Relationships between the variational like inequality problem and the vectorial optimization problem between Banach spaces

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Abstract

In this work, we will establish some relations between variational-like inequalities and vectorial optimization problem between Banach spaces under invexity hypotheses.

Key words: Vectorial optimization problem, variational like inequality problem, Duality, Weak efficiency, Pseudoinvex functions.

1 INTRODUCTION

Variational inequalities appearing naturally in problems from Physics, Economics, Optimization and Control, Elasticity and the Applied Sciences (see for instance, [1], [2], [3]).

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In the scalar case, Mancino and Stampacchia [4] obtained the following result: if $F : S_0 \subset \mathbb{R}^n \to \mathbb{R}^n$ is the gradient of a convex function $\theta : S \to \mathbb{R}^n$ and S is an open and convex set, then the variational inequality problem $\widetilde{VIP}$ is equivalent to the optimization problem (MP), where:

$(\widetilde{VIP})$ Find $\overline{x} \in S$ such that

$$(y - \overline{x})^T F(\overline{x}) \geq 0, \forall y \in S.$$

and the problem $(\widetilde{MP})$ is:

$$\left.\begin{array}{l} \text{Minimize } \ \theta(x) \\ \text{subject to } x \in S \end{array}\right\} \tag{MP}$$

An extension of the variational inequality problem is the variational like inequality problem $(\widetilde{VLIP})$.

Let S be a nonempty subset of $\mathbb{R}^n$ and we will consider two functions $F : S \to \mathbb{R}^\ltimes$ and $\eta : S \times S \to \mathbb{R}^\ltimes$. The variational like inequality problem $(\widetilde{VLIP})$ is:

$(\widetilde{VLIP})$: Find $\overline{x} \in S$ such that

$$\eta(y, \overline{x})^T F(\overline{x}) \geq 0, \forall y \in S.$$

Parida et. al. in [5] studied the existence of the solution of the variational like inequality $(\widetilde{VLIP})$ y los problemas de programación convexa. Ruiz et. al en [6] proved that the solutions of $(\widetilde{VLIP})$ are coincident with the solutions of a certain mathematical programming problem $(\widetilde{MP})$ under certain hypotheses of the generalized invexity and monotonicity.

In this work, we extend the above results for the case which the functions are defined between infinite dimensional Banach spaces.

Let E_1, E_2 be two Banach spaces, $f : E_1 \to E_2$ a given function and S nonempty subset of E_2. Let $Q \subset E_2$ a pointed closed, convex cone with nonempty interior and different of E_2.

The notions of efficiency that we will consider are the following:

Definition 1.1 ***(a)*** *We say that $\overline{x} \in S$ is **efficient** if no exists $y \in S$ such that*

$$f(y) - f(\overline{x}) \in -Q \setminus \{0\};$$

(b) *We say that* $\overline{x} \in S$ *is* ***weakly efficient*** *if no exist* $y \in S$ *such that*

$$f(y) - f(\overline{x}) \in -intQ.$$

We denote by $E(f;S)$ the set of the efficient points and $WE(f;S)$ the set of the weakly efficient points.

Obviously, we have : $E(f;S) \subset WE(f;S)$.

We will consider the following vectorial optimization problems:

$$\left.\begin{array}{ll} \text{V-min} & f(x) \\ \text{subject to } x \in S & \end{array}\right\} \qquad (VOP)$$

whose resolution consists of the determination of the set $E(f,S)$ and the problem

$$\left.\begin{array}{ll} \text{W-min} & f(x) \\ \text{subject to } x \in S & \end{array}\right\} \qquad (WVOP)$$

whose resolution consists of the determination of the set $WE(f;S)$.

We will prove that under generalized invexity hypotheses is possible characterize the solutions of the vectorial problems (VOP) and (WVOP) through of the solutions of some variational inequalities, that we will define later.

Let $S \subset E_1$ a nonempty, $\eta : S \times S \to E_1$ and $F : S \to \mathcal{L}(E_1, E_2)$ two given functions (we denote by $\mathcal{L}(E_{1,}E_2)$ the space of the continuous linear operator between the Banach spaces E_1 and E_2).

The **Vectorial variational like inequality problem (VVLIP)** is:

(VVLIP): Find a point $\overline{x} \in S$ such that

$$F(\overline{x})\eta(y, \overline{x}) \notin -Q \setminus \{0\}, \forall y \in S \qquad (1)$$

(where, we denote by $F(\overline{x})\eta(y, \overline{x})$ the value of the function $F(\overline{x})$ applied in the vector $\eta(y, \overline{x})$).

The **Weak vectorial variational like inequality problem (WVVLIP)** is:

(WVVLIP): Find a point $\overline{x} \in S$ such that

$$F(\overline{x})\eta(y,\overline{x}) \notin -\text{int}Q, \forall y \in S. \tag{2}$$

We observe that the finite-dimensional case, i.e., $E_1 = \mathbb{R}^n$, $E_2 = \mathbb{R}^m$ and $Q = \mathbb{R}^m_+$) was studied by Ruiz-Garzón et. al. [7] and we will generalize such results for the infinite-dimensional case.

2 RELATIONSHIPS BETWEEN THE VARIATIONAL LIKE INEQUALITIES AND THE VECTORIAL OPTIMIZATION PROBLEMS

In [8], Yang and Goh proved that, under convexity assumptions, if $F = \nabla f$, then every solution of (VVLIP) is a weakly efficient solution of (VOP) and reciprocally. Therefore, they proved that the resolution of (WVOP) is equivalently to the resolution of (WVVLIP). Similar results can be find in Lee and Kum [9].

We will generalize the results obtained by Lee and Kum [9] and Yang and Goh [8] for the pseudoinvex functions defined between Banach spaces. We observe that Ansari and Siddiqi [10], Kazmi [11] and Yang [12] the weakly efficient points are identified with the solutions of the weak variational like inequalities under the hypothesis of pre-invexity (a class more restrictive those of the pseudoinvex functions, which we will define later).

The notions of the generalized invexity que will use were introduced by Osuna et. al. [13] in finite-dimensional context and can be generalized as follows:

Definition 2.1 *Let S be a nonempty subset of E_1 and $f : S \to E_2$ a given function, Fréchet differentiable (or, differentiable) at $x \in intS$.*

(a) *We say that f is **invex** (IX) at $x \in S$ iff there exist a vectorial function $\eta : S \times S \to E_1$ such that*

$$f(y) - f(x) + Df(x)\eta(y,x) \in Q, \forall y \in S;$$

(b) *The function f is called **estrictly invex** (SIX) at $x \in S$ iff, there exists a vectorial function $\eta : S \times S \to E_1$ such that*

$$f(y) - f(x) + Df(x)\eta(y,x) \in intQ, \forall y \in S, y \neq x.$$

(c) *The function f is called **pseudoinvex** (PIX) at $x \in S$ iff, there exists a* *vectorial function $\eta : S \times S \to E_1$ such that*

$$f(y) - f(x) \in -intQ \Rightarrow Df(x)\eta(y,x) \in -intQ, \forall y \in S.$$

(where $Df \in L(E_1, E_2)$ is the Fréchet derivative of f)

Follows easily from definitions:

$$\textbf{(SIX)} \Rightarrow \textbf{(IX)} \Rightarrow \textbf{(PIX)}$$

It is well known that in the case $E_2 = \mathbb{R}$ and $Q = \mathbb{R}^+$, the class of invex functions is exactly equal to pseudoinvex functions, but it is possible to prove that is false in the vectorial case (en [7]).

Theorem 2.2 *Let $f : S \subset E_1 \to E_2$ a differentiable function and invex at $\overline{x} \in intS$, respect to η. If $F = \nabla f$ and if $\overline{x}$ is a solution of the inequality (VVLIP) respect to η, then $\overline{x}$ is a efficient solution of (VOP).*

PROOF. We shall assume that $\overline{x}$ is a solution of the inequality (VVLIP) and that it is not a efficient solution of (VOP). Then, there exists $y \in S$ such that

$$f(y) - f(x) \in -Q \setminus \{0\}. \tag{3}$$

¿From the invexity hypothesis on f we obtain

$$Df(\overline{x})\eta(y, \overline{x}) \in f(y) - f(\overline{x}) - Q. \tag{4}$$

¿From (3) and (4) we have

$$Df(\overline{x})\eta(y, \overline{x}) \in -Q \setminus \{0\}. \tag{5}$$

Obviously, $Df(\overline{x})\eta(y, \overline{x}) \in -Q$. If we shall have $\nabla f(\overline{x})\eta(y, \overline{x}) = 0$, from (4) we shall obtain that$-[f(y) - f(\overline{x})] \in -Q$, but is a contradiction with (3), because Q is a pointed cone. □

Consequently, under invexity hypothesis, the solutions of the variational like inequality problem (VVLIP) are efficient solutions.

To show the reciprocal of the above theorem, we set some conditions more strong. In fact, we have:

Theorem 2.3 *Let $f : S \subset E_1 \to E_2$ be a differentiable function at $\overline{x} \in intS$. Assume that $F = Df$ and that $-f$ is strictly invex respect to η. If $\overline{x}$ is a solution of (WVOP), then $\overline{x}$ also is a solution of (VVLIP).*

PROOF. We will prove the Theorem by absurd. We shall assume that $\overline{x}$ is solution of (WVOP) but that it is not a solution of the inequality (VVLIP). Then, there exists $y \in S$ such that

$$Df(\overline{x})\eta(y,\overline{x}) \in -Q \setminus \{0\}. \tag{6}$$

By other hand, $-f$ is strictly invex, consequently

$$f(y) - f(\overline{x}) \in -Df(\overline{x})\eta(y,\overline{x}) + \text{int}Q \subset Q + \text{int}Q \subset \text{int}Q \tag{7}$$

therefore $\overline{x}$ is not a weakly efficient solution of (WVOP). Thus, $\overline{x}$ is a solution of the inequality (VVLIP).□

Theorem 2.4 *Let $f : S \subset E_1 \to E_2$ be a differentiable function at $\overline{x} \in intS$ and $F = Df$.*

(i) *If $\overline{x}$ is a weakly efficient solution of (WVOP), then $\overline{x}$ is a solution of the weak variational like inequality (WVVLIP).*

(ii) *If f is a pseudoinvex function respect to η at $\overline{x}$ and if $\overline{x}$ is a solution of the weak variational like inequality (WVVLIP) respect to η, then $\overline{x}$ is a weakly efficient solution of (WVOP).*

PROOF.

(i) Assume that $\overline{x}$ is a weakly efficient solution of (WVOP). Let $y \in S$. Since $\overline{x} \in \text{int}S$, then, for each $t > 0$ sufficiently small, the point $\overline{x} + t\eta(y,\overline{x})$ belongs to S. By other hand, being $\overline{x}$ a weakly efficient point of (WVOP), follows that

$$f(\overline{x} + t\eta(y,\overline{x})) - f(\overline{x}) \notin -\text{int}Q \tag{8}$$

and, since $-\text{int}Q$, is a cone, for such t, we have

$$\frac{1}{t}[f(\overline{x} + t\eta(y,\overline{x})) - f(\overline{x})] \notin -\text{int}Q. \tag{9}$$

Taking in (9) $t \downarrow 0$ and recalling that $(-\text{int}Q)^c$ is closed, follows that

$$D\nabla f(\overline{x})\eta(y,\overline{x}) \notin -\text{int}Q, \forall y \in S \tag{10}$$

and $\overline{x}$ is solution of the inequality (WVVLIP).

(ii) We shall assume that $\overline{x} \in S$ is a solution of the inequality (WVVLIP) but that it is not a weakly efficient solution of (WVOP). Consequently, there exists $y \in S$ such that

$$f(y) - f(x) \in -\text{int}Q \tag{11}$$

and, since f is pseudoinvex, we have

$$Df(\overline{x})\eta(y, \overline{x}) \in -\text{int}Q \tag{12}$$

which contradicts the optimality of $\overline{x}$ of the problem (WVVLIP). $\square$

Theorem 2.5 *Let $f : S \subset E_1 \to E_2$ be a differentiable function at point $\overline{x}$. Assume that $F = Df$ and that f is strictly invex respect to η at $\overline{x}$. If $\overline{x}$ is a solution of (WVOP), then is also solution of (VOP).*

PROOF. We shall assume that $\overline{x}$ is a weakly efficient solution of(WOP) and it is not a solution of (VOP) and we will exhibit a contradiction. Consequently, there exists $y \in S$ such that

$$f(y) - f(\overline{x}) \in -Q \setminus \{0\}. \tag{13}$$

and, by other hand, being f strictly invex at $\overline{x}$, we have

$$f(y) - f(\overline{x}) - \nabla f(\overline{x})\eta(y, \overline{x}) \in \text{int}Q. \tag{14}$$

Then, from (13) and (14), we obtain

$$Df(\overline{x})\eta(y, \overline{x}) \in f(y) - f(\overline{x}) - \text{int}Q \subset -Q \setminus \{0\} - \text{int}Q \subset \text{int}Q. \tag{15}$$

So, from the last equation follows that $\overline{x}$ is not a solution of (WVVLIP) and, by using Theorem 2.4, we obtain that $\overline{x}$ is not a weakly efficient solution, which contradicts the hypothesis. $\square$

When $C \subset E_2$ is a cone, we define the **dual cone** of C as follows

$$C^* := \{\xi \in E_2^* : \langle \xi, x \rangle \geq 0, \forall x \in C\} \tag{16}$$

where E_2^* denote the topological dual of E_2 and $\langle \cdot, \cdot \rangle$ is the canonical pairing duality betweenE_2^* and E_2.

Definition 2.6 *We say that $\overline{x} \in S$ is a* ***vectorial critical point*** *(VCP) if there exists a functional $\lambda^* \in C^* \setminus \{0\}$ such that $\lambda^* \circ \nabla f(\overline{x}) = 0$.*

In [14], Craven proved that every vectorial critical point is a necessary condition for the weak efficiency of (WVOP). Next, we will proved, under hypotheses, the inverse affirmation. Before, we recall some necessary results.

Lemma 2.7 *Let F be a Banach space and $C \subset F$ a closed, convex cone with C and $intC \neq \varnothing$. If $x \in intC$ and $\xi \in C^* \setminus \{0\}$, then $\langle \xi, x \rangle > 0$.*

The following result is a generalization of the classical alternative Farkas'theorem for the infinite-dimensional spaces, see [15].

Lemma 2.8 *Let X, Y, V be three normed spaces, $A \in \mathcal{L}(X, V), M \in \mathcal{L}(X, Y)$ two continuous linear operators, $T \subset V, Q \subset Y$ convex cones, $b \in -T$, $s \in -Q$. Assume that the set $[A\ b]^T(T^*)$ is $w^* - closed. Then, the following system$*

$$\begin{cases} Ax + b \in -T \\ Mx + s \in -intQ \end{cases} \tag{17}$$

has not solution $x \in X$, iff there exist $\tau \in Q^* \setminus \{0\}$ y $\lambda \in T^*$ such that

$$\begin{cases} \tau M + \lambda A = 0 \\ \langle \lambda, b \rangle = 0 \\ \langle \tau, s \rangle = 0. \end{cases} \tag{18}$$

Proposition 2.9 *All the vectorial points critical are solution of (WVOP) iff the function f is pseudoinvex.*

PROOF. Let f be a pseudoinvex function and $\overline{x} \in S$ a vectorial critical point. We assume that $\overline{x}$ is not a weakly efficient solution of (WVOP) and exhibit a contradiction. Then, there exists $x \in S$ such that

$$f(x) - f(\overline{x}) \in -\text{int}Q \tag{19}$$

and, by other hand, there exists $\lambda^* \in Q^* \setminus \{0\}$ such that

$$\lambda^* \circ Df(\overline{x}) = 0. \tag{20}$$

Since f is pseudoinvex follows from (19)

$$Df(\overline{x})\eta(x,\overline{x}) \in -\text{int}Q \tag{21}$$

and, by using Lemma 2.7,

$$\lambda^* \circ Df(\overline{x})\eta(x,\overline{x}) < 0 \tag{22}$$

which contradicts (20).

Now, we will prove the other implication. We assume that all vectorial point critical is a weakly efficient solution of (WVOP). We fix $\overline{x} \in S$ and we consider the systems:

$$f(x) - f(\overline{x}) \in -\text{int}Q \tag{23}$$

and

$$Df(\overline{x})u \in -\text{int}Q. \tag{24}$$

We will prove that the system (24) has a solution $u \in E_1$ when the system (23) has a solution $x \in S$.

In fact, if the system (23) has a solution $x \in S$, then $\overline{x}$ is not a weakly efficient solution of (WVOP) and, by hypotheses, is not a vectorial critical point, i.e., does not exist $\lambda^* \in Q^* \setminus \{0\}$ such that $\lambda^* \circ Df(\overline{x}) = 0$.

Since $\overline{x}$ is not a vectorial critical point, we have that not exist $\tau \in Q^* \setminus \{0\}$, $\lambda \in Q^*$ such that

$$\begin{aligned} \tau M + \lambda A &= 0 \\ \langle \lambda, b \rangle &= 0 \\ \langle \tau, s \rangle &= 0. \end{aligned} \tag{25}$$

where:

$$\begin{aligned} &A := 0 \in \mathcal{L}(E_1, E_2) \\ &M := Df(\overline{x}) \in \mathcal{L}(E_1, E_2) \\ &b := 0 \in E_1 \\ &s := 0 \in E_2. \end{aligned} \tag{26}$$

From Lemma 2.8, there exists $u \in E_1$ such that

$$\begin{cases} Au + b = 0 \in -T \\ Mu + s = Df(\overline{x})u \in -\text{int}Q \end{cases} \tag{27}$$

and, in particular, the system (24) has solution $u \in E_1$.

It is sufficient put $\eta(x, \overline{x}) := u$ and we obtain that f is pseudoinvex. □

From Theorem 2.4 and Proposition 2.9, we can to relate the vectorial critical points, the weakly efficient solutions of (WVOP) and the solutions of the variational like inequality problem (WVVLIP). Saying it in a more precise form:

Corollary 2.10 *Assume that S is an open subset and $F = Df$. If f is pseudoinvex respect to η, then the vectorial critical points, the weakly efficient points of (WVOP) and the solutions of (WVVLIP) are coincident.*

The results obtained in this paper can be described in the following diagram:

$$\begin{array}{ccccc}
\textbf{(VVLIP)} & \Rightarrow f(IX), F = Df & \textbf{(VOP)} & & \\
& \Leftarrow -f(SIX), F = Df & & & \\
\Downarrow & & \Downarrow\Uparrow f(SIX) & & \\
\textbf{(WVVLIP)} & \Rightarrow f(PIX), F = Df & \textbf{(WVOP)} & \Leftarrow f(PIX), F = Df & \textbf{(VCP)} \\
& \Leftarrow F = Df & & \Rightarrow &
\end{array}$$

3 CONCLUSIONS

In Ruiz et. al. [6] it is proved that the solutions of the variational-like inequality problem (VLIP) in the scalar case are equivalent to the ninima of the mathematical programming problem in invex environments. In [7] it is proved that these results can be generalized to the vectorial problem between Euclidian spaces. In this work, we have extend these results to the vectorial optimization problems between Banach spaces, when the domination structure is defined by convex cones. Under the condition of pseudoinvexity, we have seen the relationship that exists between vector variational-like problems and vector optimization problems and managed to identify the weakly efficient points, the solutions of the weak vector variational-like inequality problems (WVVLIP) and the vector critical points.

References

[1] S. Dafermos, Exchange Price Equilibrium and Variational Inequalities, Mathematical Programming 46 (1990), 391-402;

[2] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and their applications, Academic Press, London, 1980;

[3] P. T. Harker and J. S. Pang, Finite Dimensional Variational Inequality and Nonlinear Complementarity Problems: a Survey of Theory, Mathematical Programming 48 (1990), 161-220;

[4] O. G. Mancino and G. Stampacchia, Convex Programming and Variational Inequalities, Journal Optimization Theory and Applications, 9 (1972) 3-23;

[5] J. Parida, M. Sahoo e A. Kumar, A Variational-like Inequality Problem, Bull. Austral. Math. Soc. 39 (1989) 225-231;

[6] G. Ruiz, R. Osuna and A. Rufián, Generalized Invex Monotonicity, European Journal of Operational Research, 144, (2003) 501-512;

[7] G. Ruiz-Garzón, R. Osuna-Gómez and A. Rufián-Lizana, Relationships and Vector Variational-like Inequality and Optimization Problems (*to appear*)

[8] X. Q. Yang and C. J. Goh, On Vector Variational Inequalities: Application to Vector Equilibria, Journal Optimization Theory and Applications 95 (1997) 431-443;

[9] G. M. Lee and S. Kum, Vector Variational Inequalities in Hausdorff Topological Vector Spaces, in F. Gianessi (ed.), Vector Variational Inequalities and Vector Equilibria: Mathematical Theories, Collection: Nonconvex Optimization and its Applications, vol 38, Kluwer Academic Publishers, London, 2000, pp. 307-320;

[10] Q. H. Ansari and A. H. Siddiqi, A Generalized Vector Variational-like Inequality and Optimization Over a Efficient Set, in the Collection: Fuctional Analysis with Current Applications in Science, Technology and Industry (Aligargh, 1995), Pitman Res. Notes Math. Ser., 377, 1995, 177-191;

[11] K. R. Kazmi, Some Remarks on Vector Optimization Problems, Journal Optimization Theory and Applications 96 (1) (1998) 133-138;

[12] X. Q. Yang, On Some Equivalent Conditions of Vector Variational Inequalities and Vector Equilibria, in F. Gianessi (ed.), Vector Variational Inequalities and Vector Equilibria: Mathematical Theories, Colection: Nonconvex Optimization and its Applications, vol. 38, Kluwer Academic Publishers, london, 2000, pp. 423-432;

[13] R. Osuna, A. Rufián and P. Ruiz, Invex Functions and Generalized Convexity in Multiobjective Programming, Journal Optimization Theory and Applications, 98 (1998) 651-661;

[14] B. D. Craven, Lagrangean Conditions and Quasiduality, Bulletin of the Australian Mathematical Society, vol. 16 (1977) 325-339.

[15] B. D. Craven, Mathematical Programming and Control Theory, Chapman and Hall (1978).