Duality and the Poincaré-Hopf inequalities

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Abstract

In this article we consider flows on locally compact manifolds and invariants sets which need not be compact. We obtain a duality result which is used to generalize the Poincaré-Hopf inequalities.

1 Introduction

In this paper we want to analyze the most general conditions under which the Poincaré-Hopf inequalities hold. These inequalities were presented in [1], [2] and [3]. It was shown in [1] that the Poincaré-Hopf inequalities hold for isolating blocks N with exit set N^- and entering set N^+ of a maximal isolated invariant set Λ of a continuous flow ϕ_t on a smooth compact manifold M. These inequalities relate the Betti numbers of the components of N^+ and N^- to the ranks of the $\mathbb{Z}_2$ Conley homology index of the isolated invariants sets of the flow.

The Conley index of (Λ, φ_t) is defined as the homotopy type of the space N/N^- and we consider

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its homology $H_*(N/N^-)$ which is isomorphic to $H_*(N, N^-)$.[1] One can also consider the Conley index for the reverse flow (Λ, φ_{-t}) and its homology $H_*(N/N^+)$ which is isomorphic to $H_*(N, N^+)$.

One might ask how these homologies $H_*(N, N^-)$ and $H_*(N, N^+)$ are related in general?[2] In [6] a Poincaré-Lefschetz duality result for the homology Conley index is proven: for N an orientable locally compact manifold with boundary. The Poincaré-Hopf inequalities for isolating blocks depend on a specific duality of the Conley homology index which implies that $\operatorname{rank} H_j(N, N^-) = \operatorname{rank} H_{n-j}(N, N^+)$.

In this article we prove a duality result, $\check{H}^p(N^*, (N^+)^*; \mathbb{Z}_2) \simeq H_{n-p}(N, N^-; \mathbb{Z}_2)$, in Theorem 2.3 for N a locally compact manifold and $\partial N = N^+ \sqcup N^-$ disjoint union, where N, N^+ and N^- need not be compact and N^* represents the Alexandroff one-point compactification of N.

Using this duality result we also show in Theorem 3.1 a generalized version of the Poincaré-Hopf inequalities which hold in a more general setting, we need only to assume that the phase space M be a manifold with finitely generated homology and cohomology groups. We consider a pair of spaces (N, N^-) of a maximal isolated invariant set Λ, $\Lambda = \operatorname{inv}(N, \varphi) \subset \operatorname{int} N$. We may consider Λ a non compact maximal isolated invariant set, where $\partial N = N^+ \coprod N^-$ where N^+ and N^- are the entering and exit sets for the flow respectively and may be empty. These inequalities relate the Betti numbers of the components of N^+ and N^- to the ranks of the $\mathbb{Z}_2$-homology of the pair, $H_*(N, N^-; \mathbb{Z}_2)$ as in Section 3, which in the compact case is the rank of the classical Conley homology index.

2 Poincaré duality Results

In this section we prove a duality result $\check{H}^p(N^*, (N^+)^*; \mathbb{Z}_2) \simeq H_{n-p}(N, N^-; \mathbb{Z}_2)$.

We begin by presenting a version of the classical Poincaré duality theorem considering homology and cohomology with compact supports.

Theorem 2.1 (Poincaré duality for manifolds with boundary) *Let N be an oriented n-manifold with boundary ∂N. Then the homomorphisms $H^q_c(N) \to H_{n-q}(N, \partial N)$ and $H^q_c(N - \partial N) \to H_{n-q}(N)$ defined by $x \to \mu \cap x$ and $y \to i_*(\mu \cap y)$ respectively are isomorphisms (here $i_* : H_{n-q}(N - \partial N) \to H_{n-q}(N)$ is the homomorphism induced by inclusion).*

For a connected nonorientable n-manifold, $H_n(N, Z) = 0$ hence these theorems can not be

[1] Namely for flows on locally compact manifolds M where neither M nor the maximal invariant sets need be compact.

[2] Throughout this article we use $\mathbb{Z}_2$ coefficients.

true as stated in the nonorientable case. However, by using homology and cohomology mod 2 it is possible to state a weaker form of Poincaré duality which is true for all manifolds, whether orientable or not. For a manifold with boundary, the mod 2 fundamental class is defined to be that of its interior. With this understanding, Theorem 2.1 remains true if the word "oriented" is deleted from the hypothesis.

The following results will be used to prove Theorem 2.3.

Proposition 2.1 (Alexander duality) *Let (X, A) a compact pair such that $X - A$ is a topological manifold. Then $\check{H}^p(X, A) \simeq H_{n-p}(X - A)$, where $\check{H}^*(\cdot)$ represents the Čech homology.*

Remark 2.2 *If N is a manifold then $H_p(N) \simeq H_p(N - \partial N)$.*

Lemma 2.1 *Let N be a manifold with boundary ∂N. Let N^*, $(\partial N)^*$ be the Alexandroff one-point compactification. Then $\check{H}^p(N^*, (\partial N)^*)$ is isomorphic to $H_{n-p}(N)$.*

Proof: By Proposition 2.1 $\check{H}^p(N^*, (\partial N)^*))$ is isomorphic to $H_{n-p}(N - \partial N)$. Since N is a manifold, $H_{n-p}(N - \partial N)$ is isomorphic to $H_{n-p}(N)$. ∎

Lemma 2.2 *Let N be a manifold with boundary $\partial N = N^+ \coprod N^-$ closed. Then $\check{H}^p(N^*, (N^+)^* \cap (N^-)^*)$ is isomorphic to $H_{n-p}(N, \partial N)$.*

Proof: We have that $\check{H}^p(N^*, (N^+)^* \cap (N^-)^*)$ is isomorphic to $\check{H}^p(N^*, point)$. Also, since (X, A) is a locally compact pair then $\check{H}^p_c(X - A)$ is isomorphic to $\check{H}^p_c(X, A)$. Hence, $\check{H}^p(N^*, point)$ is isomorphic to $H^p_c(N)$ (Note that $N^* - point = N$). Using Theorem 2.1 we have that $H^p_c(N)$ is isomorphic to $H_{n-p}(N, \partial N)$. ∎

Theorem 2.3 *Suppose N is a connected manifold with $\partial N = N^+ \coprod N^-$, the disjoint union of two closed spaces. Then*

$$\check{H}^p(N^*, (N^+)^*) \xrightarrow{\approx} H_{n-p}(N, N^-)$$

and

$$\check{H}^p(N^*, (N^-)^*) \xrightarrow{\approx} H_{n-p}(N, N^+).$$

Proof: Using Mayer-Vietoris we have:

$$\begin{array}{ccccccccc}\to & \check{H}^p(N^*,(N^+)^*\coprod(N^-)^*) & \to & \check{H}^p(N^*,(N^+)^*) & \oplus & \check{H}^p(N^*,(N^-)^*) & \to & \check{H}^p(N^*,(N^+)^*\cap(N^-)^*) & \to \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \to & H_{n-p}(N) & \to & H_{n-p}(N,N^-) & \oplus & H_{n-p}(N,N^+) & \to & H_{n-p}(N,N^+\coprod N^-) & \to\end{array}$$

Using Lemmas 2.1 and 2.2 the first and the last arrow are isomorphism.

By the Five Lemma we have that

$$\check{H}^p(N^*,(N^+)^*) \xrightarrow{\approx} H_{n-p}(N,N^-)$$

and

$$\check{H}^p(N^*,(N^-)^*) \xrightarrow{\approx} H_{n-p}(N,N^+)$$

are isomorphism.

Observe that the middle arrows are constructed in this way because of the naturality of the cap product. ■

The following result is a simpler version of Theorem 2.1 and can be used to prove Corollary 2.1.

Proposition 2.2 *Let $(N,\partial N)$ be a compact orientable n-manifold with fundamental class $\mu \in H_n(N,\partial N)$. Then the duality maps*

$$H^k(N,\partial N) \to H_{n-k}(N) \text{ and } H^k(N) \to H_{n-k}(N,\partial N)$$

given by taking the cap product with μ, are both isomorphisms.

Corollary 2.1 *Suppose N is a compact connected topological space with $\partial N = N^+ \cup N^-$, the disjoint union of two closed codimension one subspaces. If $\mu \in H_n(N,\partial N)$ is a fundamental class, then there is a suitably defined cap product which yields an isomorphism*

$$H^{n-p}(N,N^+) \xrightarrow{\approx} H_p(N,N^-)$$

given by capping with μ.

Proof: The proof is similar by considering Proposition 2.2. ■

Theorem 2.4 *Given a non compact manifold N with boundary $\partial N = N^+ \sqcup N^-$ we have that*

$$h_{n-p}(N^*, (N^-)^*) = h_{n-p}(N, N^-)$$

and

$$h_{n-p}(N^*, (N^+)^*) = h_{n-p}(N, N^+),$$

where N^* is the Alexandroff one-point compactification of N and $h_{n-p}(N, N^\pm) = \operatorname{rank} H(N, N^\pm)$.

Proof: By Theorem 2.3 we have that

$$\check{H}^p(N^*, (N^+)^*) \xrightarrow{\approx} H_{n-p}(N, N^-)$$

and

$$\check{H}^p(N^*, (N^-)^*) \xrightarrow{\approx} H_{n-p}(N, N^+)$$

are isomorphisms. Using the Universal Coefficient Theorem we have that

$$h_p(N^*, (N^+)^*) = \operatorname{rank} \check{H}^p(N^*, (N^+)^*) \simeq \operatorname{rank} H_{n-p}(N, N^-) = h_{n-p}(N, N^-)$$

and

$$h_p(N^*, (N^-)^*) = \operatorname{rank} \check{H}^p(N^*, (N^-)^*) \simeq \operatorname{rank} H_{n-p}(N, N^+) = h_{n-p}(N, N^+).$$

Since N^* is compact, we have that

$$h_{n-p}(N^*, (N^-)^*) = h_p(N^*, (N^+)^*)$$

and

$$h_{n-p}(N^*, (N^+)^*) = h_p(N^*, (N^-)^*).$$

Hence,

$$h_{n-p}(N^*, (N^-)^*) = h_p(N^*, (N^+)^*) = h_{n-p}(N, N^-)$$

and

$$h_{n-p}(N^*, (N^+)^*) = h_p(N^*, (N^-)^*) = h_{n-p}(N, N^+).$$

This means that given a non compact manifold N with boundary $\partial N = N^+ \sqcup N^-$, we have that

$$h_{n-p}(N^*, (N^-)^*) = h_{n-p}(N, N^-)$$

and

$$h_{n-p}(N^*, (N^+)^*) = h_{n-p}(N, N^+).$$

where N^* is the Alexandroff one-point compactification of N. ■

3 Poincaré-Hopf inequalities

In this section we present the Poincaré-Hopf inequalities for a pair of spaces (N, N^-) of a maximal isolated invariant set Λ for a flow defined on a locally compact manifold M, where N may be a non compact space, N^- the exit set for the flow and N^+ the exit set for the reverse flow are closed spaces. These inequalities were discussed in [1] for isolating blocks which are compact manifolds with boundary and were obtained for closed smooth manifolds M in [2] and [3].

By considering the duality results developed in the previous section we get the duality of the relative homology even in the non compact case. This means that $h_j = \operatorname{rank} H_j(N, N^-)$ is the same as $h_{n-j} = \operatorname{rank} H_j(N, N^+)$. Because of this duality the Poincaré-Hopf inequalities for non compact manifolds with boundary contains the Poincaré-Hopf inequalities for compact manifolds (isolating blocks). This can be easily seen by using the Poincaré duality of the Betti numbers of N^+ and N^- in the compact case, i.e., $B_j^{\pm} = B_{n-j-1}^{\pm}$.

Theorem 3.1 (Poincaré-Hopf Inequalities) *Let* $h_j = \operatorname{rank} H_j(N, N^-)$, $h_{n-j} \operatorname{rank} H_j(N, N^+)$ *and* $\operatorname{rank}(H_j(N^{\pm})) = B_j^{\pm}$. *The Poincaré-Hopf inequalities for a maximal invariant set* Λ *in* N *with entering set for the flow* N^+ *and exiting set for the flow* N^- *are:*

Even case, $n = 2i$.

$$
\left\{
\begin{array}{l}
\left\{ \; -(B^+_{n-1} - B^-_{n-1}) + (B^+_{n-2} - B^-_{n-2}) - \ldots \pm (B^+_i - B^-_i) \pm \ldots \pm (B^+_0 - B^-_0) = 0 \right. \\
\\
\left\{ \begin{array}{l} h_1 \geq -(B^+_{n-1} - B^-_{n-1}) - (h_n - h_0)) \\ h_{n-1} \geq -[-(B^+_{n-1} - B^-_{n-1}) - (h_n - h_0)] \end{array} \right. \\
\\
\left\{ \begin{array}{l} h_2 \geq (B^+_{n-1} - B^-_{n-1}) - (B^+_{n-2} - B^-_{n-2}) - (h_{n-1} - h_1) + (h_n - h_0) \\ h_{n-2} \geq -[(B^+_{n-1} - B^-_{n-1}) - (B^+_{n-2} - B^-_{n-2}) - (h_{n-1} - h_1) + (h_n - h_0)] \end{array} \right. \\
\\
\qquad\qquad \vdots \\
\\
\left\{ \begin{array}{l} h_i \geq -(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \pm (B^+_0 - B^-_0) \\ \qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \ldots \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm (h_{2i} - h_0) \\ \\ h_i \geq - \big[-(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \pm (B^+_0 - B^-_0) \\ \qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \ldots \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm (h_{2i} - h_0)] \end{array} \right. \\
\\
\qquad\qquad \vdots \\
\\
\left\{ \begin{array}{l} h_j \geq -(B^+_{j-1} - B^-_{j-1}) + (B^+_{j-2} - B^-_{j-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \pm (B^+_0 - B^-_0) \\ \qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \ldots \pm (h_{n-1} - h_1) \pm (h_n - h_0) \\ \\ h_{n-j} \geq - \big[-(B^+_{j-1} - B^-_{j-1}) + (B^+_{j-2} - B^-_{j-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \pm (B^+_0 - B^-_0) \\ \qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \ldots \pm (h_{n-1} - h_1) \pm (h_n - h_0) +] \end{array} \right. \\
\\
\qquad\qquad \vdots \\
\\
\left\{ \begin{array}{l} h_2 \geq -(B^+_1 - B^-_1) + (B^+_0 - B^-_0) - (h_{n-1} - h_1) + (h_n - h_0) + \\ h_{n-2} \geq - \big[-(B^+_1 - B^-_1) + (B^+_0 - B^-_0) - (h_{n-1} - h_1) + (h_n - h_0) \big] \end{array} \right. \\
\\
\left\{ \begin{array}{l} h_1 \geq -(B^+_0 - B^-_0) - (h_6 - h_0) \\ h_{n-1} \geq -[-(B^+_0 - B^-_0) - (h_6 - h_0)] \end{array} \right.
\end{array}
\right. \tag{1}
$$

Odd case, $n = 2i + 1$.

$$
\left\{
\begin{array}{l}
\left\{ \; -(B^+_{n-1} - B^-_{n-1}) + (B^+_{n-2} - B^-_{n-2}) - \ldots \pm (B^+_i - B^-_i) \pm \ldots \pm (B^+_0 - B^-_0) = 2\sum_{j=0}^{2i+1} (-1)^{j+1} h_j \right. \\[2ex]
\left\{ \begin{array}{l} h_1 \geq -(B^+_{n-1} - B^-_{n-1}) - (h_n - h_0)) \\ h_{n-1} \geq -[-(B^+_{n-1} - B^-_{n-1}) - (h_n - h_0)] \end{array} \right. \\[2ex]
\left\{ \begin{array}{l} h_2 \geq (B^+_{n-1} - B^-_{n-1}) - (B^+_{n-2} - B^-_{n-2}) - (h_{n-1} - h_1) + (h_n - h_0) \\ h_{n-2} \geq -[(B^+_{n-1} - B^-_{n-1}) - (B^+_{n-2} - B^-_{n-2}) - (h_{n-1} - h_1) + (h_n - h_0)] \end{array} \right. \\
\qquad\qquad \vdots \\
\left\{ \begin{array}{l} h_i \geq -(B^+_{2i} - B^-_{2i}) + (B^+_{2i-1} - B^-_{2i-1}) + - \ldots \pm (B^+_{i+1} - B^-_{i+1}) \\ \qquad - (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i-1} - h_2) \pm (h_{2i} - h_1) \pm (h_{2i+1} - h_0) \\[2ex] h_{i+1} \geq - \left[-(B^+_{2i} - B^-_{2i}) + (B^+_{2i-1} - B^-_{2i-1}) + - \ldots \pm (B^+_{i+1} - B^-_{i+1}) \right. \\ \qquad \left. - (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i-1} - h_2) \pm (h_{2i} - h_1) \pm (h_{2i+1} - h_0) \right] \end{array} \right. \\[2ex]
\left\{ \begin{array}{l} h_i \geq -(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \pm (B^+_0 - B^-_0) \\ \qquad - (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i-1} - h_2) \pm (h_{2i} - h_1) \pm (h_{2i+1} - h_0) \\[2ex] h_{i+1} \geq - \left[-(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \pm (B^+_0 - B^-_0) \right. \\ \qquad \left. - (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i-1} - h_2) \pm (h_{2i} - h_1) \pm (h_{2i+1} - h_0) \right] \end{array} \right. \\
\qquad\qquad \vdots \\
\left\{ \begin{array}{l} h_j \geq -(B^+_{j-1} - B^-_{j-1}) + (B^+_{j-2} - B^-_{j-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \pm (B^+_0 - B^-_0) \\ \qquad -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \ldots \pm (h_{n-1} - h_1) \pm (h_n - h_0) \\[2ex] h_{n-j} \geq - \left[-(B^+_{j-1} - B^-_{j-1}) + (B^+_{j-2} - B^-_{j-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \pm (B^+_0 - B^-_0) \right. \\ \qquad \left. -(h_{n-(j-1)} - h_{j-1}) + (h_{n-(j-2)} - h_{j-2}) + - \ldots \pm (h_{n-1} - h_1) \pm (h_n - h_0) + \right] \end{array} \right. \\
\qquad\qquad \vdots \\
\left\{ \begin{array}{l} h_2 \geq -(B^+_1 - B^-_1) + (B^+_0 - B^-_0) - (h_{n-1} - h_1) + (h_n - h_0) + \\ h_{n-2} \geq - \left[-(B^+_1 - B^-_1) + (B^+_0 - B^-_0) - (h_{n-1} - h_1) + (h_n - h_0) \right] \end{array} \right. \\[2ex]
\left\{ \begin{array}{l} h_1 \geq -(B^+_0 - B^-_0) - (h_6 - h_0) \\ h_{n-1} \geq -[-(B^+_0 - B^-_0) - (h_6 - h_0)] \end{array} \right.
\end{array}
\right. \tag{2}
$$

The Poincaré-Hopf inequality in the case $n = 2$ is

$$\left\{ h_1 - h_2 - h_0 + 2 - (B_0^+ + B_0^-) \geq 0 \quad \text{and even.}^3 \right. \tag{3}$$

Proof: Note that (N, N^-) is a pair for Λ and (N, N^+) is a pair for the isolated invariant set of the reverse flow, Λ' where N^- and N^+ are the exit sets for the flow φ_t and the reverse flow φ_{-t}.

Consider the long exact sequences for the pairs (N, N^-) and (N, N^+):

$$\begin{gathered} 0 \to H_n(N^-) \xrightarrow{i_n} H_n(N) \xrightarrow{p_n} H_n(N, N^-) \xrightarrow{\partial_n} H_{n-1}(N^-) \xrightarrow{i_{n-1}} H_{n-1}(N) \xrightarrow{p_{n-1}} \\ \to H_{n-1}(N, N^-) \xrightarrow{\partial_{n-1}} H_{n-2}(N^-) \xrightarrow{i_{n-2}} H_{n-2}(N) \xrightarrow{p_{n-2}} H_{n-2}(N, N^-) \xrightarrow{\partial_{n-2}} \dots \\ \xrightarrow{\partial_4} H_3(N^-) \xrightarrow{i_3} H_3(N) \xrightarrow{p_3} H_3(N, N^-) \xrightarrow{\partial_3} H_2(N^-) \xrightarrow{i_2} H_2(N) \xrightarrow{p_2} H_2(N, N^-) \xrightarrow{\partial_2} \\ \to H_1(N^-) \xrightarrow{i_1} H_1(N) \xrightarrow{p_1} H_1(N, N^-) \xrightarrow{\partial_1} H_0(N^-) \xrightarrow{i_0} H_0(N) \xrightarrow{p_0} H_0(N, N^-) \to 0 \end{gathered} \tag{4}$$

$$\begin{gathered} 0 \to H_n(N^+) \xrightarrow{i'_n} H_n(N) \xrightarrow{p'_n} H_n(N, N^+) \xrightarrow{\partial'_n} H_{n-1}(N^+) \xrightarrow{i'_{n-1}} H_{n-1}(N) \xrightarrow{p'_{n-1}} \\ \to H_{n-1}(N, N^+) \xrightarrow{\partial'_{n-1}} H_{n-2}(N^+) \xrightarrow{i'_{n-2}} H_{n-2}(N) \xrightarrow{p'_{n-2}} H_{n-2}(N, N^+) \xrightarrow{\partial'_{n-2}} \dots \\ \xrightarrow{\partial'_4} H_3(N^+) \xrightarrow{i'_3} H_3(N) \xrightarrow{p'_3} H_3(N, N^+) \xrightarrow{\partial'_3} H_2(N^+) \xrightarrow{i'_2} H_2(N) \xrightarrow{p'_2} H_2(N, N^+) \xrightarrow{\partial'_2} \\ H_1(N^+) \xrightarrow{i'_1} H_1(N) \xrightarrow{p'_1} H_1(N, N^+) \xrightarrow{\partial'_1} H_0(N^+) \xrightarrow{i'_0} H_0(N) \xrightarrow{p'_0} H_0(N, N^+) \to 0 \end{gathered} \tag{5}$$

It is an elementary result that given a long exact sequence of vector spaces,

$$\xrightarrow{h} A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \dots\dots \to D \to 0$$

rank $\operatorname{Im} h$ + rank $\operatorname{Im} i$ = rank A. This follows from the fact that $\operatorname{rank} A = \operatorname{rank} \operatorname{Im} i + \operatorname{rank} \ker i$ and from the exactness of the sequence $\ker i = \operatorname{Im} h$. Hence,

$$\operatorname{rank} h = \operatorname{rank} A - \operatorname{rank} B + \operatorname{rank} C - + \dots \pm \operatorname{rank} D \geq 0.$$

[3]If M is non-orientable the Poincaré-Hopf inequality is the same, however the expression on the right hand side of the inequality (3) need not be even.

Applying these arguments to the long exact sequences of the pairs (N, N^-) and (N, N^+), and considering that $H_n(N) = H_n(N^\pm) = 0$, $\operatorname{rank} H_j(N, N^-) = h_j$, $\operatorname{rank} H_j(N, N^+) = h_{n-j}$ and $\operatorname{rank}(H_j(N^\pm)) = B_j^\pm$. Hence, the following equation is obtained:

$$\begin{aligned}\operatorname{rank} p_n = h_n - B_{n-1}^- + \operatorname{rank}(H_{n-1}(N)) - h_{n-1} + B_{n-2}^- - \operatorname{rank}(H_{n-2}(N)) + h_{n-2} \dots \\ \pm B_3^- \mp \operatorname{rank}(H_3(N)) \pm h_3 \mp B_2^- \pm \operatorname{rank}(H_2(N)) \mp h_2 \\ \pm B_1^- \mp \operatorname{rank}(H_1(N)) \pm h_1 \mp B_0^- \pm \operatorname{rank}(H_0(N)) \mp h_0 \qquad (6)\end{aligned}$$

The first equality is obtained by analyzing p_n and p'_n and we refer to it as the top equality.

Since $\operatorname{rank} p_n = 0$, it follows from (6) that:

$$B_0^- - B_1^- + B_2^- - + \dots \pm B_{i-1}^- \pm B_i^- \pm \dots \pm B_{n-1}^- - h_n + h_{n-1} - h_{n-2} + - \dots - h_3 + h_2 - h_1 + h_0 =$$

$$= \sum_{j=0}^{n-1} (-1)^j \operatorname{rank}(H_j(N)) \qquad (7)$$

Similarly, using the long exact sequence of the pair (N, N^+) and using the duality of the relative homology, $h_j(\Lambda) = h_{n-j]}(\Lambda')$ the following equation holds:

$$B_0^+ - B_1^+ + B_2^+ - + \dots \pm B_{i-1}^+ \pm B_i^+ \pm \dots \pm B_{n-1}^+ - h_0 + +h_1 - h_2 + - \dots - h_{n-2} + h_{n-1} - h_n =$$

$$= \sum_{j=0}^{n-1} (-1)^j \operatorname{rank}(H_j(N)) \qquad (8)$$

Subtracting (7) from (8), the following equation holds when $n = 2i + 1$:

$$(B_0^+ - B_0^-) - (B_1^+ - B_1^-) + - \dots \pm (B_{i-1}^+ - B_{i-1}^-) \pm (B_i^+ - B_i^-) \pm \dots (B_{2i}^+ - B_{2i}^-)$$

$$+2h_{2i+1} - 2h_{2i} + 2h_{2i-1} - + \dots \pm h_i \pm \dots - 2h_2 + 2h_1 - 2h_0 = 0 \qquad (9)$$

Subtracting (7) from (8), the following equation holds when $n = 2i$:

$$-(B_{2i-1}^+ - B_{2i-1}^-) + (B_{2i-2}^+ - B_{2i-2}^-) - \dots \pm (B_i^+ - B_i^-) \pm \dots \pm (B_0^+ - B_0^-) = 0,$$

since the h_j's cancel.

The inequalities will be obtained by analyzing two pairs of maps of (4) and (5) at a time,

$$\left\{\left[\left(p_{n-1},\partial'_{n-1}\right),\left(p'_{n-1},\partial_{n-1}\right)\right],\ldots\left[\left(p_1,\partial'_1\right),\left(p'_1,\partial_1\right)\right]\right\}.$$

The first half of the set of inequalities are further simplified by using the top equality. Note that the analysis of these pairs provide the inequalities in both the odd-dimensional case, $n=2i+1$, as well as the even dimensional case, $n=2i$. The analysis of these pairs of maps in the long exact sequence is always the same and for clarity we will describe it in the middle dimensional cases. In the mid-dimension the analysis should be divided in two cases, $n=2i+1$ and $n=2i$.

Middle dimensional analysis, $n=2i+1$

We analyze the middle dimensional inequalities by considering the ranks of p_i in (4). The following holds:

$$\operatorname{rank} p_i = h_i - B^-_{i-1} + \operatorname{rank} H_{i-1}(N) - h_{i-1} + B^-_{i-2} - \operatorname{rank} H_{i-2}(N) + h_{i-2} - + \ldots$$

$$\pm B^-_2 \mp \operatorname{rank} H_2(N) \pm h_2 \mp B^-_1 \pm \operatorname{rank} H_1(N) \mp h_1 \pm B^-_0 \mp \operatorname{rank} H_0(N) \pm h_0 \geq 0$$

$$\Rightarrow h_i \geq B^-_{i-1} - B^-_{i-2} + - \ldots \pm B^-_2 \mp B^-_1 \pm B^-_0 + h_{i-1} - h_{i-2} + - \ldots \pm h_2 \mp h_1 \pm h_0$$

$$-\operatorname{rank} H_{i-1}(N) + \operatorname{rank} H_{i-2}(N) - + \ldots \pm \operatorname{rank} H_2(N) \mp \operatorname{rank} H_1(N) \pm \operatorname{rank} H_0(N) \tag{10}$$

Similarly, by considering rank ∂'_i in (5), the following inequality holds:

$$-\operatorname{rank} H_{i-1}(N) + \operatorname{rank} H_{i-2}(N) - + \ldots \pm \operatorname{rank} H_2(N) \mp \operatorname{rank} H_1(N) \pm \operatorname{rank} H_0(N) \geq$$

$$-B^+_{i-1} + B^+_{i-2} - + \ldots \pm B^+_2 \mp B^+_1 \pm B^+_0 - h_{i+2} - + \ldots \mp h_{2i-1} \pm h_{2i} \mp h_{2i+1} \tag{11}$$

Substituting (11) in (10) the following inequality holds:

$$h_i \geq -\left(B^+_{i-1} - B^-_{i-1}\right) + \left(B^+_{i-2} - B^-_{i-2}\right) + - \ldots \pm \left(B^+_2 - B^-_2\right) \mp \left(B^+_1 - B^-_1\right) \pm \left(B^+_0 - B^-_0\right)$$

$$-\left(h_{i+2} - h_{i-1}\right) + \left(h_{i+3} - h_{i-2}\right) + - \ldots \pm \left(h_{2i-1} - h_2\right) \mp \left(h_{2i} - h_1\right) \pm \left(h_{2i+1} - h_0\right) \tag{12}$$

Analogously analyzing p'_i and ∂_i as above and using the duality of the indices, the following inequality is obtained:

$$h_{i+1} \geq -\left[-\left(B^+_{i-1} - B^-_{i-1}\right) + \left(B^+_{i-2} - B^-_{i-2}\right) + - \ldots \pm \left(B^+_2 - B^-_2\right) \mp \left(B^+_1 - B^-_1\right) \pm \left(B^+_0 - B^-_0\right)\right.$$

$$- (h_{i+2} - h_{i-1}) + (h_{i+3} - h_{i-2}) + - \ldots \pm (h_{2i-1} - h_2) \mp (h_{2i} - h_1) \pm (h_{2i+1} - h_0)] \qquad (13)$$

Note that in equation (12) h_i is greater than or equal to an integer number and in equation (13), h_{i+1} is greater than or equal to the opposite of that number. Obviously, since h_i and h_{i+1} are non-negative integers, one of the inequalities is redundant.

Middle dimensional analysis, $n = 2i$.

In this section, the inequalities are obtained in the same fashion, that is by analyzing the long exact sequences of the pairs $(N, N^{\pm})$ where $\dim N = 2i$. Hence we obtained

$$\begin{cases} h_i \geq -(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \pm (B^+_0 - B^-_0) \\ \qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \ldots \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm (h_{2i} - h_0) \\ \\ h_i \geq - \big[-(B^+_{i-1} - B^-_{i-1}) + (B^+_{i-2} - B^-_{i-2}) + - \ldots \pm (B^+_2 - B^-_2) \pm (B^+_1 - B^-_1) \pm (B^+_0 - B^-_0) \\ \qquad -(h_{i+1} - h_{i-1}) + (h_{i+2} - h_{i-2}) + - \ldots \pm (h_{2i-2} - h_2) \pm (h_{2i-1} - h_1) \pm (h_{2i} - h_0)] \end{cases} \qquad (14)$$

Similarly one of these inequalities is redundant.

In the case $n = 2$ the Poincaré-Hopf inequality (3) in the case $n = 2$ appears in [1] and [5]. ■

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