Existence of Weak Efficient Solutions in Vector Optimization

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Abstract

In this paper, we present an existence result for weak efficient solution for vector optimization problem. The result is stated for invex strongly compactly Lipschitz functions.

Key words: Vector optimization, Efficiency, Generalized convexity, Clarke generalized gradient.

1 Introduction

In the last year many efforts has been done to study the vectorial optimization problem (for short: (VP)). The works devoted to this problems include questions such as existence of solutions, necessary and sufficient conditions for optimality. Traditionally, these questions have been treated for smooth functions defined between finite-dimensional spaces (see for instance,[15], [3]). Nowadays, this problem have been studied for the case where the functions are nonsmooth and are defined between infinite-dimensional spaces (see for instance [2], [4], [8], [11] and references therein.

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We have interest in to study the existence of solutions for non-convex, non-smooth functions defined between infinite dimensional Banach spaces.

For the achieve this objective, we will consider the strongly compactly Lipschitz functions. This definition was introduced by Thibault [17] for the study of necessary conditions in vectorial optimization problem [8], [18] and will permit us extended the non-smooth analysis of Clarke [4] for vector functions between abstract spaces see [17].

Our objective in this work is to establish the existence of weakly efficient solutions for (VP). To done this, we will make some hypotheses of generalized convexity on f. We will prove that, when f is invex and strongly compactly Lipschitz, the solutions of (VP) coincident with the solutions of a variational like inequality and using this characterization together with KKM-Fan Theorem, we prove the existence of solutions for Problem (VP). The results obtained generalized those obtain early by Chen and Craven [2] and Kazmi [11].

The paper has the following structure: in Section 2 we fix some basic notation and terminology, in Section 3 we will establish our main result.

2 Preliminaries

Let X, Y be a two real Banach spaces. We denote by $\|\cdot\|$ the norm in Y. Let $K \subset X$ be a nonempty $P \subset Y$ a pointed convex cone such that $\text{int}P \neq \varnothing$. Let $f : X \to Y$ be a given function. We will consider the following vectorial optimization problem:

$$\left.\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & \\ & x \in K \end{array}\right\} \qquad \text{(VP)}$$

The notion of optimality that we will consider it is the weak efficiency: we say that $x_0 \in K$ is **weakly efficient solution** for (VP) iff

$$f(x) - f(x_0) \notin -\text{int}P, \ \forall x \in K.$$

Now, we recall some notions and results from nonsmooth analysis. The **Clarke generalized directional derivative** of a local Lipschitz function ϕ from X

into $\mathbb{R}$ at $\overline{x}$ in the direction d, denoted by $\phi^o(\overline{x}, d)$ (see [4]) is given by:

$$\phi^o(\overline{x}; d) = \limsup_{\substack{x \to \overline{x} \\ t \downarrow 0}} \frac{\phi(y + tv) - \phi(y)}{t}.$$

The **Clarke generalized gradient** of ϕ at $\overline{x}$ is given by

$$\partial\phi(\overline{x}) = \{x^* \in X^* : \phi^o(\overline{x}; d) \geq \langle x^*, d\rangle, \ \ \forall d \in X\},$$

where X^* denotes the topological dual of X. Let C be a nonempty subset of X and consider its **distance function**, that is, the function $\delta_C(\cdot) : X \to \mathbb{R}$ defined by

$$\delta_C(x) = \inf\{\|x - c\| : c \in C\}.$$

The distance function is not everywhere differentiable but is globally Lipschitz. Let $\overline{x} \in C$. A vector $d \in X$ is said to be *tangent* to C at $\overline{x}$ if $\delta_C^o(\overline{x}; d) = 0$. The set of tangent vectors to C at $\overline{x}$ is a closed convex cone in X, called the tangent cone to C at $\overline{x}$ and denoted by $T_C(\overline{x})$. By polarity, we define the *normal cone* to C at $\overline{x}$:

$$N_C(\overline{x}) := \{\xi \in X^* : \langle \xi, v\rangle \leq 0, \forall v \in T_C(\overline{x})\}.$$

We recall that $N_C(\overline{x})$ is a weak*-closed, convex cone.

Definition 2.1 *A mapping $h : X \to Y$ is said to be strongly compactly Lipschitz at $\overline{x} \in X$ if there exist a multifunction $R : X \to Comp(Y)$, where $Comp(Y)$ denotes the set of all norm compact subsets of Y, and a function $r : X \times X \to \mathbb{R}_+$ satisfying*

(i) $\lim_{x \to x, d \to 0} r(x, d) = 0$*;*
(ii) There exists $\alpha > 0$ such that

$$t^{-1}[h(x + td) - h(x)] \in R(d) + \|d\| r(x, t) B_Y,$$

for all $x \in x + \alpha B_Y$ and $t \in]0, \alpha[$ (Here B_Y denotes the closed unit ball around the origin of Y);
(iii) $R(0) = \{0\}$ and R is upper semicontinuous.

Remark 2.2 *If Y is finite-dimensional, then h is strongly compactly Lipschitzian at $\overline{x}$ if and only if it is locally Lipschitz near $\overline{x}$. If h is strongly compactly Lipschitz, then for all $u^* \in Y^*, (u^* \circ h)(x) = \langle u^*, h(x)\rangle$ is locally Lipschitz. For more details about strongly compactly Lipschitz mappings we refer the reader to [8].*

3 Existence of weakly efficient solutions

We recall some definitions of generalized convexity for functions between Banach spaces. Given the cone $P \subset Y$, we define the dual cone of P by

$$P^* := \{\xi \in Y^* : \langle \xi, x \rangle \geq 0, \forall x \in P\}.$$

Definition 3.1 (1) *(Phoung, Sach and Yen [14]) We say that the function $\theta : K \subset X \longrightarrow \mathbb{R}$ locally Lipschitz on K is* ***invex*** *respect to η on K if for each $x, y \in K$ there exists a vector $\eta(x_1, x_2) \in T_K(y)$ such that*

$$\theta(x) - \theta(y) - \theta^0(y, \eta(x, y)) \geq 0;$$

(2) *(Brandão, Rojas-Medar and Silva [1]) We say that the function $f : K \subset X \longrightarrow Y$ is P-* ***invex*** *respect to η on K if $\omega^* \circ f : K \longrightarrow \mathbb{R}$ is invex, for each $\omega^* \in P^*$.*

(3) *The function $f : K \subset X \longrightarrow Y$ is P-* ***preinvex*** *respect to η, if $\eta : K \times K \longrightarrow X$ is such that for each $\alpha \in (0, 1)$ and each $x, y \in K$ we have $\eta(x, y) \in T_K(y)$, $y + \alpha\eta(x, y) \in K$ and*

$$\alpha f(x) + (1 - \alpha) f(y) - f(y + \alpha\eta(x, y)) \in P.$$

When $y \in \text{int}K$ or, more generality, if K is open, we have $T_K(y) = X$ and in this case the definition above it is coincident with those given by Weir and Jeyakumar [16]. The set K is called **invex** with respect to η, if satisfied $y + \alpha\eta(x, y) \in K$ for any $x, y \in K$. We recall the following results:

Lemma 3.2 *If P is a closed, convex cone with nonempty interior $y \in Y$ is such that $\omega^*(y) \geq 0$ $\forall \omega^* \in P^*$, then $y \in P$. And, moreover, if $y \in intP$, then, $\omega^*(y) > 0$, $\forall \omega^* \in P^* \setminus \{0\}$.*

Lemma 3.3 *If P is a closed, convex cone with nonempty interior then $P^* \neq \{0\}$.*

The proof of the above Lemmas can be see in [5] and [10], respectively.

Proposition 3.4 *Let $f : K \subset X \longrightarrow Y$ a function strongly compactly Lipschitz on K. Si f is P-preinvex, then f is P-invex.*

PROOF. We would like to prove that, for each $\omega^* \in P^*$, The composition $\omega^* \circ f : K \longrightarrow \mathbb{R}$ is invex. Since f is P-preinvex, from Lemma 3.2, we obtain

$$\alpha(\omega^* \circ f)(x) + (1 - \alpha)(\omega^* \circ f)(\tilde{y}) - (\omega^* \circ f)(\tilde{y} + \alpha\eta(x, \tilde{y})) \geq 0 \qquad (2)$$

for each $\omega^* \in P^*$, $\alpha \in (0,1)$ and $x, \widetilde{y} \in K$. Since f is P-preinvex, $\eta(x,y) \in T_K(y), \forall y \in K$. From (2),

$$\frac{1}{\alpha}[\omega^* \circ f(\widetilde{y} + \alpha\eta(x,\widetilde{y})) - \omega^* \circ f(\widetilde{y})] \leq \omega^* \circ f(x) - \omega^* \circ f(\widetilde{y}). \qquad (3)$$

We fix $\varepsilon > 0$. Then:

$$\sup_{\substack{0<\left\|x-\widetilde{x}\right\|<\varepsilon \\ 0<\alpha<\varepsilon}} \frac{1}{\alpha}[\omega^* \circ f(\widetilde{y} + \alpha\eta(x,\widetilde{y})) - \omega^* \circ f(\widetilde{y})] \geq$$

$$\sup_{\substack{0<\left\|x-\widetilde{x}\right\|<\varepsilon \\ 0<\alpha<\varepsilon}} \frac{1}{\alpha}[\omega^* \circ f(\widetilde{y} + \alpha\eta(x,y)) - \omega^* \circ f(\widetilde{y})] \qquad (4)$$

and taking the limit $\varepsilon \downarrow 0$ in (4),

$$\limsup_{\substack{\alpha\downarrow 0 \\ \widetilde{y}\longrightarrow y}} \frac{1}{\alpha}[\omega^* \circ f(\widetilde{y} + \alpha\eta(x,\widetilde{y})) - \omega^* \circ f(\widetilde{y})] \geq (\omega^* \circ f)^0(y, \eta(x,y)) \quad (5)$$

Taking lim sup in (3) when $\widetilde{y} \longrightarrow y$ and $\alpha \downarrow 0$ and observing that $\omega^* \circ f$ is continuous, we obtain from (5) the following inequality

$$(\omega^* \circ f)^0(y, \eta(x,y)) \leq \omega^* \circ f(x) - \omega^* \circ f(\widetilde{y}).$$

As ω^* is arbitrary, f is P-invex. ■

Theorem 3.5 *Let $f : K \subset X \longrightarrow Y$ be function strongly compactly Lipschitzian P-invex and let K be a invex set respect to η. Then, all local weak efficient solution is global.*

PROOF. We show this Theorem by reduction ad absurd. We assume that there exists $x_0 \in K$ that is a local weak efficient solution, but that is not global. Then, there exists a neighborhood U of x_0 such that

$$f(x) - f(x_0) \notin -\text{int}P, \ \forall x \in U \cap K, \ x \neq x_0. \qquad (6)$$

If x_0 is not a global weak efficient solution, there exists $x \in K$ such that

$$f(x) - f(x_0) \in -\text{int}P.$$

¿From Lemma 3.2, we have

$$\omega^* \circ f(x) - \omega^* \circ f(x_0) < 0, \forall \omega^* \in P^* \setminus \{0\}$$

and since f is P-invex,

$$(\omega^* \circ f)^0(x_0, \eta(x, x_0)) < 0, \forall \omega^* \in P^* \setminus \{0\}$$

that is,

$$\lim_{\varepsilon \downarrow 0} \sup_{\substack{0<\alpha<\varepsilon \\ 0<\|y-x_0\|<\varepsilon}} \frac{(\omega^* \circ f)(y + \alpha\eta(x, x_0)) - (\omega^* \circ f)(y)}{\alpha} < 0.$$

Then, for $\omega^* \in P^* \setminus \{0\}$ and $\varepsilon > 0$ sufficiently small

$$(\omega^* \circ f)(x_0 + \alpha\eta(x, x_0)) - (\omega^* \circ f)(x_0) < 0 \tag{7}$$

for each $0 < \alpha < \varepsilon$. Moreover, $x := x_0 + \alpha\eta(x, x_0) \in U \cap K$ if α is sufficiently small. Then (6)

$$f(x) - f(x_0) \notin -\text{int}P$$

and, consequently, by using Lemma 3.2

$$\exists \omega^* \in P^* \setminus \{0\},\ \omega^* \circ f(x) - \omega^* \circ f(x_0) \geq 0$$

this is a contradiction with (7), because in (7), $\omega^* \in P^*$ is arbitrary. Thus, x_0 is a global weak efficient solution of (VP). ■

Now, we consider the following variational-like inequality: **Vectorial variational-like inequality**: To find $x_0 \in K$ such that for each $x \in K$ there exist $\omega^* \in P^* \setminus \{0\}$ such that

$$(\omega^* \circ f)^0(x_0, \eta(x, x_0)) \geq 0. \tag{VI}$$

We observe that under the hypotheses done on cone P, by using Lemma 3.3, the dual cone $P^* \neq 0$.

Theorem 3.6 *Let K be an invex set wit respect to η and $f : K \subset X \longrightarrow Y$ a function P-invex respect to η. Then, the vectorial optimization problem (VP) and the vectorial variational-like inequality (VI) have the same solutions.*

PROOF. Necessity. Suppose that $x_0 \in K$ is a weak efficient solution of (VP). Let $x \in K$. Since K is invex, $x_0 + \alpha\eta(x, x_0) \in K$, for each $\alpha > 0$ sufficiently small. Moreover, for these α

$$\frac{1}{\alpha}(f(x_0 + \alpha\eta(x, x_0)) - f(x_0)) \notin -\text{int}P.$$

Then, Lemma 3.2 implies that there exists $\omega^* \in P^* \setminus \{0\}$ such that

$$\frac{1}{\alpha}(\omega^* \circ f(x_0 + \alpha\eta(x, x_0)) - \omega^* \circ f(x_0)) \geq 0.$$

By other hand:

$$\begin{aligned}(\omega^* \circ f)^0(x_0, \eta(x, x_0)) &= \lim_{\varepsilon \downarrow 0} \sup_{\substack{0<\alpha<\varepsilon \\ 0<\left\|\widehat{x}-x_0\right\|<\varepsilon}} \frac{\omega^* \circ f(\widehat{x} + \alpha\eta(x, x_0)) - \omega^* \circ f(\widehat{x})}{\alpha} \\ &\geq \lim_{\varepsilon \downarrow 0} \sup_{0<\alpha<\varepsilon} \frac{\omega^* \circ f(x_0 + \alpha\eta(x, x_0)) - \omega^* \circ f(x_0)}{\alpha} \\ &\geq 0\end{aligned}$$

and, therefore, x_0 is a solution of (VI) . Sufficiency. We assume that x_0 is not a weak efficient solution of (VP) and that is a solution for inequality (VI) . Then, there exists $x \in K$ such that

$$f(x) - f(x_0) \in -\text{int}P.$$

By other hand, there exist $\omega^* \in P^* \setminus \{0\}$ such that $(\omega^* \circ f)^0(x_0, \eta(x, x_0)) \geq 0$. Since f is P -invex,

$$\omega^* \circ f(x) - \omega^* \circ f(x_0) \geq (\omega^* \circ f)^0(x_0, \eta(x, x_0)) \geq 0.$$

Nevertheless: Lemma 3.2 implies that $\omega^* \circ f(x) - \omega^* \circ f(x_0) < 0$. This is a contradiction with the above inequality. ■

Remark 3.7 *We observe that in the proof of Theorem 3.6, we have not used the fact that K is an invex set respect to η to conclude that the weak efficient solutions of (VP) are solutions of the inequality (VI) .*

The following Lemma will be very useful to obtain our main result:

Lemma 3.8 *(KKM-Fan Theorem [9]) Let X be a topological vectorial space, $E \subset X$ nonempty and $F : E \rightrightarrows X$ a set-valued mapping such that for each*

$x \in E$, the set $F(x)$ is closed and nonempty and, moreover, there exist $x \in E$ such that $F(x)$ is compact. If for each finite subset of E, $\{x_1, ..., x_n\}$, is satisfied

$$co\{x_1, ..., x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$$

then,

$$\bigcap_{x \in E} F(x) \neq \emptyset.$$

(where $co\{x_1, ..., x_n\}$ is the convex hull of $\{x_1, ..., x_n\}$).

Now, we give our existence result for (VP).

Theorem 3.9 *Let X be a reflexive Banach space and K a closed, convex, bounded subset of X. Let $f : K \longrightarrow Y$ a function strongly compactly Lipschitz and P-invex respect to η. We assume that, for each $x \in K$ and each $\omega^* \in P^* \setminus \{0\}$, the sets*

$$\Phi(x, \omega^*) := \{y \in K : (\omega^* \circ f)^0(x, \eta(y, x)) < 0\}$$

are convex. Furthermore, we assume that η is continuous and that satisfied $\eta(x, x) = 0$, for each $x \in K$. Then, the vectorial optimization problem (VP) has a weak global minimum $x_0 \in K$.

PROOF. For each $y \in K$ and $\omega^* \in P^*$, we define

$$F(y, \omega^*) := \{x \in K : (\omega^* \circ f)^0(x, \eta(y, x)) \geq 0\}.$$

By using Theorem 3.6 and Remark 3.7, it is sufficient to prove that vectorial variational-like inequality has a solution $x_0 \in K$. That is, we would like to show that

$$\bigcap_{y \in K} \bigcup_{\omega^* \in P^* \setminus \{0\}} F(y, \omega^*) \neq \emptyset.$$

Then, we will check that the set-valued mapping $G : K \rightrightarrows X$ given by

$$G(y) := \bigcup_{\omega^* \in P^* \setminus \{0\}} F(y, \omega^*)$$

satisfies the hypotheses from KKM-Fan Theorem. Obviously, $G(y) \neq \emptyset$, $\forall y \in K$. Because $P^* \neq \{0\}$ [10]. Then, for each $\omega^* \in P^* \setminus \{0\}$, by hypotheses, we have $\eta(y, y) = 0$ and, furthermore, $y \in F(y, \omega^*)$. Consequently, $y \in G(y)$. For each $\{x_1, ..., x_n\} \subset K$, $\mathrm{co}\{x_1, ..., x_n\} \subset \bigcup_{i=1}^n G(x_i)$. In the contrary case, it would exist $\alpha_i \geq 0$, $i = 1, ..., n$ such that $\sum_{i=1}^n \alpha_i = 1$ and

$$x := \sum_{i=1}^n \alpha_i x_i \notin \bigcup_{i=1}^n G(x_i)$$

or, equivalently, for each $i = 1, ..., n$ and each $\omega^* \in P^* \setminus \{0\}$,

$$(\omega^* \circ f)^0(x, \eta(x_i, x)) < 0,$$

that is, $x_i \in \Phi(x, \omega^*)$. By using the hypotheses, $\Phi(x, \omega^*)$ is convex. Then, $x \in \Phi(x, \omega^*)$, or

$$(\omega^* \circ f)^0(x, \eta(x, x)) < 0,$$

this is a contradiction with $\eta(x, x) = 0$. For each $y \in K$, the set $G(y)$ is closed. To done this, it is sufficient to prove that the sets $F(y, \omega^*)$ are closed, for each $y \in K$ and each $\omega^* \in P^* \setminus \{0\}$. Let $(x_k) \subset F(y, \omega^*)$ a sequence, $x_k \longrightarrow x$. Then,

$$(\omega^* \circ f)^0(x_k, \eta(y, x_k)) \geq 0, \ \forall k \in \mathbb{N}$$

and, thus

$$\limsup_k (\omega^* \circ f)^0(x_k, \eta(y, x_k)) \geq 0. \tag{9}$$

By other hand, the function $(\omega^* \circ f)^0(\cdot, \cdot)$ is upper semicontinuous (see [4], p. 25) and, since η is continuous, we have

$$0 \leq \limsup_k (\omega^* \circ f)^0(x_k, \eta(y, x_k)) \leq (\omega^* \circ f)^0(x, \eta(y, x)) \tag{10}$$

where the first inequality follows of (9).

¿From (10) we have $x \in F(y, \omega^*)$ and, therefore, $G(y)$ is closed. Being K a convex, bounded subset of X, and X is reflexive, we have that K is a weakly compact.

By other hand, for each $y \in K$, $G(y)$ is a closed subset and, consequently, $G(y)$ is weakly compact. Thus, the set-valued mapping G satisfies all the hypotheses of Lemma 3.8, and, therefore, there exist $x_0 \in \bigcap_{y \in K} G(y)$. ■

Conclusions: In this paper, we obtain an existence theorem for weak efficient solutions for a vectorial optimization problem between Banach spaces whose objective function is invex strongly compactly Lipschitz. This result is obtained in a way similar to the one given by Kazmi [11]. We characterize the solutions of the vectorial problem in terms of the solutions of a variational-like inequality and, by applying this characterization and using the KKM-Fan Theorem, we establish our main result. Our results extend those obtained early by Chen and Craven [2] and Kazmi [11].

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