Optimality Conditions and Duality for the Nonsmooth Multiobjective Fractional Programming

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Abstract

In this work, we establish optimality conditions for the nonsmooth multiobjective fractional programming. Also, we give some duality results.

Key words: Vectorial fractional programming, Clarke generalized gradient, Duality, Weak efficiency, Optimality conditions.

1 Introduction

In this work, we study the so called **fractional optimization problem** whose formulation is the following:

$$\left.\begin{array}{ll} \text{Minimize} & \frac{f(x)}{g(x)} := \left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)}\right) \\ \text{subject to} & \\ & h_j(x) \leq 0, j = 1, ..., m \\ & x \in S \end{array}\right\} \qquad (VFP)$$

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where $f_i, g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $h_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ ($i = 1, ..., p$ y $j = 1, ..., m$), S is a nonempty subset of $\mathbb{R}^n$ and the functions g_i satisfy $g_i(x) > 0$ for each $x \in S$.

A fractional programming problem arise whenever the optimization of ratios such as performance/ cost, income/ investment and cost/ time is required and then, various real-life problems admit this formulation. For more details on applications of fractional programming, we suggest [1] and the references therein. In the theoretical point of view, this problem also have been extensively studied, for instance, in [2], [2], [4], [5] and [6].

One of the main approach used is the so-called parametric approach, see [7] and [8]. This approach was recently used in [9]. In this last work the authors using the parametric approach, convexity generalized and differentiability hypotheses characterized completely the solutions. Also some duality results are established.

Our main goal, in this work it is show that these results again are true, if we consider only Lipschitz continuous functions. To done this, we use the techniques from nonsmooth analysis. This article have the following structure: In Section 2, we present some results of nonsmooth analysis, which we will use in the following sections. In Section 3, we establish some optimality conditions for the nonsmooth multiobjective fractional problem and, finally, in Section 4, we use the results of the previous section to obtain some duality results.

2 Preliminaries

In this Section, we recall some notions and results from nonsmooth analysis. The **Clarke generalized directional derivative** of a local Lipschitz function ϕ from $\mathbb{R}^n$ into $\mathbb{R}$ at $\overline{x}$ in the direction d, denoted by $\phi^o(\overline{x}, d)$ ([10]) is given by:

$$\phi^o(\overline{x}; d) = \limsup_{\substack{x \to \overline{x} \\ t \downarrow 0}} \frac{\phi(y + tv) - \phi(y)}{t}.$$

The **Clarke generalized gradient** of ϕ at $\overline{x}$ is given by

$$\partial\phi(\overline{x}) = \{x^* \in \mathbb{R}^n : \phi^o(\overline{x}; d) \geq \langle x^*, d\rangle, \ \ \forall d \in \mathbb{R}^n\}.$$

Let C be a nonempty subset of $\mathbb{R}^n$ and consider its **distance function**, that is, the function $\delta_C(\cdot) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\delta_C(x) = \inf\{\|x - c\| : c \in C\}.$$

The distance function is not everywhere differentiable, but is globally Lipschitz. Let $\overline{x} \in C$. A vector $d \in \mathbb{R}^n$ is said to be *tangent* to C at $\overline{x}$ if $\delta_C^o(\overline{x}; d) = 0$. The set of tangent vectors to C at $\overline{x}$ is a closed convex cone in X, called the tangent cone to C at $\overline{x}$ and denoted by $T_C(\overline{x})$. By polarity, we define the *normal cone* to C at $\overline{x}$:

$$N_C(\overline{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, v\rangle \leq 0, \forall v \in T_C(\overline{x})\}.$$

We recall that $N_C(\overline{x})$ is a closed, convex cone. We consider the following multiobjective optimization problem:

$$\left.\begin{array}{ll} \text{Minimize} & F(x) \\ \text{subject to} & \\ & h(x) \in -\mathbb{R}^n_+ \\ & x \in S \end{array}\right\} \qquad (P)$$

where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are two given functions and S is a nonempty subset of $\mathbb{R}^n$. The concept of solution that we will consider is the following:

Definition 2.1 *We say that a point $x_0 \in \mathcal{F} := \{x \in S : h(x) \in -\mathbb{R}^n_+\}$ is a weakly efficient solution of (P) if there is not $x \in \mathbb{R}^n$ such that $F(x) - F(x_0) \in -int\mathbb{R}^n_+$.*

The following Proposition gives us a necessary condition for the weak efficiency (see [10], pp. 230):

Proposition 2.2 *If $x_0 \in \mathcal{F}$ is a weak efficient solution (P), then, there exist $\mu \in \mathbb{R}^n_+$ and $\lambda \in \mathbb{R}^n_+$, not all zero and $k > 0$ such that*

$$0 \in \partial_x(\mu \circ F + \lambda \circ h + kd_S)(x_0)$$

$$\langle \lambda, h(x_0)\rangle = 0.$$

Remark 2.3 *If $S \subseteq \mathbb{R}^n$ is open, the tangent cone $T_S(x_0) = \mathbb{R}^n$, for x_0 arbitrary. Further, this fact together with Proposition 2.1 implies that the results obtained in this work extends those given in [9].*

3 Optimality conditions

In this Section we will establish optimality conditions for (VFP). We will denote by X the feasible set of (VFP), that is,

$$X := \{x \in S : g_j(x) \leq 0, j = 1, ..., m\}.$$

By using the parametric approach, Dinkelbach [7] and Jagannathan [8] consider the following problem of optimization associated to (VFP), $(VFP)_v$ for $v = (v_1, ..., v_p) \in \mathbb{R}^p$:

$$\left.\begin{array}{ll} \text{Minimize} & (f_1(x) - v_1 g(x), ..., f_p(x) - v_p g_p(x)) \\ \text{subject to} & \\ & h_j(x) \leq 0, \forall j = 1, ..., m \\ & x \in S \end{array}\right\} \qquad (VFP_v)$$

We will use the following notations: Given $x, y \in \mathbb{R}^n$,

$x \leqq y \Leftrightarrow x_i \leq y_i, \forall i = 1, ..., n$
$x \leq y \Leftrightarrow x \leqq y$ and $x \neq y$
$x < y \Leftrightarrow x_i < y_i, \forall i = 1, ..., n.$

In [9], the following result is proved:

Lemma 3.1 *$\overline{x} \in X$ is a weakly efficient solution of (VFP) iff $\overline{x}$ is a weakly efficient solution of $(VFP)_{\overline{v}}$, con $\overline{v} = \frac{f(\overline{x})}{g(\overline{x})}$.*

Geoffrion [11] characterized the solutions of the multiobjective problems through scalarized problems. We adopt this way and we consider the associated pondered problem for $(VFP)_{\overline{v}}$:

$$\left.\begin{array}{ll} \text{Minimize} & \sum_{i=1}^{n} \omega_i (f_i(x) - \overline{v}_i g_i(x)) \\ \text{subject to} & \\ & h_j(x) \leq 0, j = 1, ..., m \\ & x \in S. \end{array}\right\} \qquad ((VFP)_{\overline{v}}(\omega))$$

where $\omega \in W := \{\omega \in \mathbb{R}^p_+ : \sum_{i=1}^{p} \omega_i = 1\}$.

By applying the classical scalarization techniques [11] to the problem $(VFP)_{\overline{v}}(\omega)$ and by using Lemma 3.1, we can enunciate:

Proposition 3.2 *If* $\overline{x} \in X$ *and* $\overline{v} = \frac{f(\overline{x})}{g(\overline{x})}$ *is a solution for the pondered problem* $(VFP)_{\overline{v}}(\omega)$ *for some* $\omega \in W$*, then* $\overline{x}$ *is solution for problem* (VFP)*.*

We can establish a reciprocal of Proposition 3.2 by using a generalized convexity notion called *KT-invexity*:

Definition 3.3 *We say that the problem* $(VFP)_v$ *is KT-invex on the feasible set respect to* η *if for each* $x_1, x_2 \in X$*, there exists a vector* $\eta(x_1, x_2) \in T_S(x_2)$ *such that*

$$\phi_{v,i}\ (x_1) - \phi_{v,i}(x_2) \geq \phi^0_{v,i}(x_2, \eta(x_1, x_2)), \forall i = 1, ..., p$$
$$h^0_j(x_2, \eta(x_1, x_2)) \leq 0, \forall j \in J(x_2) := \{j : h_j(x_2) = 0\}.$$

We note that definition coincides with that given by Martin [12] when the problem $(VFP)_{\overline{v}}$ is differentiable. Furthermore, this definition allow us to study problems that possess an abstract constraint. The definition presented in [12] consider only the case in which the problem have only inequality constraints.

Since it is usual in mathematical programming, is necessary to consider some supplementary condition under which is possible establish a multiplier rule that holds in normal case, that is, the multiplier associated to the objective function is nonzero. We consider the following constraint qualification:

We say $(VFP)_{\overline{v}}$ satisfies the **constraints qualification** on $\overline{x} \in X$ if there exists a vector $d_0 \in T_S(\overline{x})$ such that $h^0_j(\overline{x}; d_0) < 0, \forall j \in J(\overline{x})$.

Theorem 3.4 *We assume that* $\overline{x}$ *is a weakly efficient solution of* (VFP) *and that problem* (VFP) *satisfy the constraints qualification in* $\overline{x}$ *and the problem* $(VFP)_{\overline{v}}$ *is KT-invex, where* $\overline{v} = \frac{f(\overline{x})}{g(\overline{x})}$*. Then* $\overline{x}$ *is a solution for pondered problem* $(VFP)_{\overline{v}}(\omega)$*, for some* $\omega \in W$*.*

PROOF. We assume that $\overline{x} \in X$ is a weakly efficient solution (VFP). Then, using Lemma 3.1, $\overline{x}$ is a weakly efficient solution of $(VFP)_{\overline{v}}$. Therefore, applying Proposition 2.1 and Remark 2.2, there exists nonzero pair $(\theta, \lambda) \in \mathbb{R}^p_+ \times \mathbb{R}^m_+$ such that

$$[\textstyle\sum_{i=1}^p \theta_i(f_i - \overline{v}_i g_i) + \sum_{j=1}^m \lambda_j h_j]^0(\overline{x}, d) \geq 0, \forall d \in T_S(\overline{x})$$
$$\lambda_j h_j(\overline{x}) = 0, j = 1, ..., m.$$

The above inequality implies

$$[\sum_{i=1}^{p}\theta_i(f_i-\overline{v}_i g_i)]^0(\overline{x},d)+\sum_{j=1}^{m}\lambda_j h_j^0(\overline{x},d)\geq 0, \forall d\in T_S(\overline{x}).$$

In particular, for each x feasible of (VFP), we have

$$[\sum_{i=1}^{p}\theta_i(f_i-\overline{v}_i g_i)]^0(\overline{x},\eta(x,\overline{x}))+\sum_{j=1}^{m}\lambda_j h_j^0(\overline{x},\eta(x,\overline{x}))\geq 0$$

that is,

$$\begin{aligned}[\sum_{i=1}^{p}\theta_i(f_i-\overline{v}_i g_i)]^0(\overline{x},\eta(x,\overline{x})) &\geq -\sum_{j=1}^{m}\lambda_j h_j^0(\overline{x},\eta(x,\overline{x})) \\ &= \sum_{j\in J(\overline{x})}\lambda_j h_j^0(\overline{x},\eta(x,\overline{x}))\geq 0\end{aligned} \quad (1)$$

and using the hypothesis of KT-invexity follows that

$$\begin{aligned}&\sum_{i=1}^{p}[\theta_i(f_i(x)-\overline{v}_i g_i(x))]-\sum_{i=1}^{p}[\theta_i(f_i(\overline{x})-\overline{v}_i g_i(\overline{x}))] \\ &\qquad \geq [\sum_{i=1}^{p}\theta_i(f_i-\overline{v}_i g_i)]^0(\overline{x},\eta(x,\overline{x}))\geq 0.\end{aligned} \quad (2)$$

We claim that $\theta\neq 0$. For if it did have $\theta=0$, then $\lambda\geq 0$ would satisfy

$$h_j^0(\overline{x},d_0)<0, \forall j\in J(\overline{x}) \quad (3)$$

where we use the constraint qualification. By other hand (3) implies

$$\sum_{j\in J(\overline{x})}\lambda_j h_j^0(\overline{x},d_0)<0$$

which contradicts (1). So, we can assume that $\theta\geq 0$, with $\sum_i\theta_i=1$. Making $\omega=\theta$, it follows directly from equation (1) that $\overline{x}$ is solution of $(\text{VFP})_{\overline{v}}(\omega)$. □

As a straightaway consequence of the Proposition 2.1, we obtain the following necessary optimality condition:

Theorem 3.5 *Let $\overline{x}$ a weakly efficient solution of (VFP) and $\overline{v}=\frac{f(\overline{x})}{g(\overline{x})}$. Then* *there exist* $(\overline{\lambda}, \overline{\mu}) \geq 0$ *such that*

$$[\sum_{i=1}^{p} \overline{\lambda}_i(f_i - \overline{v}_i g_i) + \sum_{j=1}^{m} \overline{\mu}_j h_j]^0(\overline{x}, d) \geq 0, \forall d \in T_S(\overline{x})$$
$$\overline{\mu}_j h_j(\overline{x}) = 0, \forall j = 1, ..., m.$$

Since it is usual in mathematical programing, by assuming a constraint qualification in $\overline{x}$, we can assure that the multiplier associated to the objective function in $(VFP)_{\overline{v}}$ is nonzero:

Theorem 3.6 *Assume that* $\overline{x}$ *is a weakly efficient solution of* (VFP) *and such that* (VFP) *satisfies a constraint qualification in* $\overline{x}$. *Let* $\overline{v} = \frac{f(\overline{x})}{g(\overline{x})}$. *Then there exist* $\overline{\lambda} \geq 0$ *and* $\overline{\mu} \geqq 0$ *Such that*

$$[\sum_{i=1}^{p} \overline{\lambda}_i(f_i - \overline{v}_i g_i) + \sum_{j=1}^{m} \overline{\mu}_j h_j]^0(\overline{x}, d) \geq 0, \forall d \in T_S(\overline{x})$$
$$\overline{\mu}_j h_j(\overline{x}) = 0, \forall j = 1, ..., m. \tag{4}$$

PROOF. From Theorem 3.5 we have that there exist $\overline{\lambda}, \overline{\mu} \geq 0$ satisfying (4). We assume that $\overline{\lambda} = 0$ and exhibit a contradiction. Then, from Theorem 3.5 we have that $\overline{\mu} \geq 0$ and (4) implies

$$\sum_{j=1}^{m} \overline{\mu}_j h_j^0(\overline{x}, d) \geq 0, \forall d \in T_S(\overline{x}). \tag{5}$$

By other hand, from the constraint qualification hypothesis there exist $d_0 \in T_S(\overline{x})$ such that $h_j^0(\overline{x}, d_0) < 0, \forall j \in J(\overline{x})$, and therefore

$$\sum_{j} \overline{\mu}_j h_j^0(\overline{x}, d_0) < 0$$

which contradicts (5). Hence, $\overline{\lambda} \geq 0$. □

Under generalized convexity hypothesis, we obtain the following reciprocal of Theorem 3.6:

Theorem 3.7 *If* $\overline{x} \in X$ *is such that verifies (4) with* $\overline{\lambda} \geq 0$, $\overline{\mu} \geqq 0$ *and the problem* $(VFP)_{\overline{v}}$ *with* $\overline{v} = \frac{f(\overline{x})}{g(\overline{x})}$ *is KT- invex on the feasible set, then* $\overline{x}$ *is a weakly efficient solution of* (VFP).

PROOF. We shall assume that $\overline{x}$ is not a weakly efficient solution of (VFP) and exhibit a contradiction. From Lemma 3.1, we have that $\overline{x}$ is not a weakly efficient solution of $(VFP)_{\overline{v}}$. Consequently, there exists a feasible solution x such that

$$f_i(x) - \overline{v}_i g_i(x) < f_i(\overline{x}) - \overline{v}_i g_i(\overline{x}), \forall i = 1, ..., p. \tag{6}$$

By other hand, (4) implies

$$[\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i) + \sum_{j=1}^{m} \overline{\mu}_j h_j]^0(\overline{x}, \eta(x, \overline{x})) \geq 0 \forall x \in X$$

and therefore

$$[\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i)]^0(\overline{x}, \eta(x, \overline{x})) + \sum_{j=1}^{m} \lambda_j h_j^0(\overline{x}, \eta(x, \overline{x})) \geq 0. \tag{7}$$

Since $(VFP)_{\overline{v}}$ is KT-invex, we have $\sum_{j=1}^{m} \lambda_j h_j^0(\overline{x}, \eta(x, \overline{x})) \leq 0$. Moreover, (7) implies

$$[\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i)]^0(\overline{x}, \eta(x, \overline{x})) \geq 0.$$

Then, from KT-invexity, we obtain

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i(x) - \overline{v}_i g_i(x)) - \sum_{i=1}^{p} \overline{\lambda}_i (f_i(\overline{x}) - \overline{v}_i g_i(\overline{x})) \geq 0. \tag{8}$$

By other hand, (6) implies

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i(x) - \overline{v}_i g_i(x)) - \sum_{i=1}^{p} \overline{\lambda}_i (f_i(\overline{x}) - \overline{v}_i g_i(\overline{x})) < 0$$

which contradicts (8). Hence $\overline{x}$ is a weakly efficient solution of (VFP). □

4 Duality

We will consider the following dual problem of maximum a (VFP), analogous to the proposed by Jagannathan [8] and Schaible [13]:

$$\left.\begin{array}{l} \text{Maximize } (v_1, ..., v_p) \\ \text{subject to} \\ 0 \in \partial(\sum_{i=1}^p \lambda_i(f_i - v_i g_i) + \sum_{j=1}^m \mu_j h_j + kd_S)(u) \\ \sum_{i=1}^p \lambda_i(f_i(u) - v_i g_i(u)) \geq 0 \\ u \in S,\ \lambda = (\lambda_1, ..., \lambda_p) \geq 0,\ \mu \geqq 0. \end{array}\right\} \qquad (DF)$$

We denote by Y the set of the feasible solutions of (DF). Now, we prove some duality results for the (VFP) and (DF), under KT-invexity hypotheses.

Theorem 4.1 *(weak duality) Let $x \in X$ and $(u, \lambda, \mu, v) \in Y$ given. If the problem $(VFP)_v$ is KT-invex, then*

$$\frac{f(x)}{g(x)} \nless v.$$

PROOF. Since $(VFP)_v$ is KT-invex,

$$\sum_{i=1}^p \lambda_i(f_i(x) - v_i g_i(x)) \geq \sum_{i=1}^p \lambda_i(f_i(u) - v_i g_i(u)) + \sum_{i=1}^p \lambda_i(f_i - v_i g_i)^0(u, \eta(x, u)) \qquad (9)$$

From of the feasibility, we have

$$\sum_{i=1}^p \lambda_i(f_i(u) - v_i g_i(u)) \geq 0. \qquad (10)$$

From (9) and (10) follows

$$\sum_{i=1}^p \lambda_i(f_i(x) - v_i g_i(x)) \geq \sum_{i=1}^p \lambda_i(f_i - v_i g_i)^0(u, \eta(x, u)). \qquad (11)$$

From of the feasibility, we have:

$$0 \in \partial(\sum_{i=1}^p \lambda_i(f_i - v_i g_i) + \sum_{j=1}^m \mu_j h_j + kd_S)(u)$$

that is,

$$0 \leq (\sum_{i=1}^{p} \lambda_i(f_i - v_i g_i) + \sum_{j=1}^{m} \mu_j h_j)^0(u, d), \ \forall d \in T_S(x),$$

in particular,

$$0 \leq (\sum_{i=1}^{p} \lambda_i(f_i - v_i g_i))^0(u, \eta(x, u)) + \sum_{j=1}^{m} \mu_j h_j^0(u, \eta(x, u)). \tag{12}$$

Furthermore, (12) and KT-invexity imply

$$\begin{aligned} \sum_{i=1}^{p} \lambda_i(f_i(x) - v_i g_i(x)) &\geq -\sum_{j=1}^{m} \mu_j h_j^0(u, \eta(x, u)) \qquad (13) \\ &= \sum_{j \in J(u)} \mu_j h_j^0(u, \eta(x, u)) \geq 0. \end{aligned}$$

We assume that $\frac{f(x)}{g(x)} < v$ and we will show a contradiction. Since $\lambda \geq 0$, we have

$$\sum_{i=1}^{p} \lambda_i(f_i(x) - v_i g_i(x)) < 0.$$

But, this is a contradiction with (13). Consequently $\frac{f(x)}{g(x)} \not< v$. □

Theorem 4.2 *(Strong duality) We assume that $(VFP)_v$ is KT-invex for each $v \in \mathbb{R}^p$ such that there exist (u, λ, μ) satisfying $(u, \lambda, \mu, v) \in Y$. Moreover, we assume that $\overline{x} \in X$ is a weakly efficient solution of (VFP) and that it is verified the constraint qualification in $\overline{x}$. Then, there exist $(\overline{\lambda}, \overline{\mu}, \overline{v})$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v})$ is a weakly efficient solution for (DF).*

PROOF. Let $\overline{v} = \frac{f(\overline{x})}{g(\overline{x})}$. If $\overline{x} \in X$ is a weakly efficient solution of (VFP), then, from Theorem 3.5, there exist $\overline{\lambda} \geq 0$ and $\overline{\mu} \geqq 0$ such that

$$(\sum_{i=1}^{p} \overline{\lambda}_i(f_i - \overline{v}_i g_i) + \sum_{j=1}^{m} \overline{\mu}_j h_j)^0(\overline{x}, d) \geq 0, \forall d \in T_S(\overline{x}) \tag{14}$$

$$\sum_{j=1}^{p} \overline{\mu}_j h_j(\overline{x}) = 0.$$

Since (14) is equivalent to

$$0 \in \partial(\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i) + \sum_{j=1}^{m} \overline{\mu}_j h_j + k d_S)(\overline{x})$$

and,

$$f_i(\overline{x}) - \overline{v}_i g_i(\overline{x}) = 0,\ \forall i = 1, ..., m$$

then, $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}) \in Y$. We assume that $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v})$ is not weakly efficient solution of (DF) and we will exhibit a contradiction. Then, there exist (x, λ, μ, v) such that

$$v_i \ > \overline{v}_i,\ \forall i = 1, ..., m$$

that is,

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} < v_i,\ \forall i = 1, ..., m$$

which contradicts the weak duality established in Theorem 4.1. □

Theorem 4.3 *(Inverse duality) Let $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{v}) \in Y$ and $(VFP)_{\overline{v}}$ a KT-invex problem. If $\overline{v} = \frac{f(\overline{x})}{g(\overline{x})}$ for $\overline{x} \in X$, then $\overline{x}$ is a weakly efficient solution of (VFP). If, moreover, for each $(u, \lambda, \mu, v) \in Y$ the problem is KT-invex on the feasible set, then $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{v})$ is a weakly efficient solution for (DF).*

PROOF. Let $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{v}) \in Y$. If $\overline{x}$ is not a weakly efficient solution of (VFP) then, from Lemma 3.1, we have that $\overline{x}$ is not a weakly efficient solution for $(VFP)_{\overline{v}}$, that is, there exists $x \in X$ such that

$$f_i(x) - \overline{v}_i g_i(x) < f_i(\overline{x}) - \overline{v}_i g_i(\overline{x}) = 0, \forall i = 1, ..., m$$

or, equivalently,

$$\frac{f(x)}{g(x)} < \overline{v}.$$

This is a contradiction with Theorem 4.1 on weak duality. Now, we prove the second affirmation. By absurd. We assume that $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{v})$ is not a weakly efficient solution for (DF). Then, there exist $(u, \lambda, \mu, v) \in Y$ such that

$$v_i > \frac{f_i(\overline{x})}{g_i(\overline{x})} = \overline{v}_i, \forall i = 1, ..., m$$

but, it is newly a contradiction with Theorem 4.1, since $(VFP)_v$ is KT-invex. □

CONCLUSIONS: In this work, we obtain necessary and sufficient conditions of optimality (in the sense of weak efficiency) for some nonsmooth multiobjective fractional problem. Also, we obtain some theorems of duality. These results were obtained by the parametric approach of Dinkelbach [7] and by usual techniques of scalarization [11] and extends those obtained by Osuna-Gómez et al. [9] to the nonsmooth problem with abstract constraints.

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