# $f$–Structures on the Classical Flag Manifold which Admit (1,2)–Symplectic Metrics

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**Abstract**

*We characterize the $f$-structures $\mathcal{F}$ on the classical maximal flag manifold $\mathbb{F}(n)$ which admit (1,2)–symplectic metrics. This provides a sufficient condition for the existence of $\mathcal{F}$–harmonic maps from any cosymplectic Riemannian manifold onto $\mathbb{F}(n)$.*

*In the special case of almost–complex structures, our analysis extends and unifies two previous approaches: a paper of A.E. Brouwer 1980 on locally transitive digraphs, involving unpublished work by P.J. Cameron; and work by Mo, Paredes, Negreiros, Cohen and San Martin on cone–free digraphs. We also discuss the construction of (1,2)-symplectic metrics and calculate their dimension. Our approach is entirely graph theoretic.*

**Key words:** Flag manifolds, (1,2)–simplectic structures, directed graphs.

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# 1 Introduction

Gray [7] and Lichnerowicz [10] were among the first to observe the relevance of (1,2)–symplectic structures, not necessarily invariant or Kahler, in Hermitian geometry and harmonic maps, respectively. Originally, almost complex structures were considered, but there is interest in studying the more general case of $f$–structures [21], [1].

Here we consider the special case of the maximal flag manifold $\mathbb{F}(n)$ associated with $\mathfrak{sl}(n, \mathbb{C})$, endowed with an *invariant* $f$–structure $\mathcal{F}$. Following Burstall and Salamon [3] and Black [1], there is interest in analyzing the conditions under which $\mathcal{F}$ admits an invariant metric $ds^2$ on $\mathbb{F}(n)$ which is (1,2)–symplectic. In this paper we discuss such a condition and describe the set of (1,2)–symplectic metrics $ds^2$ admitted by $\mathcal{F}$.

The pair $(\mathbb{F}(n), \mathcal{F})$ defines in a natural way a digraph (oriented graph) $\mathcal{G} = (V, E)$, while the metric $ds^2$ provides a weighting $\lambda_e > 0,\ \ e \in E$. The (1,2)–symplectic conditions constitute a simple system of linear homogeneous restrictions on the weights $\lambda_e$. The issue is, therefore, finding a necessary and sufficient condition for the consistency of this system.

A special case of interest is when the invariant structure $\mathcal{F}$ is *almost Hermitian*. Here, the digraph $\mathcal{G}$ is complete, i.e. a tournament digraph. It was suggested by Mo and Negreiros [11] that $\mathcal{F}$ admits (1,2)–symplectic metrics if and only if $\mathcal{G}$ is *cone–free*, namely omits certain sub–graphs. This has been verified in some cases by Paredes [14], [15], and demonstrated in the general case by Cohen, Negreiros and San Martin [4],[5]. Up to permutation, the incidence matrix of such a digraph has a stair–shaped form which is preserved under the cyclic shift in $n$ indices [4],[5].

Another class of digraphs, also preserved by the cyclic shift, called *locally transitive* digraphs, has been studied earlier in A.E. Brouwer's paper [2]. We show that the cone–free and locally–transitive conditions are, in fact, equivalent and define the same family of digraphs. We thank Brendan McKay (ANU Canberra, Australia) for bringing [2] to our attention.

For *complete* locally transitive digraphs, the following results are available: (i) the enumeration of essentially different graphs of this type with $n$ vertices [2], (ii) a description of the full set of (1,2)–symplectic metrics associated with such a digraph [4],[5].

The last result is relevant for almost complex structures on the flag manifold. Its extension to $f$-structures, which amounts to admitting *non-complete* locally transitive digraphs, is performed in the present paper. The extension of Brouwer's enumeration in [2] remains an interesting open problem, and it seems that his technique does not extend to the non-complete case. It would also be interesting to connect the results obtain here with the existence of harmonic maps into $\mathbb{F}(n)$.

## 2 Flag preliminaries

Consider the classical maximal flag manifold $\mathbb{F}(n) = U(n)/T$, where $U(n)$ is a unitary group and $T$ is a maximal torus in $U(n)$ (we shall follow the definitions and notation of [5]). If $b$ stands for the origin in $\mathbb{F}(n)$, the tangent space at $b$ identifies naturally with the subspace $\mathfrak{q} \subset \mathfrak{u}(n)$ spanned by $A_{jk}, iS_{jk}$, where $A_{jk} = E_{jk} - E_{kj}$ and $S_{jk} = E_{jk} + E_{kj}$. Here $E_{jk}$ is the matrix with 1 in entry $jk$ and zeros otherwise.

By classical theory, an invariant metric $ds^2$ on $\mathbb{F}(n)$ can be identified with an inner product in $\mathfrak{q}$ of the form $X, Y \to tr((\Lambda \circ X)Y)$ where $\Lambda = \{\lambda_{jk}\}$ is a real symmetric matrix with positive off–diagonal entries and $\circ$ is the Hadamard (i.e. entrywise) product. As a special case, the Cartan–Killing inner product $tr(YX)$ is induced by the *Cartan–Killing metric* corresponding to $\lambda_{jk} \equiv 1 \ \ (j < k)$.

An $f$–structure (see [21]) on $\mathbb{F}(n)$ is a section $\mathcal{F}$ of the bundle $End(T\mathbb{F}(n))$ which satisfies $\mathcal{F}^3 + \mathcal{F} = 0$. We shall assume that $\mathcal{F}$ is *invariant*, namely, commutes with the adjoint action of $T$ on $\mathfrak{q}$. We call $\mathcal{F}$ *almost complex* if it satisfies $\mathcal{F}^2 + \mathcal{F} = 0$.

Every invariant almost complex structure $\mathcal{F}$ on the flag manifold assumes in the canonical basis the form $X \to i\varepsilon X$ where $\varepsilon = \{\varepsilon_{jk}\}$ is an anti–symmetric (1,-1)–matrix (we denote by $i$ the complex unit $\sqrt{-1}$). As a natural extension, every invariant $f$–structure $\mathcal{F}$ on the flag manifold is represented by an anti–symmetric (0,1,-1)–matrix $\varepsilon$. Every matrix in the canonical basis $E_{jk}$ is an eigenvector, with eigenvalue $0, 1$ or $-1$, for the Hadamard product $X \to \varepsilon \circ X$. We may therefore split $\mathfrak{q}$ as the direct sum of three eigenspaces: $\mathfrak{q}_0, \mathfrak{q}_+, \mathfrak{q}_-$.

In the sequel we shall allow some abuse of notation and identify the $f$–structure $\mathcal{F}$ and the metric $ds^2$ with the matrices $\varepsilon$ and $\Lambda$.

## 3 Graph theoretic preliminaries

A *digraph* is a finite oriented graph $\mathcal{G} = (V, E)$ . If $v, w \in V$ then an arrow $v \to w$ indicates that $vw \in E$; while $v \leftrightarrow w$ indicates either $vw \in E$ or $wv \in E$ . Furthermore, we define the $v$–loser and $v$–winner sets

$$\mathcal{G}_L(v) = \{w \in V : \quad wv \in E\}, \qquad \mathcal{G}_W(v) = \{w \in V : \quad vw \in E\},$$

considered as sub-digraphs of $\mathcal{G}$. This is analogous to the concept of neighbor set used in unoriented graphs. Finally, we say that $v$ is a *winner* (resp. *loser*) in $\mathcal{G}$ if $\mathcal{G}_L(v)$ or $\mathcal{G}_W(v)$ equals $V \setminus \{v\}$.

We now specialize to the problem at hand. Through the incidence matrix $\varepsilon = \{\varepsilon_{jk}\}$ we may identify an $f$–structure $\mathcal{F}$ on $\mathbb{F}(n)$ with a digraph $\mathcal{G} = (V, E)$ with $V = \{1, \cdots, n\}$. Similarly, through the matrix $\Lambda = \{\lambda_{jk}\}$ we may identify an invariant metric $ds^2$ on $(\mathbb{F}(n) \ , \mathcal{F})$ with a positive weighting on the edge set $E$ of the digraph. Note that if $\varepsilon_{jk} = 0$ the weight $\lambda_{jk}$ may be ignored since $jk \notin E$. According to [1], the (1,2)–symplecticity conditions imposed by $\varepsilon$ on the metric $\Lambda$ amount to the following three rules:

$$\text{If } \ k \to j, \quad k \to l, \quad j \not\leftrightarrow l \ \text{ then } \ \lambda_{jk} = \lambda_{kl}; \tag{1}$$

$$\text{If } \ j \to k, \quad l \to k, \quad p \not\leftrightarrow l \ \text{ then } \ \lambda_{jk} = \lambda_{kl}; \tag{2}$$

$$\text{If } \ k \to j, \quad j \to l, \quad k \to l \ \text{ then } \ \lambda_{kl} = \lambda_{jk} + \lambda_{jl}. \tag{3}$$

These restrictions apply to any 3-vertex sub-digraph of $G$ of the types given in Figure 1.

## 4 Locally transitive $f$–structures

As stated in the introduction, our main problem is the characterization of $f$–structures which admit (1,2)–symplectic metrics. In graph–theoretic terms, we wish to characterize the digraphs $\mathcal{G} = (V, E)$ which admit positive weights $\Lambda$ which satisfy properties (1)-(3). It is this version of the problem which we shall consider in the rest of the paper.

The following definitions will be crucial for our main result.

**Definition 4.1.** *A digraph $\mathcal{G}' := (V', E')$ is called: (i)* trivial *if* $|E'| = 0$; *(ii)* transitive *if the relation* ” $\to$ ” *is transitive (i.e. for* $i, j, k \in V'$, $i \to j \to k$ *implies* $i \to k$); *(iii)* relatively connected *if for all* $i, j, k \in V'$ $i \to j$ *implies* $i \leftrightarrow k$ *or* $j \leftrightarrow k$.

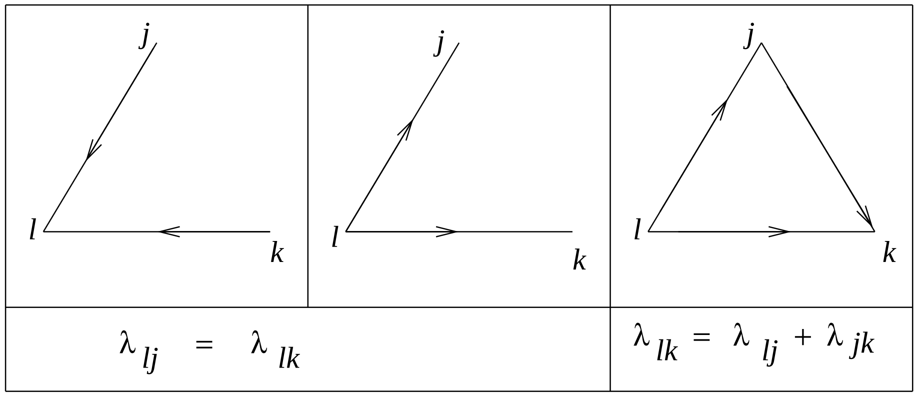


Figure 1: 3–vertex sub–digraphs associated with local transitivity.

Transitivity for *complete* digraphs may be characterized by the absence of cycles, and the incidence matrix of such digraphs is permutation–similar to the canonical matrix $\varepsilon_{jk} = 1$ $(j < k)$ [12]. We shall be more interested in the following *local* version of this property.

**Definition 4.2.** *We call the digraph $\mathcal{G} = (V, E)$* locally transitive *(in short, LT) if for all $v \in V$ each of the sub-digraphs $\mathcal{G}_L(v)$ and $\mathcal{G}_W(v)$ is transitive and relatively connected.*

Several remarks are in order:

(i) Local transitivity means that the digraphs $\mathcal{G}_L(v), \mathcal{G}_W(v)$ omit certain 3–vertex sub-digraphs, namely the ones whose edges form a non–empty subset of a 3–cycle (compare with Figure 2).

(ii) In case $\mathcal{G}$ is complete, local transitivity implies that both $\mathcal{G}_W(v), \mathcal{G}_L(v)$ are (complete and) transitive. This way we recover the original definition introduced for complete digraphs by P.J. Cameron and discussed in [2].

(iii) If $\forall v \in V \;\; max\{|\mathcal{G}_W(v)|, |\mathcal{G}_L(v)|\} \leq 2$ then $\mathcal{G}$ is LT.

(iv) All the digraphs of size $\leq 3$ are LT. As to $n = 4$, simple analysis shows that up to digraph isomorphism there exist 42 digraphs with 4 vertices, six of which are not LT (see Figure 2).

(v) According to (iv), a non–LT 4–vertex digraph must have a winner or a loser, but not both. If $v$ is the winner/loser then $V_W(v)$ (resp. $V_L(v)$) is a non–trivial sub-digraph of a 3–cycle.

**Lemma 4.3.** *$\mathcal{G}$ is LT if and only if every 4–vertex sub-digraph of $\mathcal{G}$ is LT.*

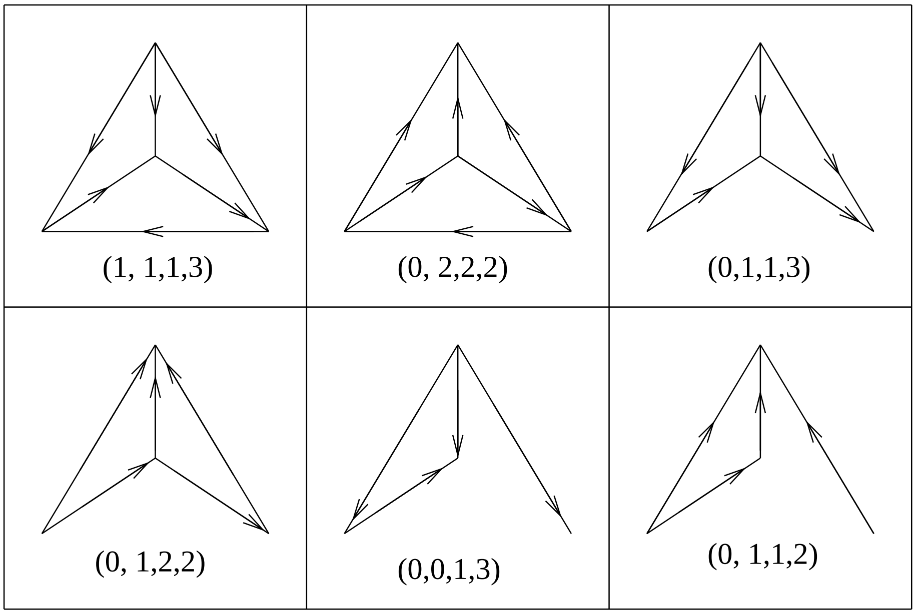


Figure 2: 4–vertex digraphs which are not LT

**Proof:** If $\mathcal{G}$ is LT then any sub-digraph of $\mathcal{G}$, including all the 4–vertex sub-digraphs, is LT. It remains to show the converse direction. Assume that $\mathcal{G}$ is not LT. Then we have two cases, both leading to the existence of a non–LT 4–vertex sub-digraph, completing the proof.

**Case 1:** There exists $v \in V$ such that one of the sets $\mathcal{G}_L(v), \mathcal{G}_W(v)$ is not transitive. Namely, in this set there exist $j, k, l$ such that $jk, kl \in E$ but $jl \notin E$. It can be checked against Figure 2 that whether $lj \in E$ or not, the sub-digraph of $\mathcal{G}$ supported on $\{v, j, k, l\}$ is not LT.

**Case 2:** There exists $v \in V$ such that one of the sets $\mathcal{G}_L(v), \mathcal{G}_W(v)$ is neither trivial nor relatively connected. Namely, this set contains $j, k, l$ such that $jk \in E$ but $jl, kl, lj, lk \notin E$. Here too, the sub-digraph supported on $\{v, j, k, l\}$ is not LT. $\square$

The case of *complete* digraphs is of special interest as it corresponds to almost–complex structures. Exactly two of the six 4–vertex digraphs in Figure 2 are complete: those which contain a winner/loser and a 3–cycle. In [11] these two digraphs were called "cones", and in [4],[5] a complete digraph $\mathcal{G}$ which omitted them was called "cone–free". Lemma 4.3 states, therefore, that $\mathcal{G}$ is LT if and only if it is cone–free. As a result, the two families of complete digraphs studied separately in [11],[4],[5],[15] (coneless digraphs) and in [2] (LT digraphs) are one and the same.

# 5 Completely Non–Transitive Digraphs

Here we study the structure of (1,2)-symplectic metrics on a special class of LT digraphs, namely the completely non-transitive digraphs.

**Definition 5.1.** *(i) A* transitive triangle *is a transitive digraph* $\mathcal{G}_t = (V_t, E_t)$ *with* $|V_t| = 3$*. Assuming* $V_t = \{u, v, w\}$ *and* $E = \{uv, vw, uw\}$*, we shall refer to* $uv, vw$ *as* sides *and to* $uw$ *as* basis*.*

*(ii) We shall call a digraph* $\mathcal{G}' = (V', E')$ completely non-transitive *if it does not contain any transitive triangle.*

A completely non-transitive digraph is LT. Indeed, the sets $\mathcal{G}'_W(v)$ and $\mathcal{G}'_L(v)$ are trivial in the sense of Definition 4.1 (also, the digraph is cone free since every cone contains a transitive triangle).

At the same time, a completely non-transitive digraph admits (1,2)-symplectic metrics, namely positive weightings $\{\lambda_e > 0, \;\; e \in E'\}$ which respect the identities (1-3). Indeed, due to the absence of transitive triangles, system (1-3) has no identities of type (3); hence the *Cartan-Killing metric* $\lambda \equiv 1$, which automatically satisfies (1-2), is (1,2)–symplectic (in general, $\mathcal{G}'$ admits non-constant (1,2)–symplectic metrics along with the Cartan-Killing metric).

We observe that the Cartan–Killing metric $\lambda \equiv 1$ on a digraph $\mathcal{G}'$ is (1,2)–symplectic if and only if $\mathcal{G}'$ is completely non-transitive. In the special case of complete digraphs, this implies that the Cartan–Killing metric is (1,2)–symplectic only if $|V| < 3$, as observed in [4].

Define the following equivalence relation between edges in $E'$: $e \sim e'$ if for some $v, v', u \in V$ we have either $e = vu$ and $e' = v'u$, or $e = uv$ and $e' = uv'$. A metric on $\mathcal{G}'$ is (1,2)–symplectic if and only if it is constant on every equivalence class in $E'$. Thus, the dimension of the space of (1,2)–symplectic metrics is equal to $\beta$, the number of equivalence classes in $E'$.

How can $\beta$ be calculated from $\mathcal{G}'$ directly? We do not know the answer, but a promising observation is that $\beta$ is the number of connected components in a "spanning forest" for $\mathcal{G}'$, assuming every vertex in the forest is a winner or a loser.

# 6 The General Case

In studying (1,2)–symplectic metrics on a general LT digraph $\mathcal{G}$, our approach will be reduction to an associated completely non-transitive digraph $\mathcal{G}'$ with the same vertex set, based on the following "edge deletion lemma".

**Lemma 6.1.** *Let $\mathcal{G} = (V, E)$ be a LT digraph which is not completely non-transitive (see definition 5.1). Then $E$ contains an edge $e$ which is a base but not a side. In this case, the sub-digraph $\tilde{\mathcal{G}} := (V, E \setminus \{e\})$ is LT.*

**Proof:** Let $\mathcal{G}_* = (V_*, E_*)$ be a maximal subgrah of $\mathcal{G}$ which is complete and transitive, and $|V_*| \geq 3$. The assumption guarantees the existence of at least one such a sub-digraph. Then $\mathcal{G}_*$ has a single base $e \in E_*$ which is not a side, namely the arrow $e$ which connects the winner and loser in $\mathcal{G}_*$.

The edge $e$ is therefore a base in $\mathcal{G}$. We claim that $e$ cannot be a side in some transitive triangle in $\mathcal{G}$. Assume to the contrary that such a triangle $\mathcal{G}_t = (V_t, E_t)$ does exist. Note that $V_t \not\subset V_*$ since $e$ is not a side in $\mathcal{G}_*$. Therefore, the sub-digraph $\mathcal{G}^*$ of $\mathcal{G}$ supported on $V_* \cup V_t$ strictly contains $\mathcal{G}^*$. Local transitivity of $\mathcal{G}$ implies that $\mathcal{G}^*$ is again complete and transitive, contradicting the maximality of $\mathcal{G}_*$.

Next we show that $\tilde{\mathcal{G}}$ is LT. By Lemma 4.3 it suffices to show that every 4-subdigraph $\hat{\mathcal{G}} \subset \tilde{\mathcal{G}}$ is LT. As long as $e$ is not in $\hat{\mathcal{G}}$ there is nothing to prove since $\hat{\mathcal{G}}$ is a subdigraph of $\mathcal{G}$. Otherwise, if $e = uw$ then $u, w$ are vertices in $\hat{\mathcal{G}}$. Suppose $\hat{\mathcal{G}}$ is not LT. According to 4 $\hat{\mathcal{G}}$ contains a winner or a loser, namely, $v$. Since $e$ is not in $\hat{\mathcal{G}}$, $v \neq u, w$. Whether $v$ is a winner or a loser $e$ is a side in the transitive triangle $\{u, v, w\}$, which is impossible by the first part of the lemma. $\square$

The following central result follows.

**Theorem 6.2.** *The digraph $\mathcal{G} = (V, E)$ admits (1,2)–symplectic metrics if and only if it is LT.*

**Proof:** For $n < 4$, $\mathcal{G}$ is always LT, and verification of the Theorem is an easy exercise. For $n = 4$, verification is easy, based on the digraphs in Figure 2. So, assume $n > 4$.

If $\mathcal{G}$ admits (1,2)–symplectic metrics then by restriction every 4–vertex sub-digraph of $\mathcal{G}$ admits (1,2)–symplectic metrics, hence (as just observed) is LT. But then by Lemma 4.3 $\mathcal{G}$ is LT.

Conversely, assume that $\mathcal{G}$ is LT. We argue by induction: If $\mathcal{G}$ is completely non-transitive then the existence of (1,2)–symplectic metrics was guaranteed in the previous section. Otherwise, by Lemma 6.1 we may delete an edge $e$ from $\mathcal{G} = (V, E)$, obtaining another LT digraph $\tilde{\mathcal{G}} = (V, \tilde{E})$. By the induction argument, $\tilde{\mathcal{G}}$ has (1,2)–symplectic metrics. We extend each such metric to a metric on $\mathcal{G}$ by defining $\lambda_e = \lambda_{e'} + \lambda_{e''}$, where $e', e''$ are the sides corresponding to the base $e$. This is the only extension for which $\Lambda$ is (1,2)–symplectic on the triangle in question, hence the only extension which *might* be (1,2)–symplectic for the whole digraph. We want to show that, in fact, it is.

**Step 1.** We show that the extension is well defined. Namely, assume that $e = uw$ is simultaneously basis for two transitive triangles, say $\{uv, vw, uw\}$ and $\{uz, zw, uw\}$ with $\{u, v, w, z\} \subset V$. We need to show that a priori

$$\lambda_{uz} + \lambda_{zw} = \lambda_{uv} + \lambda_{vw}. \tag{4}$$

There are two cases to consider. If $v \leftrightarrow z$, we may assume for definiteness that $v \to z$. In this case, by (3) we have a priori $\lambda_{vz} = \lambda_{uz} - \lambda_{uv}$ and $\lambda_{vz} = \lambda_{vw} - \lambda_{zw}$, implying (4). Otherwise, by (1-2) we have a priori $\lambda_{uv} = \lambda_{uz}$ $\lambda_{vw} = \lambda_{zw}$, again implying (4).

**Step 2.** We show that the extended metric is (1,2)–symplectic. Every conflict within the constraint system (1)-(3) should involve the deleted edge $e$, since $\tilde{\mathcal{G}}$ is assumed to satisfy these restrictions. By Lemma 6.1, $e$ is not a side in $\mathcal{G}$, hence any conflict with (3) is of the type already discussed in Step 1.

A conflict with (1) implies that $\lambda_e \neq \lambda'_e$ where, say, $e = uw$ and $e' = tw$. This can occur only if $t \not\leftrightarrow u$. Now it can be easily seen that independently of the relation between $v$ and $t$, $\mathcal{G}$ and $\tilde{\mathcal{G}}$ cannot both be LT since one of the two contains one of the non-LT digraphs of Figure 2. This is a contradiction to our assumptions.

A conflict with (2) leads to a similar contradiction, and so the proof is complete. $\square$

Assume that the LT digraph is reduced, via edge deletion, to a completely non-transitive digraph $\mathcal{G}'$. Theorem 6.2 shows that every (1,2)–symplectic metric on $\mathcal{G}'$ extends uniquely to a (1,2)–symplectic metric on $\mathcal{G}$. Thus, the dimension of the cone of (1,2)–symplectic metrics is equal in both digraphs (in the previous section it was denoted by $\beta$). It is not clear how to calculate $\beta$ directly from the original digraph $\mathcal{G}$. One approach is to represent (1-3) as a homogeneous linear system and calculate the dimension of its kernel.

We end this section with several remarks.
(i) The completely non-transitive digraph $\mathcal{G}'$ obtained by edge deletion from $\mathcal{G}$ does not depend on the order of the edges deleted;
(ii) A completely non-transitive digraph $\mathcal{G}'$ can be the outcome of a non-void edge deletion of a LT digraph if and only if the union of all the sub-digraphs of $\mathcal{G}'$ of type $\{uv, vw, zw\}$ and $\{wz, wv, vu\}$ does not contain every sub-digraph of type $\{uv, vw\}$;
(iii) It seems that a completely non-transitive digraph is the result of a edge deletion of a transitive digraph if and only if it is a union of open directed paths whose terminal vertices are exclusively winners or losers.

In [2] the author applied a nice counting argument in order to enumerate the *complete* LT digraphs with $n$ vertices. On first reading it appears that his method is not adequate for the enumeration of *all* the LT digraphs with $n$ vertices. One possible attack on the problem would be to enumerate first the completely non-transitive ones, and then to figure out how many LT digraphs edge-delete into a given completely non-transitive digraph.

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