# The Use of Unitary Functions in the Behaviour Analysis of Elliptic Integrals

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**Abstract**

We show how unitary functions can simplify the analysis of some elliptic integrals.

## 1. Introduction

The Theory of Riemann Surfaces is one of the main sources of elliptic integrals. This theory is largely applicable to Minimal Surfaces, and their connection was first established by two important works from Osserman [2], Meeks and Rosenberg [1]. In the analysis of real functions given by elliptic integrals, it happens frequently that the main purposes do not rely on accurate evaluations, but just on some general information about these functions as monotonicity, bounds and limits. In these cases, a subtable choice of the integration path or change of variables can greatly simplify the analysis. However, these procedures are not general and depend on each specific case, without guarantee of success.

If it is possible to get an integrand containing *unitary functions*, chances will then increase to simplify the integral analysis. A *unitary function* is the exponential of a pure imaginary function, which implies several properties explained in this paper. In fact, our present work is devoted to elliptic integrals of this kind. We shall exemplify the use of unitary functions in some special cases, but the procedures explained herein are structurally general and likely to be helpful in many other circumstances.

## 2. Examples

In this section we present a practical application involving three elliptic integrals. Their integrands contain two free parameters, and so the related to integrals are functions of two variables. First of all, consider a natural $n \geq 2$, a positive real $x > 1$ and the complex variable $h$ in the first quadrant $Q := \{z : 0 < Arg(z) < \frac{\pi}{2}\}$ of $\mathbb{C}$. The Möbius transformations $\frac{x-h}{x+h}$, $\frac{1-h}{1+h}$ and their inverses are such that this quadrant is always brought to open regions of $\mathbb{C}$ which exclude the real negative semi-axis. Therefore, the branch of the $n$-th root given by $\sqrt[n]{e^{it}} := e^{it/n}$ is well defined and continuous on these regions. We fix <u>this</u> branch and define

$$F(x,h) := \left(\frac{x-h}{x+h}\right)^{1-\frac{1}{n}} \cdot \left(\frac{1-h}{1+h}\right)^{\frac{1}{n}}. \tag{1}$$

Let $\gamma$ be an integration path in $Q$ connecting the segments $]0,1[$ and $]x,\infty[$ (see Figure 1). The function $F$ will turn out to be unitary on some special paths homotopic to $\gamma$. They will be discussed later.

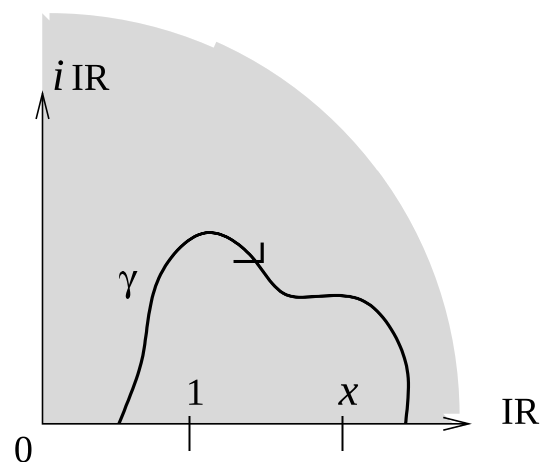


Figure 1: The first quadrant $Q$ with the curve $\gamma$.

Now define the following integrals:

$$a(n,x) := Re \int_\gamma \frac{(F+F^{-1})dh}{ih^2(1-h^2)}; \tag{2}$$

$$b(n,x) := 2Re\int_\gamma \frac{(F-F^{-1})dh}{ih(1-h^2)}, \text{ and} \tag{3}$$

$$c(n,x) := Re\int_\gamma \frac{(F+F^{-1})dh}{i(1-h^2)}. \tag{4}$$

We are going to prove Proposition 2.1, which summarizes some important information about the behaviour of the functions $a$, $b$ and $c$:

PROPOSITION 2.1. *The above defined functions $a$, $b$ and $c$ satisfy the following properties:*

*a) $b$ is negative and increasing with $x$;*

*b) $0 < -b < 2a$, for every $x > 1$;*

*c) $a$ is positive and decreasing with $x$;*

*d) $c$ is positive and increasing with $x$;*

*e) $a$, $b$ and $c$ are continuous at $x=1$. Moreover, $c|_{x=1} = 1 + \frac{b+c}{a}|_{x=1} = 0$;*

*f) The function $a+b+c$ is positive and increasing with $x$.*

PROOF

(a) An easy calculation shows that

$$\lim_{h\to 0}\frac{F-F^{-1}}{h} = \left.\frac{\partial F}{\partial h}\right|_{h=0} - \left.\frac{\partial F^{-1}}{\partial h}\right|_{h=0}.$$

Thus, the limit exists and is finite. Therefore, the differential in the integrand of (3) is holomorphic at the origin. This implies that $b(n,x)$ will be the same if we integrate this differential on the following curve: $h(t) = it, 0 < t < \infty$. A simple calculation leads to

$$b = 2\int_0^\infty \frac{((F-F^{-1})\circ h)(t)dt}{it(1+t^2)}. \tag{5}$$

Since $|F\circ h(t)| \equiv 1$, in this case we have $F - F^{-1} = 2iIm(F)$. Let us analyse the imaginary part of $F\circ h(t)$. The Möbius transformation $\frac{x-h}{x+h}$ brings the curve $h(t)$ to the lower unitary semi-circumference in $\mathbb{C}$. This implies that $-\pi < Arg(F\circ h(t)) < 0$ and therefore, $Im(F\circ h(t))$ is negative. By applying these conclusions to (5), one has that $b$ is negative.

To prove that $b$ is increasing with $x$, it will be sufficient to show that $\frac{\partial b}{\partial x} > 0$. We are going to make use of the following formulas:

$$\frac{\partial F}{\partial x} = 2(1-1/n)\cdot\frac{h}{x^2-h^2}\cdot F \quad \text{and} \tag{6}$$

$$\frac{\partial F^{-1}}{\partial x} = -2(1-1/n)\cdot\frac{h}{x^2-h^2}\cdot F^{-1}. \tag{7}$$

By applying (6) and (7) to (5) we get

$$\frac{\partial b}{\partial x} = 4(1-1/n)\int_0^\infty \frac{((F+F^{-1})\circ h)(t)dt}{(1+t^2)(x^2+t^2)}. \tag{8}$$

Since $|F\circ h(t)|\equiv 1$, in this case we have $F+F^{-1}=2Re(F)$. A simple calculation shows that the real part of $\dfrac{x-it}{x+it}$ is increasing with $x$. Since $Im(F)$ is negative, from (6) we conclude that $\dfrac{\partial}{\partial x}(Re(F))>0$. Because of that, $Re(F)$ is also increasing with $x$. Using the same arguments which will be discussed in item (e) of this proposition, one shows that the integrand in (8) is uniformly continuous at $x=1$. Then, if we show that $\left.\dfrac{\partial b}{\partial x}\right|_{x=1}>0$, we shall have $\dfrac{\partial b}{\partial x}>0$ for every $x>1$.

An easy calculation shows that

$$\left.\frac{\partial b}{\partial x}\right|_{x=1} = 8(1-1/n)\int_0^\infty \frac{(1-t^2)dt}{(1+t^2)^2(x^2+t^2)}.$$

By splitting the integration interval into $]0,1]$ and $[1,\infty[$, and using the change $t\to 1/t$ for the integral on $[1,\infty[$ we get:

$$\left.\frac{\partial b}{\partial x}\right|_{x=1} = 8(1-1/n)\{\int_0^1 \frac{(1-t^2)dt}{(1+t^2)^2(x^2+t^2)} - \int_0^1 \frac{(1-t^2)dt}{(1+t^2)^2(x^2+1/t^2)}\}.$$

Since $t\in]0,1[$ implies $t<1/t$, it follows $\left.\dfrac{\partial b}{\partial x}\right|_{x=1}>0$ and together with the fact that $Re(F)$ increases with $x$, we then have $\dfrac{\partial b}{\partial x}>0$ for every $x>1$. This concludes item (a) of Proposition 2.1.

(b) A simple calculation leads to

$$a+\frac{b}{2} = \int_\gamma \frac{F dh}{ih^2(1-h)} + \int_\gamma \frac{F^{-1}dh}{ih^2(1+h)}. \tag{9}$$

For the integrals in (9) we can choose their integration path to be $h(t)=t, 1<t<x$, instead of $\gamma$. In this case, we assert that

$$F\circ h(t) = \left(\frac{x-t}{x+t}\right)^{1-1/n}\cdot\left(\frac{t-1}{t+1}\right)^{1/n}\cdot e^{-i\pi/n}. \tag{10}$$

The reason for the choice $\sqrt[n]{-1} = e^{-i\pi/n}$ is due to Figure 1. To be compatible with this picture we must choose the value of $\sqrt[n]{-1}$ which makes the function continuous on $\bar{Q}$.

Now based on (10) we can easily rewrite (9) as follows:

$$a + \frac{b}{2} = \sin\frac{\pi}{n} \cdot \{\int_1^x \frac{|F(h(t))| dt}{t^2(t-1)} + \int_1^x \frac{|F(h(t))|^{-1} dt}{t^2(t+1)}\}. \tag{11}$$

Clearly, the right-hand side of (11) is positive for every $x > 1$. Together with item (a) we have $2a + b - b > -b > 0$. This concludes item (b) of Proposition 2.1.

(c) We have just proved item (b), which implies that $a$ is positive. Let us now analyse the derivative $\frac{\partial a}{\partial x}$. By applying (6) and (7) to (2) we obtain:

$$\frac{\partial a}{\partial x} = 2(1 - 1/n) \int_\gamma \frac{(F - F^{-1}) dh}{ih(1-h^2)(x^2 - h^2)}. \tag{12}$$

As we have mentioned before, $\lim_{h \to 0} \frac{F - F^{-1}}{h}$ exists and is finite. Therefore, the integrand in (12) is holomorphic at $h = 0$ and the value of the integral will be the same if we choose the integration path to be $h(t) = it$, for $0 < t < +\infty$, instead of $\gamma$. In the demonstration of item (a) we saw that, in this case $F - F^{-1} = 2iIm(F)$ and $Im(F)$ is negative. Hence, $\frac{\partial a}{\partial x}$ is positive and this concludes item (c).

(d) The integrand in (12) is holomorphic at $h = 0$ and so the integral value will be invariant if we take the integration path $h(t) = it$, $0 < t < +\infty$, instead of $\gamma$. In this case we have:

$$c = \int_0^\infty \frac{((F + F^{-1}) \circ h)(t) dt}{1 + t^2}. \tag{13}$$

As we said before, in this case $(F + F^{-1})(it) = 2Re(F(it))$, which is increasing with $x$. In the next item we shall prove that $c$ is continuous at $x = 1$. Hence, if $c|_{x=1} \geq 0$, then $c$ will be positive for every $x > 1$ and also increasing with $x$. Let us analyse the case $x = 1$. From (13) we shall have:

$$c|_{x=1} = \int_0^\infty \frac{(1 - t^2) dt}{(1 + t^2)^2}.$$

By splitting the integration interval into $]0,1]$ and $[1,\infty[$, and using the change $t \to 1/t$ for the integral on $[1,\infty[$ we get:

$$c|_{x=1} = \int_0^1 \frac{(1-t^2)dt}{(1+t^2)^2} - \int_0^1 \frac{(1-t^2)dt}{(1+t^2)^2} = 0,$$

which concludes item (d) of Proposition 2.1.

(e) Consider the quadrant $Q = \{z : 0 < Arg(z) < \frac{\pi}{2}\}$ of $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ and a compact subset $\mathcal{K} \subset \bar{Q}$. Moreover, suppose that $\mathcal{K} \cap \{0,1,x\} = \emptyset$. In this case, if $h \in \mathcal{K}$, then

$$\frac{x-h}{x+h} - \frac{x+h}{x-h} = \frac{2(x-1)h}{(x+h)(1-h)},$$

which shows that the convergence $\lim_{x\to 1} F^n(x,h) = \left(\frac{1-h}{1+h}\right)^n$ is uniform on $\mathcal{K}$. In the demonstration of item (b) we took $\sqrt[n]{-1} = e^{-i\pi/n}$. Therefore, the convergence $\lim_{x\to 1} F(x,h) = \frac{1-h}{1+h}$ is also uniform on $\mathcal{K}$. We can choose $\mathcal{K}$ such that $\{\gamma\} \subset \mathcal{K}$. Hence, the functions $a, b$ and $c$, defined by (2-4) are continuous at $x = 1$. We have just seen that $c|_{x=1} = 0$. Let us analyse $\frac{b+c}{a}$.

Given a real variable $y \in [0,1]$ we can define the parabola $P := ay^2 + by + c$ and rewrite it as

$$P = \int_\gamma \left\{ (h+y)^2 F + (h-y)^2 F^{-1} \right\} \frac{dh}{ih^2(1-h^2)}. \tag{14}$$

in the special case $x = 1$, we apply the following simplifications:

$$\frac{(h+y)^2 F}{h^2(1-h^2)} = \frac{1}{h^2}\left(\frac{h+y}{h+1}\right)^2 = \frac{a_1}{h} + \frac{b_1}{1+h} + \frac{c_1}{h^2} + \frac{d_1}{(1+h)^2} \tag{15}$$

and

$$\frac{(h-y)^2 F^{-1}}{h^2(1-h^2)} = \frac{1}{h^2}\left(\frac{h-y}{h-1}\right)^2 = \frac{a_2}{h} + \frac{b_2}{1-h} + \frac{c_2}{h^2} + \frac{d_2}{(1-h)^2}. \tag{16}$$

where

$$\begin{cases} c_1 = y^2 \\ d_1 = (1-y)^2 \\ a_1 = 2y(1-y) \\ b_1 = -2y(1-y) \end{cases} \quad \text{and} \quad \begin{cases} c_2 = y^2 \\ d_2 = (1-y)^2 \\ a_2 = -2y(1-y) \\ b_2 = -2y(1-y). \end{cases}$$

Therefore,

$$P|_{x=1} = Re\int_\gamma b_1\left(\frac{1}{1+h}+\frac{1}{1-h}\right)\frac{dh}{i}$$
$$-\underbrace{Re}_{=0}\{i[-(c_1+c_2)h^{-1}-d_1(1+h)^{-1}+d_2(1-h)^{-1}]_y^\infty\}.$$

Let us consider $h\circ\gamma(t)=y+it, 0\le t\le\infty$. Then

$$P|_{x=1} = b_1\int_0^\infty\left[\frac{1+y}{(1+y)^2+t^2}+\frac{1-y}{(1-y)^2+t^2}\right]dt$$
$$=2y(y-1)\left[\arctan\frac{t}{1+y}+\arctan\frac{t}{1-y}\right]_0^\infty = 2y(y-1)\pi.$$

This means, we have the explicit equation of the parabola in this case. It confirms that $c|_{x=1}=0$. Moreover, one has $(a+b+c)|_{x=1}=0$ as well, with $a|_{x=1}=-b|_{x=1}=2\pi$. Thus, $1+\dfrac{b+c}{a}|_{x=1}=0$ and the item (e) of Proposition 2.1 is concluded.

(f) We have just seen that $(a+b+c)|_{x=1}=0$. By calculating $y=1$ in (14) we get

$$P|_{y=1}=a+b+c=Re\int_\gamma(\tilde{F}+\tilde{F}^{-1})\cdot\frac{dh}{ih^2},\quad\text{where}$$

$$\tilde{F}:=\frac{1+h}{1-h}\cdot F=\left(\frac{x-h}{x+h}\right)^{1-\frac{1}{n}}\cdot\left(\frac{1+h}{1-h}\right)^{1-\frac{1}{n}}.$$

Regarding the derivative with respect to $x$, the functions $\tilde{F}$ and $\tilde{F}^{-1}$ follow the same rules as in (6) and (7). Therefore,

$$\frac{\partial}{\partial x}(a+b+c)=2(1-1/n)Re\int_\gamma(\tilde{F}-\tilde{F}^{-1})\cdot\frac{dh}{ih(x^2-h^2)}. \tag{17}$$

Now we proceed with the same arguments used in item (c) to prove that $\dfrac{\partial a}{\partial x}$ is negative. Namely, the limit $\lim\limits_{h\to0}\dfrac{\tilde{F}-\tilde{F}^{-1}}{h}$ exists and is finite. Then, the integration (17) can be done on the path $h(t)=it, 0<t<\infty$, instead of $\gamma$. Hence,

$$\frac{\partial}{\partial x}(a+b+c)=2(1-1/n)Re\int_0^\infty\frac{\tilde{F}-\tilde{F}^{-1}}{it}\cdot\frac{dt}{x^2+t^2}. \tag{18}$$

Similarly to $F$, one easily verifies that $Im(\tilde{F})$ is positive on $h(t)$, for any $x > 1$. Since $\tilde{F} - \tilde{F}^{-1} = 2iIm(\tilde{F})$, from (18) it follows that $a + b + c$ is increasing with $x$. Together with the fact that $(a + b + c)|_{x=1} = 0$, we finally conclude the last item of Proposition 2.1.

q.e.d.

## References


[**1**] W.H. Meeks III and H. Rosenberg, *The geometry of periodic minimal surfaces*, Comment. Math. Helv. (4), **68** (1993), 538-578.

[**2**] R. Osserman, *A survey of minimal surfaces*, Second edition, Dover Publications, New York (1986).