfies the same differential equation satisfied by $\widehat{w}$, but with different boundary conditions, since $g'(0)$ and $g'(1)$ do not necessarily vanish. But those values can be estimated. Indeed, using the 1-dimensional Sobolev inequality $|g'|_\infty^2 \le \|g'\|^2 + 2\|g'\|\|g''\|$, and (3.51), we have

$$\begin{aligned} |k||g'(0)|^2 &\le |k||g'|_\infty^2 \le |k|\|g'\|^2 + 2|k|\|g'\|\|g''\| \le CR^2\|\widehat{\mathbf{F}}\|^2 \\ |k||g'(1)|^2 &\le |k||g'|_\infty^2 \le |k|\|g'\|^2 + 2|k|\|g'\|\|g''\| \le CR^2\|\widehat{\mathbf{F}}\|^2. \end{aligned} \tag{3.53}$$

Now, let ϕ be the solution of

$$\begin{aligned} &TT_0\phi = 0 \\ &\phi(0) = \phi(1) = 0 \\ &\phi'(0) = g'(0) \\ &\phi'(1) = g'(1). \end{aligned} \tag{3.54}$$

Then, $\widehat{w} = g - \phi$, as can be easily checked. Since we already have estimates for g, estimates for ϕ will imply estimates for $\widehat{w}$. Now, note that if ϕ_1 and ϕ_2 are the solutions of

$$\begin{array}{cc} TT_0\phi_1 = 0 & TT_0\phi_2 = 0 \\ \phi_1(0) = \phi_1(1) = 0 & \phi_2(0) = \phi_2(1) = 0 \\ \phi_1'(0) = 1 & \phi_2'(0) = 0 \\ \phi_1'(1) = 0 & \phi_2'(1) = 1, \end{array} \tag{3.55}$$

then $\phi = g'(0)\phi_1 + g'(1)\phi_2$. Therefore, if for some absolute constant C we have

$$\begin{aligned} |k|\|\phi_1(k_1,k_2,\cdot,s)\|^2 \le C \quad & |k|\|\phi_2(k_1,k_2,\cdot,s)\|^2 \le C \\ \|\phi_1'(k_1,k_2,\cdot,s)\|^2 \le C \qquad & \|\phi_2'(k_1,k_2,\cdot,s)\|^2 \le C, \end{aligned} \tag{3.56}$$

then, using (3.53), we get

$$\begin{aligned} k^2\|\phi\|^2 &\le 2k^2|g'(0)|^2\|\phi_1\|^2 + 2k^2|g'(1)|^2\|\phi_2\|^2 \le CR^2\|\widehat{\mathbf{F}}\|^2 \\ |k|\|\phi'\|^2 &\le 2|k||g'(0)|^2\|\phi_1'\|^2 + 2|k||g'(1)|^2\|\phi_2'\|^2 \le CR^2\|\widehat{\mathbf{F}}\|^2. \end{aligned} \tag{3.57}$$

Since $\widehat{w} = g - \phi$, inequalities (3.51) and (3.57) imply

$$|k|\|\widehat{w}'\|^2 + k^2\|\widehat{w}\|^2 \le CR^2\|\widehat{\mathbf{F}}\|^2, \tag{3.58}$$

which proves the Theorem.

We study the solutions ϕ_1 and ϕ_2 of (3.55) numerically. These problems are suitable for a numerical approach for two main reasons: first, they are homogeneous problems, with fixed non-homogeneous boundary conditions for all values of the parameters k_1, k_2, ξ, R. Second, they need to be studied only for bounded values of k_1, k_2 and s, namely for $0 < k^2 < \dfrac{R}{\sqrt{2}}$, and $s = i\xi$, $\xi \in \mathbb{R}$, $0 \le |\xi| < 2(1+\sqrt{R})$. The results are shown in Section 4, providing evidence for the bounds (3.56).

Therefore, from the three cases studied above, we conclude that for all values of the parameters k_1, k_2, and s , we have

$$\begin{aligned} |k|\|\widehat{w}'\|^2 + k^2\|\widehat{w}\|^2 \le CR^2\|\widehat{\mathbf{F}}\|^2, \quad k^2 \ne 0 \\ \|\widehat{w}'\|^2 + \|\widehat{w}\|^2 \le CR^2\|\widehat{\mathbf{F}}\|^2, \quad k = 0. \end{aligned} \tag{3.59}$$

Having bounds for the normal velocity $\widehat{w}$, we now derive the bounds for the normal vorticity, and use them to estimate $\widehat{u}$, $\widehat{v}$, the remaining components of the velocity.

3.2 Estimates for the normal vorticity

We prove

Theorem 7 *if the estimates (3.59) hold, then*

$$\|\widehat{\eta}_3'\|^2 + k^2\|\widehat{\eta}_3\|^2 \le CR^4\|\widehat{\mathbf{F}}\|^2. \tag{3.60}$$

Moreover, inequality (3.60) implies

$$\|\widehat{u}\|^2 + \|\widehat{v}\|^2 \le CR^4\|\widehat{\mathbf{F}}\|^2. \tag{3.61}$$

Proof: The function $\widehat{\eta}_3$ is the solution of

$$T\widehat{\eta}_3 = \frac{1}{R}\widehat{\eta}_3'' - \Big(i\xi + \frac{k^2}{R} + ik_2 z\Big)\widehat{\eta}_3 = ik_1\widehat{w} + ik_2\widehat{F} - ik_1\widehat{G}$$
$$\widehat{\eta}_3(k_1,k_2,0,\xi) = \widehat{\eta}_3(k_1,k_2,1,\xi) = 0 \tag{3.62}$$

Taking the inner product of the differential equation with $\widehat{\eta}_3$, and integrating by parts the first term of the resulting equation once, we get

$$\frac{1}{R}\|\widehat{\eta}_3'\|^2 + \Big(i\xi + \frac{k^2}{R}\Big)\|\widehat{\eta}_3\|^2 + ik_2\langle\widehat{\eta}_3, z\widehat{\eta}_3\rangle = -ik_1\langle\widehat{\eta}_3,\widehat{w}\rangle - ik_2\langle\widehat{\eta}_3,\widehat{F}\rangle + ik_1\langle\widehat{\eta}_3,\widehat{G}\rangle.$$

Since $\langle\widehat{\eta}_3, z\widehat{\eta}_3\rangle \in \mathbb{R}$, taking the real part of the equation above and using the Cauchy-Schwarz inequality, we have

$$\frac{1}{R}\|\widehat{\eta}_3'\|^2 + \frac{k^2}{R}\|\widehat{\eta}_3\|^2 \le |k_1|\|\widehat{\eta}_3\|\|\widehat{w}\| + |k_2|\|\widehat{\eta}_3\|\|\widehat{F}\| + |k_1|\|\widehat{\eta}_3\|\|\widehat{G}\|. \tag{3.63}$$

If $k^2 = 0$, the desired estimates follow directly. For $k^2 \neq 0$, (3.63) implies

$$\frac{|k|}{R}\|\widehat{\eta}_3\| \le \frac{|k_1|}{|k|}\|\widehat{w}\| + \frac{|k_2|}{|k|}\|\widehat{F}\| + \frac{|k_1|}{|k|}\|\widehat{G}\| \le \|\widehat{w}\| + \|\widehat{F}\| + \|\widehat{G}\| \le CR\|\widehat{\mathbf{F}}\|,$$

where we used (3.59) to bound $\|\widehat{w}\|$. Therefore,

$$k^2\|\widehat{\eta}_3\|^2 \le CR^4\|\widehat{\mathbf{F}}\|^2. \tag{3.64}$$

Using (3.63) and (3.64), we can bound $\widehat{\eta}_3'$ by

$$\|\widehat{\eta}_3'\|^2 \le CR^4\|\widehat{\mathbf{F}}\|^2. \tag{3.65}$$

Inequalities (3.64) and (3.65) together give

$$\|\widehat{\eta}_3'\|^2 + k^2\|\widehat{\eta}_3\|^2 \le CR^4\|\widehat{\mathbf{F}}\|^2, \tag{3.66}$$

which proves the first part of the Theorem.

We now use (3.64) to bound $\widehat{u}$, $\widehat{v}$, components of the velocity. The velocity components u and v can be recovered once one knows the normal velocity w and normal vorticity η_3 by solving, with appropriate boundary conditions, the equations

$$-u_{xx} - u_{yy} = \eta_{3y} + w_{xz} \tag{3.67}$$
$$-v_{xx} - v_{yy} = w_{yz} - \eta_{3x}. \tag{3.68}$$

For the transformed functions, the equations above are

$$k^2\widehat{u} = ik_2\widehat{\eta}_3 + ik_1\widehat{w}' \tag{3.69}$$
$$k^2\widehat{v} = ik_2\widehat{w}' - ik_1\widehat{\eta}_3. \tag{3.70}$$

Using (3.59) and (3.64), the estimates

$$k^2\|\widehat{u}\| \leq CR^2\|\widehat{\mathbf{F}}\| \tag{3.71}$$

$$k^2\|\widehat{v}\| \leq CR^2\|\widehat{\mathbf{F}}\| \tag{3.72}$$

follow. □

Inequalities (3.59), (3.71), (3.72) and Theorem 2 together imply

$$\begin{aligned}\|\widehat{\mathbf{u}}(k_1,k_2,\cdot,s)\|^2 &= \|\widehat{u}(k_1,k_2,\cdot,s)\|^2 + \|\widehat{v}(k_1,k_2,\cdot,s)\|^2 + \|\widehat{w}(k_1,k_2,\cdot,s)\|^2\\ &\leq CR^4\|\widehat{\mathbf{F}}(k_1,k_2,\cdot,s)\|^2 + CR^4\|\widehat{\mathbf{F}}(k_1,k_2,\cdot,s)\|^2 + CR^2\|\widehat{\mathbf{F}}(k_1,k_2,\cdot,s)\|^2\\ &\leq CR^4\|\widehat{\mathbf{F}}(k_1,k_2,\cdot,s)\|^2,\end{aligned}$$

for all $(k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}$ and for all $s\in\mathbb{C}$, $\mathrm{Re}(s)\geq 0$. By Lemma 3, this implies the resolvent estimate

$$\|(s\mathcal{I}-\mathcal{L}_R)^{-1}\|^2 \leq CR^4\ ,\ \forall\, s\in\mathbb{C}\ ,\ \mathrm{Re}(s)\geq 0 \tag{3.73}$$

Remarks about weighted norms In [7], the authors define a weighted norm $\|\cdot\|_3$, which is given in our coordinate system by

$$\|\widetilde{\mathbf{u}}\|_3^2 := \|\widetilde{u}\|^2 + \|\widetilde{v}\|^2 + R^2\|\widetilde{w}\|^2. \tag{3.74}$$

Via direct numerical computations, they conclude that

$$\|\widetilde{\mathbf{u}}\|_3^2 \leq CR^2\|\widetilde{\mathbf{F}}\|_3^2. \tag{3.75}$$

Our analysis shows that, if one gets estimates of the type (3.27) for all values of k_1, k_2, that is, estimating the normal velocity by the normal component of the forcing only, inequality (3.75) follows. Indeed, in this case, the estimates for the normal vorticity would be

$$\|\widehat{\eta}_3'\|^2 + k^2\|\widehat{\eta}_3\|^2 \leq CR^2\|\widehat{F}\|^2 + CR^2\|\widehat{G}\|^2 + CR^4\|\widehat{H}\|^2,$$

and then, using (3.69) and (3.70),

$$\begin{aligned}\|\widehat{u}\|^2 &\leq CR^2\|\widehat{F}\|^2 + CR^2\|\widehat{G}\|^2 + CR^4\|\widehat{H}\|^2\\ \|\widehat{v}\|^2 &\leq CR^2\|\widehat{F}\|^2 + CR^2\|\widehat{G}\|^2 + CR^4\|\widehat{H}\|^2.\end{aligned}$$

Therefore,

$$\|\widehat{u}\|^2 + \|\widehat{v}\|^2 + R^2\|\widehat{w}\|^2 \leq CR^2\|\widehat{F}\|^2 + CR^2\|\widehat{G}\|^2 + CR^4\|\widehat{H}\|^2 = CR^2\|\widehat{\mathbf{F}}\|_3^2.$$

We believe this to be the case. In our argument though, we need to use all the components of the forcing $\widehat{\mathbf{F}}$ to bound $\widehat{w}$ for $0<k^2<\dfrac{R}{\sqrt{2}}$. We do not see how to overcome this technical difficulty at the moment. To get a better R growth for the perturbations using our estimates, we could define a weighted norm $\|\cdot\|_R$, scaling the components of $\widehat{\mathbf{u}}$ in the obvious way

$$\|\widehat{\mathbf{u}}\|_R^2 := \frac{1}{R^2}\|\widehat{u}\|^2 + \frac{1}{R^2}\|\widehat{v}\|^2 + \|\widehat{w}\|^2. \tag{3.76}$$

For this norm, we have

$$\|\widehat{\mathbf{u}}\|_R^2 \leq CR^2\|\widehat{\mathbf{F}}\|^2, \tag{3.77}$$

but this does not imply a resolvent estimate, since we do not have the same norms on both sides of the inequality.

4 Numerical results

The only part of the argument that relies on numerical computations are the estimates for ϕ_1 and ϕ_2, solutions of

$$\begin{gathered}
\frac{1}{R}\phi_1'''' - \Big(i\xi + \frac{2k^2}{R} + ik_2 z\Big)\phi_1'' + \Big(i\xi k^2 + \frac{k^4}{R} + ik_2k^2 z\Big)\phi_1 = 0 \\
\phi_1(k_1,k_2,0,\xi) = \phi_1(k_1,k_2,1,\xi) = 0 \\
\phi_1'(k_1,k_2,0,\xi) = 1 \\
\phi_1'(k_1,k_2,1,\xi) = 0,
\end{gathered}$$

and

$$\begin{gathered}
\frac{1}{R}\phi_2'''' - \Big(i\xi + \frac{2k^2}{R} + ik_2 z\Big)\phi_2'' + \Big(i\xi k^2 + \frac{k^4}{R} + ik_2k^2 z\Big)\phi_2 = 0 \\
\phi_2(k_1,k_2,0,\xi) = \phi_2(k_1,k_2,1,\xi) = 0 \\
\phi_2'(k_1,k_2,0,\xi) = 0 \\
\phi_2'(k_1,k_2,1,\xi) = 1,
\end{gathered}$$

for the parameter range

$$\begin{gathered}
(k_1,k_2) \in \mathbb{Z}\ ,\ 0 < k^2 = k_1^2 + k_2^2 < \frac{R}{\sqrt{2}} \\
\xi \in \mathbb{R}\ ,\ 0 \leq |\xi| < 2\big(1+\sqrt{R}\big).
\end{gathered} \tag{4.1}$$

We solved these problems using the MATLAB boundary value problem solver BVP4C, which makes use of a collocation method. For each value of R, we calculate the maximum of $|k|\|\phi_1(k_1,k_2,\cdot,\xi)\|^2$, $\|\phi_1'(k_1,k_2,\cdot,\xi)\|^2$, $\|\phi_2(k_1,k_2,\cdot,\xi)\|$, $\|\phi_2'(k_1,k_2,\cdot,\xi)\|$ for the parameter range (4.1). The results, for values of R up to 10000, are shown in figures (1), (2), (3), (4). The numerical computations were performed with different absolute and relative tolerances, using continuation in the Reynolds number for the initial guess of the solution. The results were similar in all cases. Moreover, one just needs to assure that the values of the norms above are bounded. Therefore, even though the problem is stiff for some values of the parameters, the results should be reliable. They indicate that, for all R,

$$\begin{array}{ll}
|k|\|\phi_1(k_1,k_2,\cdot,s)\|^2 \leq 1 & |k|\|\phi_2(k_1,k_2,\cdot,s)\|^2 \leq 1 \\
\|\phi_1'(k_1,k_2,\cdot,s)\|^2 \leq 1 & \|\phi_2'(k_1,k_2,\cdot,s)\|^2 \leq 1,
\end{array}$$

for $0 < k^2 < \dfrac{R}{\sqrt{2}}$, $s = i\xi$, $\xi \in \mathbb{R}$, $0 \le |\xi| < 2(1 + \sqrt{R})$.

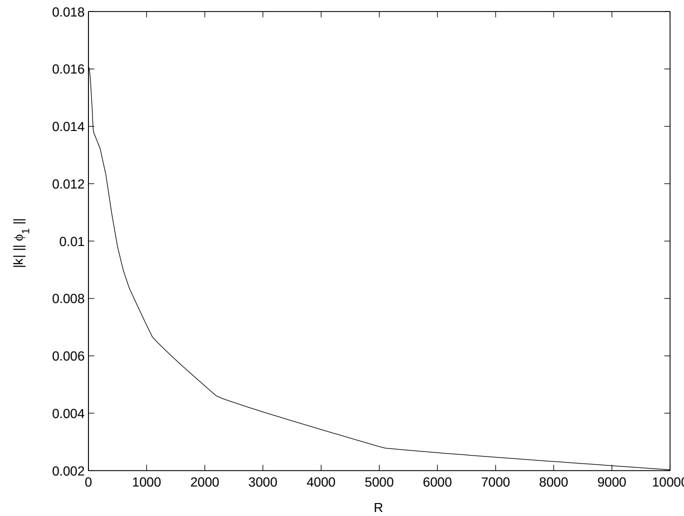

Figure 1: $\max_{k_1,k_2,\xi} |k| \|\phi_1(k_1, k_2, \cdot, i\xi)\|^2$ for $0 < k_1^2 + k_2^2 < \dfrac{R}{\sqrt{2}}$, $0 \le |\xi| < 2(1+\sqrt{R})$.

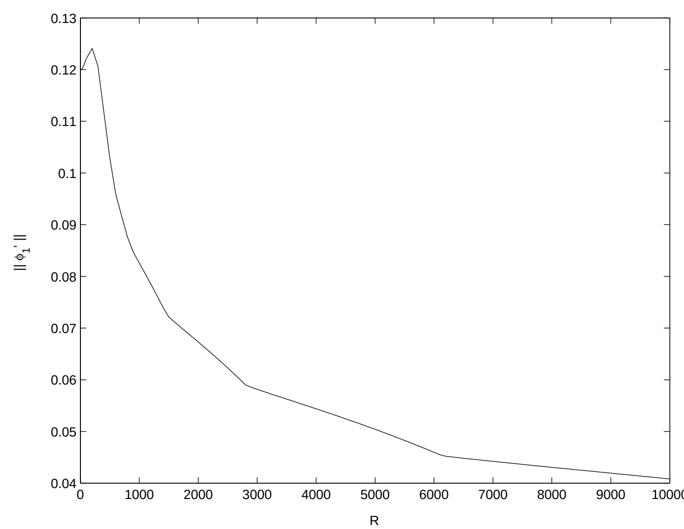

Figure 2: $\max_{k_1,k_2,\xi} \|\phi_1'(k_1, k_2, \cdot, i\xi)\|^2$ for $0 < k_1^2 + k_2^2 < \dfrac{R}{\sqrt{2}}$, $0 \le |\xi| < 2(1 + \sqrt{R})$.

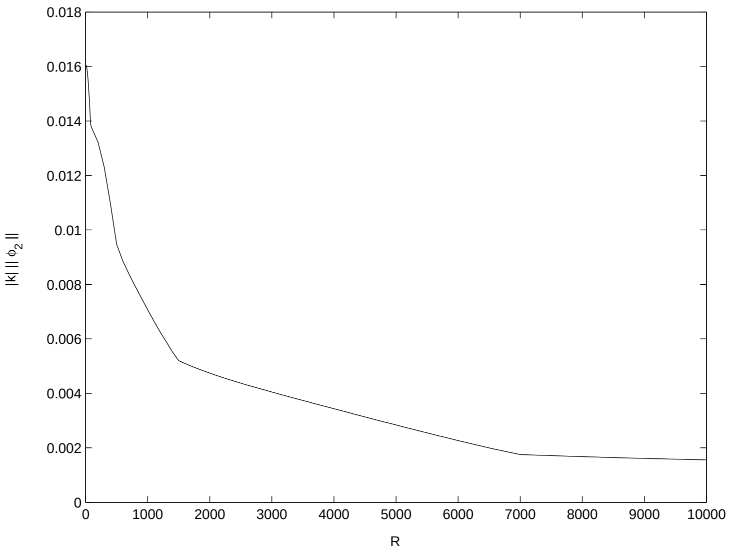

Figure 3: $\max_{k_1,k_2,\xi} |k|\|\phi_2(k_1,k_2,\cdot,i\xi)\|^2$ for $0<k_1^2+k_2^2<\dfrac{R}{\sqrt{2}}$, $0\le|\xi|<2(1+\sqrt{R})$.

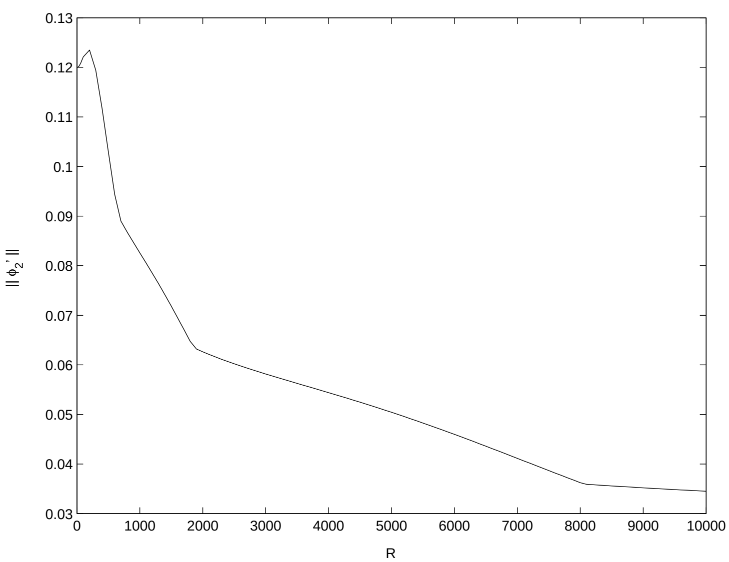

Figure 4: $\max_{k_1,k_2,\xi} \|\phi_2'(k_1,k_2,\cdot,i\xi)\|^2$ for $0<k_1^2+k_2^2<\dfrac{R}{\sqrt{2}}$, $0\le|\xi|<2(1+\sqrt{R})$.

5 Conclusions

The estimates derived here indicate the L_2 norm of the resolvent of the linear operator associated with 3 dimensional perturbations of plane Couette flow to be proportional to R^2 for the whole unstable half-plane $\mathrm{Re}(s) \geq 0$. They agree with previous numerical computations [5, 14]. In our argument though, numerical computations are used only to estimate the solutions of 4th order homogeneous linear ordinary differential equations, with nonhomogeneous boundary conditions. Deriving the estimates analytically for the entire unstable half-plane is an open problem, as far as we know. We believe that Theorem 6 may be useful towards a complete proof of the resolvent estimates. We hope to address this question in the future.

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