A recursive description of pro-p Galois groups

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Abstract

In this note we extend the results of our earlier work "Kaplansky's radical and a recursive description of pro-2 Galois groups" (Rel. Pesq. 23/01) to arbitrary prime numbers p. Although we succeed in proving the same results, the methods used in the proofs are more conceptual. To be precise, let $G_p(F)$ be the Galois group of the maximal Galois p-extension of a field F of characteristic $\neq p$. Denote by $R(F)$ the radical of the skew-symmetric bilinear pairing which associates to each pair a, b of non-zero elements of F the class of the cyclic algebra $(a, b)_F$ in the Brauer group of F. We deduce from a condition connecting $R(F)$ with valuation rings of F and also orderings of F when $p = 2$, that $G_p(F)$ can be obtained from some suitable closed subgroups using free pro-2 products and semi-direct group extension operations a finite number of times.

Key Words: K-theory; valuation; ordering; pro-p group; free pro-p product.

1 Introduction

Fix a prime number p and let F be a field of characteristic $\neq p$ containing a primitive p-th root of unity. Denote by $G_p(F)$ the Galois group of the maximal p-extension of F. It is conjectured that if $G_p(F)$ is (topologically) finitely generated, then $G_p(F)$ can be built from some "basic" pro-p groups by iterating two group theoretical operations (the so-called elementary type conjecture). The basic groups are $\mathbb{Z}_p$, Demushkin pro-p groups and $\mathbb{Z}/2\mathbb{Z}$ if $p = 2$. The operations used are free products and certain semidirect products in the category of pro-p groups. We propose to deal with a simplified version of this conjecture.

Before we state our results in more detail, let us fix some notations which are used throughout the paper.

By a localizer of F we shall mean either a valuation ring of F with residue field of characteristic $\neq p$ or, in the case $p = 2$, the positive cone of an ordering of F. A pair (F, A) is called locally closed if either A is a p-henselian valuation ring of F or F is euclidean and A is the positive cone of the ordering of F, when $p = 2$.

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Our aim is to show that $G_p(F)$ can be constructed following the process described in the first paragraph where in addition to the listed basic groups we also consider $G_p(L)$, for locally closed extensions (L, A') of F inside $F(p)$.

A particular instance of our results is the following known theorem ([17, Theorem 4.3], [5, Proposition 4.3]).

THEOREM. *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a family of localizers of F which induce different topologies on F. Fix, for every $1 \leq i \leq n$ a locally closed extension (H_i, A_i') of (F, A_i) in $F(p)$ and suppose that $H_1 \cap \cdots \cap H_n = F$. Then $G_p(F)$ decomposes into a free pro-p product $G_p(F) = G_p(H_1) * \cdots * G_p(H_n)$.*

Observe now that the Galois group $G_p(K)$ of every finite intermediate extension $K \subset F(p)$ of F inherits from $G_p(F)$ a similar decomposition. More precisely, by Kurosh's subgroup theorem (see for example [3]) $G_p(K) = G_0 * G_1 * \cdots G_m$, where G_0 is a free pro-p group and for every $1 \leq j \leq m$, G_j is the Galois group of an extension of some H_i inside $F(p)$, a locally closed extension of K. If G_0 is non-trivial, then K is not the intersection of the fixed fields of $G_1, \ldots, G_m$. Therefore we have to weaken the condition $H_1 \cap \cdots \cap H_n = F$ in order that our results remain true for the finite subextensions of $F(p)|F$. This is done by the introduction of a subgroup $R(F)$, called the *radical* of F as we explain in section 3. In fact $R(F)$ corresponds to the radical of the bilinear cup product.

Our results depend on a cohomological criterion, according to Neukirch, for a pro-p group to be a free pro-p product of closed subgroups ([28]) and the results of Merkurjev and Suslin which state the connection between Milnor's K-theory of a field F with the cohomology of the Galois group $G_p(F)$ [25]. These results are the subject of the next section.

In Section 3 we deal with the relationship between $R(F)$ and free products of subgroups of $G_p(F)$. Section 4 is dedicated to study the connection between localizers and $R(F)$. In section 5 we prove the above theorem in a more general form (Proposition 5.4). This will be the first step towards the main results which are in Section 6, where we also introduce the $\mathcal{A}$-admissible groups (Definition 3). In the last two sections we study properties of fields F for which $G_p(F)$ decomposes into a free pro-p product.

Throughout the paper every subgroup of a pro-p group is supposed to be closed and every homomorphism continuous.

We hope this paper will be a contribution to the study of the conjecture mentioned above. Moreover our theorems generalize several known results about the decomposition of $G_p(F)$ as a free pro-p product under heavier assumptions.

Finally, if F has characteristic p, then $G_p(F)$ is known to be a free pro-p group.

2 On free products

For any pro-p group G let $H^i(G)$ be the i-th continuous cohomology group of G with coefficients in $\mathbb{Z}/p\mathbb{Z}$. Recall that a pro-p group admits a decomposition $G = G_1 * \cdots * G_n$ into a free pro-p product of closed subgroups $G_1, \ldots, G_n$ if and only if the homomorphism $\mathrm{Res}^i : H^i(G) \longrightarrow H^i(G_1) \times \cdots \times H^i(G_n)$ is an isomorphism for $i = 1$ and injective for $i = 2$ [28, Satz 4.3].

For a Galois pro-p group $G_p(F)$ we want to translate this cohomological criterion into arithmetical conditions on F. This will be the subject of Proposition 2.1.

To be precise, assume that $G = G_p(F)$ for a field F and $G_i = G_p(H_i)$ for some extensions $H_i \subset F(p)$ of F, for every $i = 1, \ldots, m$. Since the map Res is induced by inclusion, Res^1 is an isomorphism (epimorphism) if and only if the homomorphism $\dot{F}/\dot{F}^p \longrightarrow \dot{H}_1/\dot{H}_1^p \times \cdots \times \dot{H}_n/\dot{H}_n^p$, induced by inclusions, is an isomorphism (epimorphism) because of the following well known facts:

- *For every pro-p group G, $H^1(G)$ is canonically isomorphic to the group of continuous homomorphisms $Hom_c(G, \langle\xi\rangle)$*, where ξ is a primitive p-th root of unity.
- *For every field F, Kummer's Theory implies that $\dot{F}/\dot{F}^p \cong Hom_c(G_p(F), \langle\xi\rangle)$.*

The other condition requires more considerations. By the Merkurjev-Suslin theorem we have an isomorphism $k_2F \longrightarrow H^2(G_p(F), \langle\xi\rangle)$, connecting Milnor K-theory and Galois cohomology. Therefore, the injectivity of Res^2 is equivalent to saying that inclusions $F \subset H_i$, $i = 1, \ldots, n$ induce an injective homomorphism $k_2F \longrightarrow k_2H_1 \times \cdots \times k_2H_n$.

Before we state our criterion for pro-p free products let us introduce some new conventions. Write $\dot{F}$ and $\dot{F}^p$ to represent the multiplicative groups of nonzero elements and nonzero p-th powers of F, respectively. Let $\widetilde{F} = \dot{F}/\dot{F}^p$ and denote by $\widetilde{a}$ the image of $a \in \dot{F}$ in $\widetilde{F}$. Observe that $\widetilde{F} \cong k_1F$ [26].

From now on $H_1, \ldots, H_n$ will always be extensions of F inside $F(p)$. For every $a \in F$ we denote by $D_F(a)$ and $D_i(a)$ the image of the norm homomorphism $N_a^F : F(\sqrt[p]{a}) \setminus \{0\} \longrightarrow \dot{F}$ and $N_a^i : H_i(\sqrt[p]{a}) \setminus \{0\} \longrightarrow \dot{H}_i$, respectively.

Proposition 2.1. *For F and $H_1, \ldots, H_n$ as above we suppose the following conditions hold:*

(I) The inclusions $F \subset H_i$ induce an isomorphism $\varphi_1 : \widetilde{F} \longrightarrow \widetilde{H}_1 \times \cdots \times \widetilde{H}_n$.

(II) For $a, b \in \dot{F}$, if $b \in D_i(a)$ for every $i = 1, \ldots, n$, then $b \in D_F(a)$.

Then $G_p(F) = G_p(H_1) * \cdots * G_p(H_n)$.

Conversely, if $G_p(F)$ admits the above decomposition, then conditions (I) and (II) are true.

Proof. According to the coments preceding the proposition, it is enough to prove that $k_2F \longrightarrow k_2H_1 \times \cdots \times k_2H_n$ is injective. As observed in the proof of Proposition 4 [24], $k_2F = \widetilde{F} \otimes \widetilde{F}/A$ and, for every $1 \le i \le n$, $k_2H_i = \widetilde{H}_i \otimes \widetilde{H}_i/A_i$, where A and A_i are the subgroups generated by $\{\widetilde{x} \otimes \widetilde{y} \mid x, y \in \dot{F} \text{ and } y \in D_F(x)\}$ and $\{\widetilde{x} \otimes \widetilde{y} \mid x, y \in \dot{H}_i \text{ and } y \in D_i(x)\}$, respectively. Let $\theta : \widetilde{F} \otimes \widetilde{F} \longrightarrow k_2F$ and $\theta_i : \widetilde{H}_i \otimes \widetilde{H}_i \longrightarrow k_2H_i$ be the canonical maps. Consider now the following diagram,

$$\begin{array}{ccccccccc}
0 & \to & \prod_{j=1}^{n} A_j & \to & \prod_{j=1}^{n} \widetilde{H}_j \otimes \widetilde{H}_j & \xrightarrow{\prod_{j=1}^{n} \theta_j} & \prod_{j=1}^{n} k_2H_j & \to & 0 \\
 & & \uparrow \Phi_0 & & \uparrow \Phi & & \uparrow \varphi_2 & & \\
0 & \to & A & \longrightarrow & \widetilde{F} \otimes \widetilde{F} & \xrightarrow{\theta} & k_2F & \longrightarrow & 0
\end{array}$$

where the vertical arrows are described as follows: Φ is the composition of the isomorphism

$$\varphi_1 \otimes \varphi_1 : \widetilde{F} \otimes \widetilde{F} \longrightarrow \left(\prod_{i=1}^{n} \widetilde{H}_i\right) \otimes \left(\prod_{i=1}^{n} \widetilde{H}_i\right)$$

with the natural projection

$$\prod_{r,s} \widetilde{H}_r \otimes \widetilde{H}_s \longrightarrow \prod_{j=1}^{n} \widetilde{H}_j \otimes \widetilde{H}_j.$$

For $x, y \in \dot{F}$ such that $y \in D_F(x)$ it follows that $y \in D_j(x)$, for every $j = 1, \ldots, n$. Hence $\Phi(A) \subset \prod_{j=1}^n A_j$ and we call Φ_0 the restriction of Φ to A and φ_2 the induced quotient homomorphism. Finally, observe that the squares are commutative.

We shall prove that φ_2 is injective by diagram chasing. To this end we shall show that kernel$\Phi \subset A$ and Φ_0 is surjective.

For every $1 \leq i \leq n$ let $S_i = \dot{F} \cap (\bigcap_{j \neq i} \dot{H}_j^p)$ and denote $\widetilde{S}_i = S_i/\dot{F}^p$ the image of S_i in $\widetilde{F}$. By (I), φ_1 maps $\widetilde{S}_i$ isomorphicaly on $\widetilde{H}_i$, for every $1 \leq i \leq n$. Consequently $\widetilde{F} = \widetilde{S}_1 \oplus \cdots \oplus \widetilde{S}_n$. Hence $\widetilde{F} \otimes \widetilde{F} = \bigoplus_{r,s} \widetilde{S}_r \otimes \widetilde{S}_s$ and then kernel$\Phi = \bigoplus_{r \neq s} \widetilde{S}_r \otimes \widetilde{S}_s$. We now claim that kernel$\Phi \subset A$. Indeed, for $x \in S_r$ and $y \in S_s$, since $x \in \dot{H}_j^p$ for every $j \neq r$, $D_j(x) = \dot{H}_j$ and so $y \in D_j(x)$. For $j = r \neq s$, $y \in \dot{H}_r^p \subset D_r(x)$. Thus, by (II), $y \in D_F(x)$ and so $\widetilde{x} \otimes \widetilde{y} \in A$. Therefore, for every $1 \leq r \neq s \leq n$, $\widetilde{S}_r \otimes \widetilde{S}_s \subset A$ and the claim is proved.

Next, we prove that Φ_0 is a surjective map. We have that $\prod_{i=1}^n A_i$ is generated by the elements $(\widetilde{x}_1 \otimes \widetilde{y}_1, \widetilde{x}_2 \otimes \widetilde{y}_2, \ldots, \widetilde{x}_n \otimes \widetilde{y}_n)$, such that $y_i \in D_i(x_i)$, for every $i = 1, \ldots, n$. By (I) there are $x, y \in \dot{F}$ satisfying $\varphi_1(\widetilde{x}) = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ and $\varphi_1(\widetilde{y}) = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$. Hence $x\dot{H}_i^p = x_i\dot{H}_i^p$ and $y\dot{H}_i^p = y_i\dot{H}_i^p$, for every $i = 1, \ldots, n$. Consequently, $D_i(x_i) = D_i(x)$ and $y \in D_i(x)$, for every $1 \leq i \leq n$. Thus $\widetilde{x} \otimes \widetilde{y} \in A$. In the other side, $\widetilde{x} \otimes \widetilde{y} = \widetilde{x}_i \otimes \widetilde{y}_i$ in $\widetilde{H}_i \otimes \widetilde{H}_i$, for every $i = 1, \ldots, n$. So $\Phi(\widetilde{x} \otimes \widetilde{y}) = (\widetilde{x}_1 \otimes \widetilde{y}_1, \ldots, \widetilde{x}_n \otimes \widetilde{y}_n)$ and then Φ_0 is surjective as required.

Once these two facts are established we can prove that φ_2 is injective. For $z \in k_2F$ such that $\varphi_2(z) = 0$ take $y \in \widetilde{F} \otimes \widetilde{F}$ with $\theta(y) = z$. Then $\prod_{i=1}^n \theta_i \circ \Phi(y) = 0$. Thus $\Phi(y) \in \prod_{i=1}^n A_i$ and by the surjectivity of Φ_0, there is $y' \in A$ such that $\Phi(y') = \Phi(y)$. As kernel$\Phi \subset A$ and $y' \in A$ it follows that $y \in A$ and so $z = \theta(y) = 0$.

The converse is seen by observing first that (I) follows from Neukirch's criterion ([28, Satz 4.3]), as we discussed at the beginning of this section. To prove (II), take $a, b \in \dot{F}$ such that $b \in D_i(a)$, for every $1 \leq i \leq n$. Then $\Phi(\widetilde{a} \otimes \widetilde{b}) \in \prod_{i=1}^n A_i$ which implies that $\varphi_2(\theta(\widetilde{a} \otimes \widetilde{b}) = 0$. The results of Neukirch and Merkurjev-Suslin imply that φ_2 is injective. consequently $\theta(\widetilde{a} \otimes \widetilde{b}) = 0$. Thus, by [27, Corollary 15.11], $b \in D_F(a)$, as desired. □

3 On the p-radical of a field and free products

In this section we study a field F for which $G_p(F)$ has a decomposition as in Proposition 2.1 with one of the factors a free pro-p group.

Definition 1. Let $R(F) = \bigcap_{a \in \dot{F}} D_F(a)$.

According to [27, Corollary 15.11], $R(F)$ is the radical of the symbol $\{,\} \ : \dot{F} \times \dot{F} \longrightarrow k_2F$ which associates to $x, y \in \dot{F} \longmapsto \{x, y\} =$ the image of $\widetilde{x} \otimes \widetilde{y}$ in k_2F. In other words $R(F) = \{r \in \dot{F} \mid \{r, x\} = 0$ for every $x \in \dot{F}\}$.

Since $\{x, y\} = -\{y, x\}$ ([26, Lemma 1.1]), for every $x, y \in \dot{F}$, the equivalence "$x \in D_F(y)$ if and only if $\{x, y\} = 0$ in k_2F" implies that $x \in D_F(y)$ if and only if $y \in D_F(x)$. Thus we may also describe $R(F)$ as $R(F) = \{r \in \dot{F} \mid D_F(r) = \dot{F}\}$.

The characterization of $R(F)$ can be made through a more concrete invariant of F if we look at cyclic algebras $(a, b)_F$, $a, b \in \dot{F}$ [31, §30]. Then $R(F)$ is the radical of the skew-symmetric paring $\dot{F} \times \dot{F} \to \mathrm{Br}_p(F)$, where $\mathrm{Br}_p(F)$ is the elementary p-primary subgroup of the Brauer group of F.

In the case $p = 2$, $R(F)$ is known as the radical of Kaplansky ([18]). For $p \neq 2$, Koenigsmann used $R(F)$ in the characterization of $G_p(F)$ for fields F such that $\dot{F}/R(F)$ has order at most 16.

Next we state three technical lemmas which connect $R(F)$ and a free pro-2 component in a free product decomposition of $G_p(F)$.

Lemma 3.1. *Let $L \subset F(p)$ be an extension of F and let $R \subset R(F)$ be a subgroup of $\dot{F}$.*

(a) If $\dot{L} = \dot{F}\dot{L}^p$, then $R(F) \subset R(L)$.

(b) If $\dot{L} = R\dot{L}^p$, then $R(L) = \dot{L}$.

(c) There exists an extension $F \subset E \subset F(p)$ such that the inclusion induces an isomorphism $R/\dot{F}^p \longrightarrow \dot{E}/\dot{E}^p$. Consequently $\dot{E} = R\dot{E}^p$.

Proof. (a) is immediate.

(b) Take $x, y \in \dot{L}$ and let $a, b \in R$ such that $xa^{-1}, yb^{-1} \in \dot{L}^p$. Since $R \subset R(F)$, $b \in D_F(a)$. As $D_F(a) \subset D_L(a) = D_L(x)$, it follows that $y \in D_L(x)$. Thus $\dot{L} \subset D_L(x)$ and $x \in R(L)$. Hence $\dot{L} = R(L)$.

(c) Take an extension E of F inside $F(2)$ such that the inclusion induces an injective map $R/\dot{F}^p \longrightarrow \dot{E}/\dot{E}^p$ and E is maximal with this property. The maximality of E implies that $R/\dot{F}^p \longrightarrow \dot{E}/\dot{E}^p$ is an isomorphism, as required. ☐

Lemma 3.2. *$G_p(F)$ is a free pro-p group if and only if $R(F) = \dot{F}$.*

Proof. By [33, Theorem 7.7.4] and [25, Theorem 11.5], $G_p(F)$ is a free pro-p group if and only if $k_2F = 0$. Since $k_2F = 0$ if and only if $\{x, y\} = 0$, for every $x, y \in \dot{F}$ the statement follows from the connection between $\{x, y\} = 0$ and $x \in D_F(y)$. ☐

Lemma 3.3. *Assume that the condition (II) of Proposition 2.1 holds for a field F and a family of intermediate extensions $F \subset H_1, \ldots, H_n \subset F(p)$. Then $\dot{H}_1^p \cap \ldots \cap \dot{H}_m^p \cap F \subset R(F)$.*

Proof. For every $r \in \dot{H}_1^p \cap \ldots \cap \dot{H}_m^p \cap F$ and $1 \leq i \leq m$, we have $D_i(r) = \dot{H}_i$. Thus by condition (II) of Proposition 2.1, $D_F(r) = \dot{F}$ and so $r \in R(F)$, showing the inclusion. ☐

In [15] the authors generalize the earlier mentioned Neukirch's criterion by taking Res^1 surjective instead of isomorphism (see also [23] and [6]). Regarding a pro-p group G and a family $G_1, \ldots, G_m$ of closed subgroups of G they state that there exists a free closed subgroup G_0 of G such that $G = G_0 * G_1 \cdots * G_m$ if and only if the homomorphism $Res^i : H^i(G) \longrightarrow H^i(G_1) \times \cdots \times H^i(G_m)$ is surjective for $i = 1$ and injective for $i = 2$ [15, Theorem 2.1]. In our next result we shall see that the free component G_0 is associated with $R(F)$.

Proposition 3.4. *For F and $H_1, \ldots, H_m$ as in Proposition 2.1 we suppose that:*

(Ia) The inclusions $F \subset H_i$ induce an epimorphism $\varphi_1 : \widetilde{F} \longrightarrow \widetilde{H}_1 \times \cdots \times \widetilde{H}_n$.

(II) For $a, b \in \dot{F}$, if $b \in D_i(a)$ for every $i = 1, \ldots, n$, then $b \in D_F(a)$.

Write $R = \dot{H}_1^p \cap \ldots \cap \dot{H}_m^p \cap F$ for short. Then there exists another intermediate extension $H_0 \subset F(2)$ such that $G_p(H_0)$ is a free pro-p group, the family $H_0, H_1, \ldots, H_m$ satisfies conditions (I) and (II) of Proposition 2.1 and $R/\dot{F}^p \cong \dot{H}_0/\dot{H}_0^p$.
Moreover,

(a) if the homomorphism in (Ia) is not injective, then $R(F) \neq \dot{F}^p$ ($R(F)$ is non-trivial).

(b) If $R(H_i) = \dot{H}_i^p$, for every $i = 1, \ldots, m$, then $R(F) = R$.

Proof. Since by Lemma 3.3 $R \subset R(F)$ we get from Lemma 3.1 an extension H_0 of F such that $R/\dot{F}^p \cong \dot{H}_0/\dot{H}_0^p$ and $R(H_0) = \dot{H}_0$. Hence Lemma 3.2 implies that $G_p(H_0)$ is a free pro-2 group.

Next, observe that $R/\dot{F}^p$ is the kernel of the homomorphism φ_1 (surjective by (Ia)). Therefore the homomorphism $\dot{F}/\dot{F}^p \longrightarrow \dot{H}_0/\dot{H}_0^p \times \dot{H}_1/\dot{H}_1^p \times \cdots \times \dot{H}_n/\dot{H}_n^p$ is injective. It remains to show that it is surjective. To prove this, take $h_i \in \dot{H}_i$ for every $0 \leq i \leq n$. From (Ia) there is $x \in \dot{F}$ such that $x\dot{H}_i^p = h_i\dot{H}_i^p$, for every $i = 1, \ldots, n$. Since $R/\dot{F}^p \cong \dot{H}_0/\dot{H}_0^p$, there is $y \in R$ such that $y\dot{H}_0^p = x^{-1}h_0\dot{H}_0^p$. Therefore, if we take $z = xy \in \dot{F}$, it follows that $z\dot{H}_i^p = h_i\dot{H}_i^p$, for every $i = 0, \ldots, n$ and the surjectivity is established. Consequently the conditions (I) and (II) of Proposition 2.1 hold for F and $H_0, H_1, \ldots, H_m$.

(a) The non-injectivity in condition (Ia) implies $R \neq \dot{F}^p$ and a fortiori $R(F) \neq \dot{F}^p$.

(b) It follows from (Ia) that $\dot{H}_i = \dot{F}\dot{H}_i^p$, $i = 1, \ldots, m$. Therefore Lemma 3.1 implies that $R(F) \subset \dot{H}_i^p \cap F$, $i = 1, \ldots, m$, proving the equality. □

The above result and Proposition 2.1 imply the truth of the following corollary.

Corollary 3.5. *[15] For F and $H_1, \ldots, H_m$ satisfying conditions (Ia) and (II) of the last proposition, it follows that there is an extension H_0, as in the previous proposition, such that*

$$G_p(F) = G_p(H_0) * G_p(H_1) * \cdots * G_p(H_m).$$

4 Localizers and radical

In this section we study the relationship between localizers of F and $R(F)$.

For every valuation ring A, denote by A^*, m_A, $k_A = A/m_A$, π_A, Γ_A and v_A the group of units of A, the maximal ideal, the residue field, the canonical homomorphism, the value group and a valuation corresponding to A, respectively. Recall that for every valuation ring A, k_A has characteristic $\neq p$ by assumption.

For each localizer A we write $\Re(A) = (1 + m_A)\dot{F}^p$ for a valuation ring A and $\Re(A) = A$ for a positive cone.

We say that a locally closed pair (L, A') is a *local closure* of (F, A) in $F(p)$ if $L \subset F(p)$, $A' \cap F = A$ and the pair (L, A') is minimal with these properties, i.e. if (K, B) is locally closed, $F \subset K \subset L$ and $B \cap F = A$, then $K = L$ and $B = A'$. A local closure (L, A') of (F, A) in $F(p)$ can also be described as follows: for a valuation ring A let C be an extension of A to $F(p)$. Then L is the decomposition field of C over F and $A' = C \cap L$ [9, p. 110]. If $p = 2$ and A is a positive cone, from Zorn's lemma there are maximal ordered extensions (L, A') of (F, A) inside $F(2)$. Due to the maximality of (L, A'), $A' = \dot{L}^2$. Therefore $\dot{L} = \dot{L}^2 \cup -\dot{L}^2$ and since $F(2) \mid L$ is a Galois 2-extension, it follows that $F(2) = L(\sqrt{-1}\,)$. Thus $|G_2(F)| = 2$ and $G_2(F)$ is one of the groups listed as basic at the beginning of the paper.

We shall need the following relative version of [9, Theorem 17.17] for valuation rings.

Theorem 4.1. *For $y \in F(p)$ denote by $f(X) \in F[X]$ the minimal polynomial of y over F. Let A be a valuation ring of F and let (H, A') be a local closure of (F, A). Then, the number of irreducible factors of a decomposition of $f(X)$ in $H[X]$ equals the number of extensions of A to $F(y)$.*

Proof. Let $(\bar{H}, \bar{A})$ be a henselization of (F, A) [9, § 17]. By [9, 15.6 c)] $H = \bar{H} \cap F(p)$ and $A' = \bar{A} \cap H$. Observe that $(\bar{H}, \bar{A})$ is also a henselization of (H, A'). Consider now a factorization $f = g_1 \cdots g_m$ of f in irreducible polynomials in $H[X]$. Since A' has only one extension to every intermediate extension $H \subset L \subset F(p)$, it follows from [9, Theorem 17.17] that each g_i is irreducible in $\bar{H}[X]$. Thus $f = g_1 \cdots g_m$ is also the factorization of f in $\bar{H}[X]$. Therefore [9, Theorem 17.17] implies that A has m extensions to $F(y)$. □

Lemma 4.2. *Let (H, A') be a local closure of (F, A) in $F(p)$.*

(a) $\dot{H} = \dot{F}\dot{H}^p$.

(b) $\dot{H}^p \cap F = \Re(A)$.

Proof. (a) In the case $p = 2$ if A is a cone, since $\dot{H} = \dot{H}^2 \cup -\dot{H}^2$, the statement is clearly true.

We now consider the case where A is a valuation ring. Denote by m', π' and v', the maximal ideal, the canonical homomorphism and a valuation corresponding to A', respectively.

Recall first that (H, A') is an immediate extension of (F, A) [9, Theorem 15.8, p. 112]. Hence, for $x \in \dot{H}$ there is $c \in \dot{F}$ such that $v'(x) = v'(c)$. We can now find $u \in A^*$ such that $\pi'(xc^{-1}) = \pi'(u)$. Thus $xc^{-1}u^{-1} \in 1 + m'$. By assumption k_A has characteristic $\neq p$. Therefore, the p-henselianity of A' implies $1 + m' \subset \dot{H}^p$. Thus $x \in F\dot{H}^p$, as desired.

(b) The statement is trivially true in the case $p = 2$ and A a cone of an ordering of F.

For the valuation ring case, take $z \in \dot{H}^p \cap F$. Let $x \in H$ such that $z = x^p$. As we get in the proof of item (a), there are $c, u \in \dot{F}$ such that $x \in cu(1 + m')$. Consequently, $z(cu)^{-p} \in F \cap (1 + m') = 1 + m_A$. Hence $z \in (1 + m_A)\dot{F}^p$. The other inclusion follows form $1 + m' \subset \dot{H}^p$. □

The last lemma has the following immediate consequence.

Corollary 4.3. *Given a valuation ring A of a field F such that $\Re(A) = \dot{F}$ it follows that a local closure (H, A') of (F, A) satisfies $H = F(p)$.*

In the next result we state the first connection between the radical $R(F)$ and localizers.

Lemma 4.4. *Let F be a field and A a localizer of F. If there is $r \in R(F)$ such that $r \notin \Re(A)$, then A is a valuation ring and the following statements are true:*

(a) $(\Gamma_A : p\Gamma_A) \leq p$.

(b) If $(\Gamma_A : p\Gamma_A) = p$, *then* $k_A = k_A(p)$ *and* $(\dot{F} : \Re(A)) = p$.

(c) If $v_A(r) = 0$, *then* $\pi_A(r) \in R(k_A) \setminus \dot{k}_A^p$.

Proof. To see the truth of the first statement observe that for every positive cone A of F, which may happen only if $p = 2$, we have $R(F) \subset A = \Re(A)$ since $R(F) \subset D_F(-1) = \{x^2 + y^2 \neq 0 \mid x, y \in F\}$.

(a) Let (H, A') be a local closure of (F, A) in $F(p)$. For $r \in \dot{F} \setminus \Re(A)$, by Lemma 4.2 $r \notin \dot{H}^p$. Consequently, the polynomial $X^p - r$ is irreducible in $H[X]$ and so, by Theorem 4.1, A has only one extension B to $K = F(\sqrt[p]{r})$. By [9, Theorem 16.2], for every $y \in \dot{K}$, $v_A(N_r^F(y)) = pv_B(y)$. Since $r \in R(F)$, the norm homomorphism N_r^F is surjective, i.e. $D_F(r) = \dot{F}$. Hence $\Gamma_A = v_A(\dot{F}) = pv_B(\dot{K}) = p\Gamma_B$. Since $K \mid F$ is a Galois extension of degree p, it follows from [9, Theorem 20.21] either $\Gamma_B = \Gamma_A$ or $(\Gamma_B : \Gamma_A) = p$. Putting the things together we get $(\Gamma_B : p\Gamma_B) \leq p$ and a simple calculation completes the proof of (a).

(b) Assume now $(\Gamma_A : p\Gamma_A) = p$. We have seen above that $\Gamma_A = p\Gamma_B$. Then $\Gamma_B \neq \Gamma_A$. By [9, Theorem 20.21] it follows that $k_B = k_A$. From [9, Theorem 19.1] we get $\sigma(y) - y \in m_B$, for every $\sigma \in G = \mathrm{Gal}(K, F)$ and every $y \in K$. Thus $\pi_B(\sigma(y)) = \pi_B(y)$, for every $y \in B$.

Take now $u \in A^*$. By assumption, there is $y \in \dot{K}$ such that $u = N_r^F(y)$. Recall that $0 = v_A(u) = pv_B(y)$. Thus $y \in B^*$ and $\pi_A(u) = \pi_A(\prod_{\sigma\in G}\sigma(y)) = \prod_{\sigma\in G}\pi_B(\sigma(y)) = \pi_B(y)^p$. Hence $\dot{k}_A = \dot{k}_A^p$. Our condition about the characteristic of residue fields implies $k_A = k_A(p)$.

Finally, $(\Gamma_A : p\Gamma_A) = p$ implies that $\dot{F} = A^*\dot{F}^p \cup xA^*\dot{F}^p \cup \cdots \cup x^{p-1}A^*\dot{F}^p$, for every $x \in \dot{F} \smallsetminus A^*\dot{F}^p$. From $k_A = k_A(p)$, we get $A^* = (1 + m_A)(A^*)^p$. Hence $A^*\dot{F}^p = \Re(A)$ and the result follows.

(c) Let K and B be as in the proof of (a). Observe that $v_B(\sqrt[p]{r}\,) = 0$ and $\pi_A(r) \notin \dot{k}_A^p$. Thus the polynomial $X^p - \pi_A(r)$ is irreducible in $k_A[X]$. Since $\pi_B(\sqrt[p]{r}\,)$ is a root of this polynomial, $k_B \neq k_A$. Hence, by [9, Theorem 20.21], $[k_B : k_A] = p$ and so $k_B = k_A(\sqrt[p]{\pi_A(r)}\,)$ ($K \mid F$ is a totally inertial extension). By [9, 19.8 b], we have an isomorphism $\sigma \mapsto \bar{\sigma}$ from $G = \mathrm{Gal}(K, F) \longrightarrow \mathrm{Gal}(k_B, k_A)$, where $\bar{\sigma}(\pi_B(u)) = \pi_B(\sigma(u))$, for every $u \in B^*$.

Take now $x \in A^*$. There is $u \in \dot{K}$ such that $x = N_r^F(u) = \prod_{\sigma\in G}\sigma(u)$. Since $x \in A^*$, it follows that $u \in B^*$. Finally, $\pi_A(x) = \pi_B(\prod_{sg\in G}\sigma(u)) = \prod_{\sigma\in G}\bar{\sigma}(\pi_B(u))$. Therefore the norm homomorphism from $\dot{k}_B$ to $\dot{k}_A$ is surjective and (c) is proved. □

The last lemma shows that $R(F) \not\subset \Re(A)$ occurs only for very particular valuation rings. Lemma 4.4 also has three consequences that we shall use later, the first one follows directly from item (c).

Corollary 4.5. *For every valuation ring A of F, $\pi_A(R(F) \cap A^*) \subset R(k_A)$.*

Corollary 4.6. *Let A be a valuation of a field F such that $R(F) \not\subset \Re(A)$ and $\Gamma_A \neq p\Gamma_A$. Let C be another valuation ring of F.*

(a) If $A \subset C$, then either $\Gamma_C = p\Gamma_C$ or $\Re(C) = \Re(A)$.

(b) If $C \subset A$, then $\Re(C) = \Re(A)$.

Proof. (a) Since $\Re(C) \subset \Re(A)$ we also have $R(F) \not\subset \Re(C)$. If $\Gamma_C \neq p\Gamma_C$, by Lemma 4.4, $(\dot{F} : \Re(C)) = p$. On the other hand $(\dot{F} : \Re(A)) = p$, too. Thus $\Re(C) = \Re(A)$.

(b) If $C \subset A$, then $\Re(A) \subset \Re(C)$ and $\Gamma_C \neq p\Gamma_C$, too. Going for a contradiction we assume that there is $x \in \Re(C) \smallsetminus \Re(A)$. It follows form Lemma 4.4 that $\dot{F} = \Re(A) \cup x\Re(A) \cdots x^{p-1}\Re(A) \subset \Re(C)$. Since $\Re(C) = \dot{F}$ implies $\dot{F} = C^*\dot{F}^p$, we get a contradiction with $\Gamma_C \neq p\Gamma_C$. □

Corollary 4.7. *For a valuation ring A such that $R(F) \not\subset \Re(A)$ and $\Gamma_A \neq p\Gamma_A$, let (H, A') be a local closure of (F, A). Then $G_p(H) \cong \mathbb{Z}_p$.*

Proof. Take an extension B of A to $F(p)$. Since the multiplicative group of $F(p)$ is p-divisible the same is true for $\dot{k}_B$ and Γ_B. It follows then from valuation theory that $k_B = k_A(p)$ and Γ_B is the p-divisible closure of Γ_A. By Lemma 4.4 $k_A = k_B$. Therefore the inertia group of B over F equals $G_p(H)$ (see [9, § 19]). Since k_A has characteristic $\neq p$ also the ramification group of B over F is trivial [9, 20.18]. Hence, by [9, Theorem 20.12], $G_p(H) \cong$ the character group of Γ_B/Γ_A. Since $(\Gamma_A : p\Gamma_A) = p$, the torsion group Γ_B/Γ_A has only one subgroup of order p. Thus, by the Pontryagin duality theorem, $G_p(H)$ is the pro-p cyclic group as required. □

5 Localizers and free products

In this section we shall recall some more facts about localizer and then prove the first step of our results on the decomposition of $G_p(F)$ (see Proposition 5.4 below).

Let A and B be localizers of a field F such that A is a valuation ring and B is the positive cone of an ordering of F. We say that A is *compatible* with B if $\Re(A) = (1+m_A)\dot{F}^p \subset B = \Re(B)$. The set of all valuation rings of F which are compatible with B forms a chain under inclusion and has a smallest element given by the convex hull of $\mathbb{Q}$ in F: $V(B) = \{x \in F \mid \text{ there is } q \in \mathbb{Q} \text{ such that } q \pm x \in B\}$ (see [20, Theorem 2.6]). The next proposition improves our knowledge of the connection between these localizers A and B.

Proposition 5.1. *Let $p = 2$, A be valuation ring of F and B be a cone of an ordering of F. Let (H, A') be a local closure of (F, A). The following conditions are equivalent:*

(a) A is compatible with B.

(b) $V(B) \subset A$.

(c) $\pi_A(B \cap A^*)$ is a positive cone of an ordering of k_A.

(d) There is a positive cone B' of an ordering of H such that $B' \cap F = B$.

Proof. The equivalence between (a) and (b) follows from [20, Theorem 2.6] and (a) and (c) are equivalent by [20, Theorem 2.1]. By [20, Proposition 3.14] and Lemma 4.2 (b), (d) implies (a). Since A' and A have the same residue field by [20, Corollary 3.11] (c) implies (d). □

A positive cone B is called *archimedean* if and only if $V(B) = F$ is the trivial valuation ring. If F admits an archimedean ordering, it is well known that there is an order preserving injective homomorphism from (F, B) into the reals $\mathbb{R}$ with its the unique ordering.

Localizers are compared as follows. We say that a localizer B is *coarser* than a localizer A (or A is *finer* than B) if either: $A \subset B$, for valuation rings A and B, B is compatible with A if A is a cone and B is a valuation ring or $A = B$ if both A and B are cones.

Remark 1. The trivial valuation is coarser than any other localizer. Note also that B coarser than A yields $\Re(B) \subset \Re(A)$. The converse is not true for valuation rings. Consider, for example, the case of a valuation ring $B \neq F$ such that $\Re(B) = \dot{F}$.

For an archimedean cone A there do not exist localizers different from A and F which are coarser than A, since $V(A) = F$.

More generally two localizers A and B are called *dependent* if there is a localizer C simultaneously coarser than A and B (*independent* otherwise).

Remark 2. (1) Two non-trivial valuation rings A and B are dependent if and only if $AB = \{xy \mid x \in A, y \in B\} \neq F$. If A is a valuation ring and B is a cone, dependence means that A and $V(B)$ are dependent valuation rings. Finally, two cones A and B, which correspond to non-archimedean orderings, are dependent if and only if $V(A)$ and $V(B)$ are dependent valuation rings and two archimedean orderings are dependent if and only if they coincide.

(2) Recall that every localizer A of F induces naturally a Hausdorff topology T_A on F, which is compatible with the field structure of F (see [30] for general facts about topological fields). It is known that localizers A and B are dependent if and only if they induce the same topology on F (see

[30]; Lemma 3.4 treats the case of valuation rings and at the beginning of §5 we find the connection between valuation rings and orderings). Consequently, the relation, *"A and B are dependent"* is an equivalence relation of the set of localizer of F.

Let us recall that a topology T, defined on a field F, is called *V-topology* if T is generated by a localizer of F or by an archimedean valuation of F (see for example [9, §1]). For every finite extension K of F and every localizer A we say that a topology T of K extends T_A if T is a V-topology whose restriction to F equals T_A. The study of the extensions of T_A to K is the subject of the next lemma.

Lemma 5.2. *Let $K = F(\sqrt[p]{a}\,)$ be a non-trivial Galois extension of a field F with Galois group G. Let A be a localizer of F and T_A the topology induced by A in F. Denote $\mathcal{O} = \{O \mid$ localizer of K such that $O \cap F = A\}$.*

(a) If A is a valuation ring, then $\mathcal{O}$ has 1 or p elements. If A is a cone, then either $\mathcal{O} = \emptyset$ or $\mathcal{O}$ has p elements. Moreover, $\mathcal{O} = \{\sigma(O) \mid \sigma \in G\}$, for every $O \in \mathcal{O}$.

(b) T_A has either 1 or p extensions to K.

(c) If A is not the cone of an archimedean ordering and T_A extends uniquely to K, then there is a valuation ring B of K such that $B \cap F$ is coarser than A and $\sigma(B) = B$ for every $\sigma \in G$ (T_B is the extension of T_A to K.).

(d) T_A has p extensions to K if and only if A has p pairwise independent extensions to K.

Furthermore, in the case (c) suppose that A has p extensions to K. Then B can be chosen such that for $C = B \cap F$, $[k_B : k_C] = p$ and if we define $\bar{A} = \pi_C(A \cap C^*)$ when A is a cone (see Proposition 5.1 (c)) and $\bar{A} = \pi_C(A)$ for a valuation ring, then $\bar{A}$ is a localizer of k_C which has p pairwise independent extensions to k_B.

Proof. (a) For a valuation ring A of F, by [9, Theorem 13.2], there is a valuation ring O of K which lies over A. Moreover, either $\mathcal{O} = \{O\}$, has just one element, or $\mathcal{O} = \{\sigma(O) \mid \sigma \in G\}$, has p elements [9, Theorem 20.21]. For a cone A, $\mathcal{O} = \emptyset$, if $a \notin A$. If $a \in A$, then there is a cone O of K such that $O \cap F = A$ and $\mathcal{O} = \{\sigma(O) \mid \sigma \in G\}$ has 2 elements, [1, Theorem 22, p.56].

(b) If the localizer A has an extension to K, then clearly any extension of A to K generates a topology on K which extends T_A. In the case where A is a cone and $a \notin, A$, if A is not archimedean, the extension of the topology is generated by a valuation ring of K which lies over $V(A)$. If A corresponds to an archimedean ordering, we can consider F as a subfield of $\mathbb{R}$ with its canonical topology. Thus, the usual topology of $\mathbb{C}$ induces on K the required extension. Hence, T_A has at least one extension to K.

(c) If A is a valuation ring, let O be an extension of A to K. If A is a cone, then O denotes an extension of $V(A)$ to K. By (a), $\{\sigma(O) \mid \sigma \in G\}$ is the set of all valuation rings of K which lie either over A or $V(A)$ according to the nature of A. Since there is just one topology on K whose restriction to F is T_A, for every $\lambda \neq \tau \in G$, the valuation rings $\lambda(O)$ and $\tau(O)$ are dependent, Remark 2 (2). Then $B = \lambda(O)\tau(O) \neq K$ is a valuation ring of K which contains $\lambda(O)$ and $\tau(O)$. Hence $\tau\lambda^{-1}(B)$ also contains $\tau(O)$. Therefore, by [9, Theorem 6.6], B and $\tau\lambda^{-1}(B)$ are comparable. As B and $\tau\lambda^{-1}(B)$ are extensions of $C = B \cap F$ to K, it follows that they are equal [9, 13.3 c]. Since $\tau\lambda^{-1} \neq 1$, Theorem 6.6 of [9] implies that the number of extensions of C to K is not p.

Consequently, by (a), B is the unique extension of C to K and so $\sigma(B) = B$ for every $\sigma \in G$, as required.

Assume now that A has p extensions to K. Then $\{\sigma(O) \mid \sigma \in G\}$ has p elements. Since we took arbitrary automorphisms $\lambda \neq \tau \in G$ in the construction of B, for every pair $O' \neq O'' \in \{\sigma(O) \mid \sigma \in G\}$ it follows that $O'O'' = B$. Therefore, by [9, Theorem 8.7], the set $\{\pi(\sigma(O)) \mid \sigma \in G\}$ has p pairwise independent elements. This set is contained in the set of all extensions of $\pi_C(A)$, or $\pi_C(V(A))$, to k_B, according to nature of A, a valuation ring or, respectively, a cone. Thus $k_B \neq k_C$. Observe now that $[k_B : k_C]$ is either $= 1$ or $= p$, by [9, Theorem 20.21]. Hence $[k_B : k_C] = p$ and the last statement of the lemma follows from [9, Theorem 13.7] applied to A or $V(A)$.

(d) If A has p pairwise independent extensions to K, each one of these extensions generates a topology of K which extends T_A.

Conversely, assume that T_A has p extensions to K. If any extension T of T_A to K is generated by an archimedean valuation of K, then T_A is generated by the restriction to F of this valuation. Thus, by [9, 1.10], A is an archimedean ordering of F. We claim that A extends to K. From the claim and (a), it follows that A has 2 extensions to K. Each one of these extensions is an archimedean ordering of K. Therefore the extensions of A to K are two independent localizers of K, as desired (Naturally they induce different topologies on K.).

We now prove the claim. Assume it is not true. Then A is an archimedean ordering with no extension to K ($a \notin A$). We may consider F as a ordered subfield of the real numbers $\mathbb{R}$. Therefore the topology induced by A on F is the topology generated by the usual archimedean valuation $|\ |$ of $\mathbb{R}$. By [9, Corollary 2.13], the restriction of $|\ |$ to F has just one extension to K. Consequently T_A extends uniquely to K, a contradiction.

Assume next that every extension of T_A to K is generated by a localizer of K which is not an archimedean ordering of K. If T_B is one of these extensions, then $B \cap F$ and A are dependent localizers of F, Remark 2 (2). Hence A is not the cone of an archimedean ordering of F, otherwise $B \cap F = A$ and then B would be the cone of an archimedean ordering of K extending A.

Now, as in item (c), let O be an extension of A to K, if A is a valuation ring and let O be an extension of $V(A)$ to K, when A is a cone.

If, for some $\sigma \in G$, O and $\sigma(O)$ are dependent, write $C = O\sigma(O)$. As C is coarser than O, C and O induce the same topology on K as well as $C \cap F$ and A induce the same topology on F. On the other hand, since C is simultaneously coarser than O and $\sigma(O)$, it follows that $\sigma(C) = C$. Therefore, (a) implies that C is the unique extension of $C \cap F$ to K. Consequently, T_C is the unique extension of T_A to K, a contradiction. □

The statement (c) of the lemma above does not remain true if we drop the assumption that A is not the cone of an archimedean ordering of F. For example, take $F = \mathbb{Q}$ with the usual ordering A and $K = F(\sqrt{-1})$. There is just one topology on K whose restriction to F is T_A and for every valuation ring B of K, $B \cap F$ is not coarser than A.

The next lemma is a rather technical result which will be crucial for handling a family of independent localizer in Proposition 5.4.

Lemma 5.3. *Let A be a localizer of F, (H, A') a local closure of (F, A) and $a \in \dot{F}$. If A is a valuation ring, for every valuation ring B of F coarser than A, suppose either $\Gamma_B = p\Gamma_B$ or $\Re(B) = \Re(A)$. Then $D_H(a) = D_F(a)\dot{H}^p$.*

Proof. Consider first the case $p = 2$ and A is a cone. If $a \notin A$, then $-a \in A \subset \dot{H}^2 (= A')$. Thus $H(\sqrt{a}) = F(2)$ and $D_H(a) = \dot{H}^2$. Since $D_F(a) = \{x^2 - ay^2 \neq 0 \mid x, y \in F\} \subset A$ the result follows in this case. If $a \in A$, then $a \in \dot{H}^2$ and $D_H(a) = \dot{H}$. As $-a \in D_F(a)$ and $\dot{H} = \dot{H}^2 \cup -\dot{H}^2$, also $D_F(a)\dot{H}^2 = \dot{H}$, as desired.

Assume now that A is a valuation ring. In the case $a \notin \Re(A)$, by Theorem 4.1, A has just one extension O to $K = F(\sqrt[p]{a})$, because $X^p - a$ is irreducible in $H[X]$. Let O' be the unique extension of A' to $L = H(\sqrt[p]{a})$. From [9, 15.6 b], (L, O') is a local closure of (K, O). Thus, Lemma 4.2 implies that $\dot{L} = \dot{K}\dot{L}^p$. Consequently, $D_H(a) = N_a^H(\dot{L}) \subset D_F(a)\dot{H}^p$. Since the other inclusion is clearly true the statement is proved in this case.

It remains to be seen the case $a \in \Re(A)$. Hence $a \in \dot{H}^p$ and so $D_H(a) = \dot{H}$. If we show that $\dot{F} = D_F(a)\Re(A)$, then the statement follows from Lemma 4.2.

Take $x \in \dot{F}$.

Let $K = F(\sqrt[p]{a})$ and denote by $G = \mathrm{Gal}(K, F)$ the Galois group. By Theorem 4.1 A has p distinct extensions to K. Take an extension O of A to K. By Lemma 5.2 (a), $\{\sigma(O) \mid \sigma \in G\}$ is the set of all extensions of A to K.

Assume first the extensions of A to K are pairwise independent valuation rings. According to Approximation Theorem [9, Theorem 11.16], there is $z \in \dot{K}$ such that $z \in x(1 + m_O)$ and for every $\sigma \in G$, $\sigma \neq 1$, $z \in 1 + \sigma(m_O)$. Recall that $\sigma(m_O)$ is the maximal ideal of $\sigma(O)$. Therefore $\sigma^{-1}(z) \in 1 + m_O$, for every $\sigma \neq 1$ which implies $N_F^a(z) = \prod_{\sigma \in G} \sigma(z) \in x(1 + m_O)$. Since $(1 + m_O) \cap F = 1 + m_A$, it follows that $N_F^a(z) \in x\Re(A)$, or equivalently $x \in D_F(a)\Re(A)$, as desired.

We now consider the case where T_A has only one extension to K. By Lemma 5.2 (c) there a valuation ring B of K satisfying the following conditions: $C = B \cap F$ is coarser than A; B is the unique extension of C to K; $[k_B : k_C] = p$ and $\bar{A} = \pi_C(A)$ is a valuation ring of k_C which has p pairwise independent extensions to k_B.

Moreover, since $a \in \Re(A) \subset A^*\dot{F}^p$ we may assume without loss of generality that $a \in C^*$. Hence $\bar{a} = \pi_C(a) \in \Re(\bar{A})$ and it is trivial to deduce $k_B = k_C(\sqrt[p]{\bar{a}})$. Therefore, the previous case applies to k_C and k_B. We next write the details of this fact.

Observe that $k_B \mid k_C$ is a normal extension with Galois group $\mathrm{Gal}(k_B, k_C) = \{\bar{\sigma} \mid \sigma \in G\}$, where $\bar{\sigma}(\pi_B(u)) = \pi_B(\sigma(u))$, for every $u \in B^*$ (see [9, §19]). Let us denote by $\bar{N}$ the norm homomorphism from $\dot{k}_B$ to $\dot{k}_C$ and by $\bar{D}$ its image. For every $u \in B^*$, we have $\pi_C(N_F^a(u)) = \bar{N}(\pi_B(u))$.

According to the previous case, $\dot{k}_C = \bar{D}\Re(\bar{A})$. By the above calculation we lift this equality to $C^* = D_F(a)(1 + m_A)(C^*)^p$.

Finally, since C contains A, either $\Gamma_C = p\Gamma_C$ or $\Re(C) = \Re(A)$ by assumption. Observe that $a \notin \Re(C)$ since B is the unique extension of C to K. Thus $\Gamma_C = p\Gamma_C$. Consequently, $\dot{F} = C^*\dot{F}^p$ and so $\dot{F} = D_F(a)\Re(A)$, completing the proof. □

To prove our main results we shall follow an induction process. The first step corresponds to a family of pairwise independent valuation rings. This is the subject of the next proposition, which improves slightly the theorem quoted in the introduction.

Proposition 5.4. *Let $A_1 \dots, A_n$ be a family of pairwise independent localizers of F and take for each $1 \leq i \leq n$ a local closure (H_i, A_i') of (F, A_i). Assume that $\Re(A_1) \cap \ldots \cap \Re(A_n) \subset R(F)$. Then there is an extension H_0 of F, as in Proposition 3.4, such that*

$$G_p(F) = G_p(H_0) * G_p(H_1) * \cdots * G_p(H_m).$$

To avoid double subscripts, we write N_a^i and $D_i(a)$ to denote, respectively, the norm map and the image of the norm map associated to the extension $H_i(\sqrt[p]{a})$.

To simplify the proof, we state first a lemma that will allow the use of Lemma 5.3. Note that by Corollary 4.3 we may assume without loss of generality that $\Re(A_i) \neq \dot{F}$, for every $1 \leq i \leq n$.

Lemma 5.5. *Let $A_1, \ldots, A_n$ be a family of pairwise independent localizers of a field F such that $\Re(A_i) \neq \dot{F}$ for every $i = 1, \ldots, n$ and $\Re(A_1) \cap \cdots \cap \Re(A_n) \subset R(F)$. Let C be a localizer of F which is coarser than A_i, for some $1 \leq i \leq n$. Then either $\Gamma_C = p\Gamma_C$ or $\Re(C) = \Re(A_i)$.*

Proof. Without loss of generality we assume that C is coarser than A_1. Then $\Re(C) \subset \Re(A_1)$ and so $\Re(C) \cap \Re(A_2) \cdots \Re(A_n) \subset R(F)$, too.

Let us consider first the case $R(F) \not\subset \Re(A_1)$. Then A_1 is a valuation ring, by Lemma 4.4 and if $\Gamma_{A_1} \neq p\Gamma_{A_1}$ the statement follows from Corollary 4.6. If A_1 has p-divisible value group, the same is true for Γ_C since this group is a quotient of the value group of A_1.

We now assume $R(F) \subset \Re(A_1)$. In this case, when A_1 is a cone, if $R(F) \not\subset \Re(C)$ we claim that $\Gamma_C = p\Gamma_C$, as desired. Going for a contradiction we assume that $\Gamma_C \neq p\Gamma_C$. Then Corollary 4.7 implies that $G_p(H) \cong \mathbb{Z}_2$, for some local closure (H, C') of (F, C). On the other side, by Proposition 5.1, H has an ordering which extends A_1. Hence $G_p(H)$ has torsion, a contradiction.

For the case A_1 a valuation ring and $R(F) \not\subset \Re(C)$, by Lemma 4.4 either $\Gamma_C = p\Gamma_C$ or Corollary 4.6 (b) implies $\Re(C) = \Re(A_1)$, as desired.

It remains to be seen the case $R(F) \subset \Re(A_1), \Re(C)$.

We permute the localizers $A_1, \ldots, A_n$, if necessary, in order to have $1 \leq r \leq n$ such that $R(F) \subset \Re(A_i)$ for every $1 \leq i \leq r$ and if $r < n$, $R(F) \not\subset \Re(A_j)$ for each $r+1 \leq j \leq n$.

The assumption $\Re(A_1) \cap \cdots \cap \Re(A_n) \subset R(F)$ implies that $\Re(A_1) \cap \cdots \cap \Re(A_n) = R(F) \cap \Re(A_{r+1}) \cap \cdots \cap \Re(A_n)$. On the other hand it is also true $\Re(C) \cap \Re(A_2) \cap \cdots \cap \Re(A_n) = R(F) \cap \Re(A_{r+1}) \cap \cdots \cap \Re(A_n)$, by the same argument. Therefore $\Re(A_1) \cap \cdots \cap \Re(A_n) = \Re(C) \cap \Re(A_2) \cap \cdots \cap \Re(A_n)$.

We now deduce from the hypothesis on the independence of $A_1, \ldots, A_n$ that $C, A_2, \ldots, A_n$ are also independent. Therefore, for $x \in \Re(A_1)$ there is $y \in \dot{F}$ such that $y \in x\Re(C)$ and $y \in \Re(A_j)$ for every $j = 2, \ldots, n$. The inclusion $\Re(C) \subset \Re(A_1)$ implies $y \in \Re(A_1)$. Thus $y \in \Re(A_1) \cap \cdots \cap \Re(A_n) = \Re(C) \cap \Re(A_2) \cap \cdots \cap \Re(A_n)$. Consequently, $y \in \Re(C)$ which implies $x \in \Re(C)$. Hence $\Re(C) = \Re(A_1)$ as desired. $\square$

We now prove Proposition 5.4.

Proof. By Corollary 3.5 the result will be true if F and $H_1, \ldots, H_n$ satisfy (Ia) and (II).

Let $T_1, \ldots, T_n$ be the topologies induced by $A_1, \ldots, A_n$ on F. By assumption they are different topologies.

From Lemma 4.2 (a) and Approximation Theorem for different topologies ([30, Theorem 4.1]) it follows that the homomorphism $\dot{F}/\dot{F}^p \longrightarrow \dot{H}_1/\dot{H}_1^p \times \cdots \times \dot{H}_n/\dot{H}_n^p$ is surjective, or equivalently, (Ia) holds.

We now prove (II) by means of Lemma 5.3. To this end, for an extension $K = F(\sqrt[p]{a})$, $a \in \dot{F} \setminus \dot{F}^p$, we will organize the extensions to K, of the localizers A_i, in a suitable way. Denote $G = \mathrm{Gal}(K, F)$. We will consider three types of localizers:

(i) For each localizer $A_i \in \mathcal{A}$ which is not archimedean and such that T_i has just one extension to K, by Lemma 5.2 (c), there is a valuation ring B_i of K such that $B_i \cap F$ is coarser than A_i and $\sigma(B_i) = B_i$ for every $\sigma \in G$.

(ii) For every localizer A_i such that T_i has p distinct extensions to K, by Lemma 5.2 (d) we know that A_i has p pairwise independent extensions to K. We then choose a localizer B_i of K

which lies over A_i. In this case, each topology of K extending T_i is generated by $\sigma^{-1}(B_i)$, for some $\sigma \in G$.

(iii) If $p = 2$, we possibly have localizers A_i which are cones of archimedean orderings of F such that $a \notin A_i$.

We are now going to sort the localizers B_i as follows: we first enumerate localizers of type (i), $B_1, \dots, B_r$, if any. Next, we also take localizers of type (ii), $B_{r+1}, \dots, B_s$, if they occur. We then consider the set

$$\mathcal{B} = \{B_1, \dots, B_r\} \cup \{\sigma^{-1}(B_{r+1}) \mid \sigma \in G\} \cup \ldots \cup \{\sigma^{-1}(B_s) \mid \sigma \in G\}.$$

By construction, this is a set of pairwise independent localizers of K.

Back to the proof of the proposition, take $a, b \in \dot{F}$ such that $b \in D_i(a)$ for every $1 \leq i \leq n$ and assume that $a \notin \dot{F}^p$. By Lemma 5.3 for every $i = 1, \dots, n$ there is $b_i \in D_F(a)$ such that $bb_i^{-1} \in \dot{H}_i^p$. Let $K = F(\sqrt[p]{a})$ and for every i choose $z_i \in \dot{K}$ such that $N_a^F(z_i) = b_i$.

Next, we make use of the Approximation Theorem for the different topologies generated by the localizers of the set $\mathcal{B}$ constructed as above. Let $z \in K$ such that:

for every $1 \leq i \leq r$, $z \in z_i(1 + m_i)$, where m_i is the maximal ideal of B_i;

for every $r + 1 \leq i \leq s$, $z \in z_i \Re(\sigma^{-1}(B_i))$, for every $\sigma \in G$.

Consequently, for every $1 \leq i \leq r$, $\sigma(z) \in \sigma(z_i)(1 + m_i)$ for each $\sigma \in G$, because $\sigma(B_i) = B_i$. Thus $N_a^F(z_i^{-1}z) \in (1+m_i) \cap F$. Since $B_i \cap F$ is coarser than A_i, it follows that $N_a^F(z) \in N_a^F(z_i)\Re(A_i)$, for every $1 \leq i \leq r$.

On the other hand, for every $r + 1 \leq i \leq s$ we also have $\sigma(z) \in \sigma(z_i)\Re(B_i)$, for each $\sigma \in G$. Hence $N_a^F(z_i^{-1}z) \in \Re(B_i) \cap F$. Now, note that (H_i, A_i') is also a local closure of (K, B_i), for every $r + 1 \leq i \leq s$, by Theorem 4.1. Thus, by Lemma 4.2, $\Re(B_i) \cap F = \Re(A_i)$, for every $r + 1 \leq i \leq s$. Putting the things together $N_a^F(z) \in N_a^F(z_i)\Re(A_i)$, for every $r + 1 \leq i \leq s$, too.

If $p = 2$ and the case (iii) occur, then $N_a^F(K) \subset A_i$, for every localizer A_i of this type. Hence $N_a^F(z) \in N_a^F(z_i)\Re(A_i)$ also in this case.

Consequently, there is $z \in \dot{K}$ such that $N_a^F(z) \in N_a^F(z_i)\Re(A_i)$, for every $1 \leq i \leq n$. Hence, for $c = N_a^F(z) \in D_F(a)$, $b_i c^{-1} \in \Re(A_i)$, for every $1 \leq i \leq n$. By Lemma 4.2, $b_i c^{-1} \in \dot{H}_i^p$, for every $1 \leq i \leq n$. Therefore, $bc^{-1} \in \Re(A_1) \cap \cdots \cap \Re(A_n)$. Thus $bc^{-1} \in R(F)$, by assumption. Since $R(F) \subset D_F(a)$ it follows that $b \in D_F(a)$. □

6 Main Results

The natural generalization of Proposition 5.4 corresponds to a family of pairwise non-comparable localizers.

Definition 2. We say that a family $\mathcal{A} = \{A_1, \ldots, A_n\}$ of distinct localizers of F is allowable if A_i coarser than A_j implies $i = j$.

Next we characterize the pro-p groups which will be suitable for this work. Fix an allowable family $\mathcal{A} = \{A_1, \ldots, A_m\}$ of localizer of F.

Definition 3. We first define the basic groups. We say that a pro-p group $\mathcal{G}$ is $\mathcal{A}$-basic if one of the following conditions holds:

- $\mathcal{G}$ is a free group or an abelian torsion free group.
- $\mathcal{G} \cong G_p(L)$ for some extension L of F inside $F(p)$ which is locally closed for a localizer A' that extends A_i for some $1 \leq i \leq n$.

We now define $\mathcal{A}$-*admissible* groups recursively:

(i) Every $\mathcal{A}$-basic group $\mathcal{G}$ is $\mathcal{A}$-admissible.

(ii) If $\mathcal{G}_1, \ldots, \mathcal{G}_m$ are $\mathcal{A}$-admissible groups, then so is $\mathcal{G}_1 * \cdots * \mathcal{G}_m$.

(iii) If $\mathcal{G} = \mathcal{G}_1 \rtimes \mathcal{G}_2$, where $\mathcal{G}_1$ is abelian and torsion free closed subgroup of $\mathcal{G}$ and $\mathcal{G}_2$ is $\mathcal{A}$-admissible, then $\mathcal{G}$ is an $\mathcal{A}$-admissible group.

Therefore the class of $\mathcal{A}$-admissible groups is the class of all pro-p groups which can be obtained from $\mathcal{A}$-basic groups by repeating the process of taking free pro-p products and semi-direct group extensions a finite number of times.

It is worth mentioning that a group $\mathcal{G}$ of the above type (iii) is realizable as $G_p(F)$, for some field F, only if $\mathcal{G}_2$ is realizable as Galois group and the action of $\mathcal{G}_2$ on $\mathcal{G}_1$ is of "cyclotomic" nature (see [8, § 1] or [10, Proposition 1.1]). In this case $\mathcal{G}$ is realizable for some p-henselian field F. Furthermore, since $\mathcal{A}$-basic groups are realizable as Galois groups and free pro-p products of realizable groups are also realizable, we can conclude that $\mathcal{A}$-admissible groups are realizable as Galois groups, under the above assumptions on groups of type (iii).

Next, we state the general case.

Theorem 6.1. *Let $\mathcal{A} = \{A_1 \ldots, A_n\}$ be an allowable family of localizers of F. If $\Re(A_1) \cap \ldots \cap \Re(A_n) \subset R(F)$, then $G_p(F)$ is $\mathcal{A}$-admissible.*

For the proof of this theorem we need some preparatory results about families of localizers. We shall first rank allowable families $\mathcal{A}$ according to the dependent relations among some suitable localizers which contain an element of $\mathcal{A}$. Let us identify them.

Definition 4. For any allowable family $\mathcal{A} = \{A_1, \ldots, A_n\}$ we consider the set $\mathcal{L}$ of all valuation rings B for which there are $1 \leq i \neq j \leq n$ such that A_i and A_j are dependent and

$$B = \begin{cases} A_i A_j, & \text{if } A_i \text{ and } A_j \text{ are valuation rings;} \\ V(A_i) A_j, & \text{if } A_i \text{ is a cone and } A_j \text{ is a valuation ring;} \\ V(A_i) V(A_j), & \text{if } A_i \text{ and } A_j \text{ are cones.} \end{cases}$$

Observe that if $i \neq j$ and A_i, A_j are cones such that $V(A_i) = V(A_j)$, then $V(A_i) = V(A_i)V(A_j) \in \mathcal{L}$. On the other side $\mathcal{A} \cap \mathcal{L} = \emptyset$.

Note that $\mathcal{L}$ is a finite set.

Definition 5. The complexity of $\mathcal{A} = \{A_1, \dots, A_n\}$, denoted by $\mathrm{cp}(\mathcal{A})$, is defined as follows: $\mathrm{cp}(\mathcal{A}) = 0$ if $\mathcal{L} = \emptyset$, otherwise $\mathrm{cp}(\mathcal{A}) = \max\{\, t \mid$ there exists a chain $B_1 \subset \cdots \subset B_t$ of distinct valuation rings from $\mathcal{L}\}$.

We shall next state preparatory results in order that we can prove Theorem 6.1 by induction on $\mathrm{cp}(\mathcal{A})$.

Lemma 6.2. *$cp(\mathcal{A}) = 0$ if and only if $A_1, \dots, A_n$ are pairwise independent localizers.*

Proof. Immediate (see Remark 1). □

Next we construct some elements that we need for the proofs.

Recall from Remark 2 (2) that dependence is an equivalence relation on the set of localizer of F and let $\mathcal{A} = \mathcal{A}_1 \dot\cup \cdots \dot\cup \mathcal{A}_m$ be the partition of $\mathcal{A}$ corresponding to this relation. We shall use this decomposition to construct a new family $\mathcal{B}$ of localizers of F.

For every $1 \le j \le m$ let B_j be the smallest valuation ring of F which is coarser than each element of $\mathcal{A}_j$. Let $\mathcal{B} = \{B_1, \dots, B_m\}$. Note that $B_j \neq F$, for every j, since $\mathcal{A}_j$ is a finite set.

Lemma 6.3. *For every $j = 1, \dots, m$, either $B_j \in \mathcal{A}$ or $B_j \in \mathcal{L}$. The first case occurs if and only if $\mathcal{A}_j$ is a singleton set. Furthermore, $cp(\mathcal{B}) = 0$.*

Proof. If $\mathcal{A}_j$ has exactly one element, then $B_j = A_r \in \mathcal{A}_j$. If $\mathcal{A}_j$ has at least 2 elements, then $\mathcal{L}_j = \{B \in \mathcal{L} \mid$ there are $A_r, A_s \in \mathcal{A}_j$ such that B is the finest localizer of F which is simultaneously coarser than A_r and $A_s\}$ is non empty. Note that $\mathcal{L}_j \subset \mathcal{L}$.

We claim that there is $B \in \mathcal{L}_j$ which is coarser than every element of $\mathcal{A}_j$. Take then $B_j = B$ and the first statement is proved. To prove the claim, take $B \in \mathcal{L}_j$ which is coarser than t elements of $\mathcal{A}_j$ where t is as big as possible. Going for a contradiction we assume that $\mathcal{A}_j$ has more than t elements.

Therefore, B is coarser than some A_r and not coarser than A_s, where $A_r, A_s \in \mathcal{A}_j$. The definition of $\mathcal{L}_j$ implies that there is $B' \in \mathcal{L}_j$ which is the finest valuation ring simultaneously coarser than A_r and A_s. Since B and B' are coarser than A_r they are comparable. As B' is coarser than A_s, the unique possibility is $B \subset B'$. Thus $B' \in \mathcal{L}_j$ is coarser than $t+1$ elements of $\mathcal{A}_j$, a contradiction.

We now prove that $\mathrm{cp}(\mathcal{B}) = 0$. Take $A_r \in \mathcal{A}_j$. Since $\mathcal{A}_j$ is the equivalence class of A_r with respect to the dependence relation, for $t \neq j$ and $A_s \in \mathcal{A}_t$, A_r and A_s are not dependent. As A_r and B_j, respectively A_s and B_t, are dependent, it follows that B_j and B_t have to be independent. Thus the statement follows from Lemma 6.2. □

Next, for $1 \le j \le m$ such that $B_j \in \mathcal{B}$ is a valuation ring let F_j and π_j be respectively a residue field of B_j and the canonical homomorphism corresponding to B_j and F_j. For every $A_r \in \mathcal{A}_j$, since B_j is coarser than A_r, it follows that $\bar{A}_r = \pi_j(A_r \cap B_j^*)$ is a cone, if A_r is a cone (see Proposition 5.1). If A_r is a valuation ring, then $A_r \subset B_j$ and $\bar{A}_r = \pi_j(A_r)$ is a valuation ring of F_j. Denote by $\bar{\mathcal{A}}_j$ the set of distinct and non-trivial $\bar{A}_r$, for $A_r \in \mathcal{A}_j$.

Lemma 6.4. *Keep the notation introduced above. If $\mathcal{A}_j$ is not a singleton set, then $\bar{\mathcal{A}}_j$ is an allowable family and $cp(\bar{\mathcal{A}}_t) < \ cp(\mathcal{A})$.*

Proof. The proof depends on Theorem 8.7 of [9, p. 58], which states that π_j induces an inclusion preserving bijective correspondence between the set of all valuation rings A of F finer than B_j and the set of all valuation rings $\bar{A}$ of F_j.

Observe first that if $\bar{A}_r$ and $\bar{A}_s$ are cones and $\bar{A}_r$ is coarser than $\bar{A}_s$, they coincide by definition. Take now $\bar{A}_r$ and $\bar{A}_s$ that are not both cones. We claim that if $\bar{A}_r$ is coarser than $\bar{A}_s$, then A_r is also coarser than A_s. Since $\mathcal{A}$ is allowable, it follows from the claim that $A_r = A_s$ and so $\bar{A}_r = \bar{A}_s$. Thus $\bar{\mathcal{A}}_j$ is allowable.

If $\bar{A}_r$ and $\bar{A}_s$ are valuation rings, the claim follows directly from the result quoted above. If $\bar{A}_r$ is a valuation ring and $\bar{A}_s$ is a cone, from Proposition 5.1 $V(\bar{A}_s) \subset \bar{A}_r$. Since $\pi_j(V(A_s)) = V(\bar{A}_s)$ the quoted result implies that $V(A_s) \subset A_r$ and the claim is also stated.

For the second statement, observe that the result mentioned at the first paragraph of the proof implies that every chain $\mathcal{O}_1 \subset \cdots \subset \mathcal{O}_\ell$ of valuation rings of F_j can be lift to the chain $\pi_j^{-1}(\mathcal{O}_1) \subset \cdots \subset \pi_j^{-1}(\mathcal{O}_\ell) \subset B_j$, which has $\ell + 1$ elements. Hence the statement is true. □

Lemma 6.5. *Keep the elements $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}_j$, F_j, π_j and $\bar{\mathcal{A}}_j$ as above. For $1 \le j \le m$ such that B_j is a valuation ring, let M_j be the maximal ideal of B_j and write $R_j = \bigcap \Re(A_i)$, where A_i ranges over $\mathcal{A}_j$. Assume that $\Re(A_1) \cap \cdots \cap \Re(A_n) \subset R(F)$. Then:*

(a) $\Re(B_1) \cap \ldots \cap \Re(B_m) \subset R(F)$.

(b) If B_j is a valuation ring, then $R_j \subset (1 + M_j)R(F)$.

(c) If $\mathcal{A}_j$ is not a singleton set, then $\bigcap \Re(\bar{A}_i) \subset R(F_j)$, where $\bar{A}_i$ ranges over $\bar{\mathcal{A}}_j$.

Proof. (a) For singleton sets $\mathcal{A}_j = \{A_t\}$, we have $B_j = A_t$ and $\Re(B_j) = \Re(A_t)$. In the other case B_j is a valuation ring coarser than every $A_i \in \mathcal{A}_j$. Therefore $\Re(B_j) \subset \Re(A_i)$, for every $A_i \in \mathcal{A}_j$. Hence $\Re(B_1) \cap \ldots \cap \Re(B_m) \subset \Re(A_1) \cap \ldots \cap \Re(A_n)$ and (a) is proved.

(b) For j such that B_j is a valuation ring we fix $r \in R_j$ and consider $x_1, \ldots, x_m \in F$, where $x_j = r$ and $x_t = 1$, for $t \neq j$. Since the localizers $B_1, \ldots, B_m$ are pairwise independent by lemmas 6.2 and 6.3, we can approximate $x_1, \ldots, x_m$ simultaneously by $s \in F$, sufficiently close to every x_t, in order that $s^{-1}x_t \in \Re(B_t)$, for every $t \neq j$, and $s^{-1}r \in 1 + M_j$. Since $x_t = 1 \in \Re(B_t)$, for every $t \neq j$, it follows that $s \in \Re(B_t)$, for every $t \neq j$.

Observe now that $1 + M_j \subset R_j$. Thus $s \in R_j$ and then

$$s \in \left(\bigcap_{t \neq j} \Re(B_t) \right) \cap R_j \subset \bigcap_{i=1}^{n} \Re(A_i).$$

Hence $s \in R(F)$. Finally, for $y = s^{-1}r \in 1 + M_j$, we get $r = sy \in R(F)(1 + M_t)$, as desired.

(c) Since $1 + M_j \subset \Re(A_i)$, for every $A_i \in \mathcal{A}_j$, it follows that π_j induces a surjective homomorphism from $\Re(A_i) \cap B_j^*$ onto $\Re(\bar{A}_i)$ and $\Re(A_i) \cap B_j^*$ is also the inverse image $\pi_j^{-1}(\Re(\bar{A}_i))$, for every $A_i \in \mathcal{A}_j$.

Moreover, $B_j \notin \mathcal{A}_j$, by Lemma 6.3, since we have assumed that $\mathcal{A}_j$ is not a singleton set. Thus $\bar{A}_i \neq F_j$ for every $A_i \in \mathcal{A}_j$.

Therefore, if $u \in B_j^*$ is such that $\pi_j(u) \in \bigcap \Re(\bar{A}_i)$, $\bar{A}_i \in \bar{\mathcal{A}}_t$, then $u \in R_j$. Hence item (b) yields $\pi_j(u) \in \pi_j(R(F) \cap B_j^*) \subset R(F_j)$, where the last inclusion is the content of Corollary 4.5. □

Next we prove Theorem 6.1

Proof. By Lemma 6.2 and Proposition 5.4 the result is true if $\mathrm{cp}(\mathcal{A}) = 0$. Assume now $\mathrm{cp}(\mathcal{A}) > 0$ and keep the elements $\mathcal{B}$, $\mathcal{A}_j$, F_j, π_j and $\bar{\mathcal{A}}_j$ with the meaning introduced above.

Lemmas 6.2, 6.3 and 6.5 show that Proposition 5.4 applies to F and $\{B_1, \dots, B_m\}$. For every $1 \le t \le m$ let (L_t, B'_t) be a local closure of (F, B_t). Then, there exists a free pro-p group G_0 such that $G_p(F) = G_0 * G_p(L_1) * \cdots * G_p(L_m)$.

Next, we shall show that $G_p(L_t)$ is $\mathcal{A}$-admissible for every $t = 1, \dots, m$. If for some t, $\mathcal{A}_t = \{A_j\}$, then $B_t = A_j$ and $G_p(L_t) = G_p(H_j)$ is $\mathcal{A}$-admissible.

In the other case B_t is a valuation ring and we shall need some facts from valuation theory. We first recall from [9, Theorem 15.8] that the residue field of B'_t equals F_t, the residue field of B_t. We also know that the canonical projection $B'_t \longrightarrow F_t$ gives rise to a canonical split short exact sequence

$$1 \longrightarrow T_t \longrightarrow G_p(L_t) \longrightarrow G_p(F_t) \longrightarrow 1,$$

where T_t is the inertia group over F of D_t, the unique prolongation of B'_t to $F(p)$. Thus $G_p(L_t) \cong T_t \rtimes G_p(F_t)$. Let us analyze the groups T_t and $G_p(F_t)$.

Since by our general assumption $\mathrm{char}\, F_t \ne p$, the ramification group of D_t over F is trivial [9, Theorem 20.18]. Hence [9, Theorem 20.12] implies that T_t is abelian. Let K_t be the fixed field of T_t. If $p \ne 2$, K_t is not formally real because of our assumption on the existence of a primitive p-th root of unity in F. If $p = 2$, recall from [9, Theorem 19.11] that $D_t \cap K_t$ has residue field $F_t(2)$. Since D_t is 2-henselinan, K_t is also not formally real, [20, Theorem 3.16]. Thus T_t is torsion free in any case and then it is an $\mathcal{A}$-admissible group. Therefore, to show that $G_p(L_t)$ is $\mathcal{A}$-admissible, it remains to be seen that $G_p(F_t)$ is $\mathcal{A}$-admissible.

Now, by Lemma 6.4, $\mathrm{cp}(\bar{\mathcal{A}}_t) < \mathrm{cp}(\mathcal{A})$. By Lemma 6.5 we can apply the induction hypothesis to F_t and $\bar{\mathcal{A}}_t$. Therefore $G_p(F_t)$ is an $\bar{\mathcal{A}}_t$-admissible group. We shall next show that we may lift this property to $\mathcal{A}$-admissibility.

The decomposition $G_p(L_t) \cong T_t \rtimes G_p(F_t)$ and Galois Theory guarantee the existence of an extension $E \subset F(p)$ of L_t such that $K_t \cap E = L_t$ and $K_t L_t = F(p)$. For this extension E, the following statements are true:

(1) $D_t \cap E$ is a p-henselian valuation ring with residue field F_t.

(2) The inertia group of D_t over E is trivial (follows from [9, 19.10 (b)]).

(3) There is a canonical isomorphism $G_p(E) \cong G_p(F_t)$ [9, Theorem 19.6].

(4) There is a bijective and inclusion-preserving correspondence between the set of all extensions of E inside $F(p)$ and the set of all extensions of F_t inside $F_t(p)$ [9, Theorem 19.13]. Moreover, fields which are in correspondence have isomorphic Galois groups where the isomorphism is induced by the isomorphism of item (3).

(5) For every locally closed extension $(\bar{H}_i, \bar{A}'_i)$ of $(F_t, \bar{A}_i)$, item (4) above yields a locally closed extension (H_i, A'_i) of (F, A_i) [2, Lemma 1.3].

Therefore, as $G_p(F_t)$ is $\bar{\mathcal{A}}_t$-admissible, the above remarks imply that $G_p(E)$ is $\mathcal{A}_t$-admissible and then also $\mathcal{A}$-admissible. □

If we look for admissible groups which are just free pro-p products we have to impose one more condition on the localizer of $\mathcal{A}$. To be precise, the next result generalizes Proposition 5.4.

We shall call a valuation ring B of F *exceptional* if there are distinct cones $A_i, A_j \in \mathcal{A}$ such that $V(A_i) = V(A_j) = B$.

Theorem 6.6. *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be an allowable family of localizers of F and for every $1 \le i \le n$ let (H_i, A_i') be a local closure of (F, A_i). Assume that $\mathcal{A}$ satisfies the conditions:*

(P1) $\Re(A_1) \cap \ldots \cap \Re(A_n) \subset R(F)$.

(P2) For every $B \in \mathcal{L}$, we have $(\Gamma_B : p\Gamma_B) \le p$ and if $(\Gamma_B : p\Gamma_B) = p$, then $p = 2$, B is exceptional and k_B is euclidean.

Then there is an extension H_0 of F, as in Proposition 3.4, such that $G_p(F) = G_p(H_0) * G_p(H_1) * \cdots * G_p(H_n)$.

Proof. The proof follows in the same lines as the proof of the last theorem. If $\mathrm{cp}(\mathcal{A}) = 0$ the statement was proved in Proposition 5.4. For $\mathrm{cp}(\mathcal{A}) > 0$ we prove by induction that each $G_p(L_t)$ decomposes into a free pro-p product (notations as in Theorem 6.1).

Consider first the case where $B_t \notin \mathcal{A}$ has non-p-divisible value group.

By assumption $p = 2$ and B_t is exceptional. As in the proof of Theorem 6.1, $G_p(L_t) \cong T_t \rtimes G_p(F_t)$. Now, the restrictions imposed by (P2) on the value group and the residue field of B_t imply that $T_t \cong \mathbb{Z}_2$ and $G_p(F_t) \cong \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Therefore $G_p(L_t)$ is the pro-2 dihedral group. For the cones A_i, A_j such that $V(A_i) = V(A_j) = B_t$, since (L_t, B_t') is an immediate extension of (F, B_t), they have distinct prolongations to L_t. Consequently, it is well known that $G_p(L_t) = G_p(H_i) * G_p(H_j)$.

If $B_t \notin \mathcal{A}$ has p-divisible value group, then the inertia group T_t is trivial and so $G_p(L_t) \cong G_p(F_t)$. Construct $\bar{\mathcal{L}}$ from $\bar{\mathcal{A}}_t$ as $\mathcal{L}$ was constructed from $\mathcal{A}$. Now, we only need to modify the argument in the proof of Theorem 6.1 by showing that the valuation rings in $\bar{\mathcal{L}}$ satisfies condition (P2), in addition to (P1), in order to use induction.

It follows from [9, Theorem 8.7] that any valuation ring $\bar{\mathcal{O}} \in \bar{\mathcal{L}}$ corresponds to a valuation ring $\mathcal{O} \in \mathcal{L}$ such that $\mathcal{O} \subset B_t$. Let us denote by Γ and Δ the value groups of $\mathcal{O}$ and $\bar{\mathcal{O}}$, respectively. From valuation theory we know that B_t has value group order isomorphic to the quotient group Γ/Δ. Since B_t has p-divisible value group it follows that $(\Delta : p\Delta) = (\Gamma : p\Gamma)$. On the other side, by [9, 8.3], $\mathcal{O}$ and $\bar{\mathcal{O}}$ have the same residue field. Therefore, every $\bar{\mathcal{O}} \in \bar{\mathcal{L}}$ satisfies (P2).

As in the proof of the previous theorem (P1) follows from lemmas 6.2 and 6.5. Consequently, by repeating the arguments (1) to (5) in the end of the proof of Theorem 6.1 with E and D_t replaced by L_t and B_t we see that $G_p(L_t)$ also decomposes into a free pro-p product of the desired type. □

Observe that in the last result each (H_i, A_i') is a local closure of (F, A_i) instead of just a locally closed extension.

7 The free pro-p product case

In this section we study fields for which $G_p(F)$ admits a decomposition into a free pro-p product of finite family of subgoups. Our aim is to show the converse of Theorem 6.6.

For the reader's convenience we recall a few facts concerning free pro-p products.

Remark 3. Let G be a pro-p group and $G_1, \dots, G_n$ be a family of subgroups such that $G = G_1 * \cdots * G_n$.

(1) If $g \in G$ has finite order, then there are $1 \le i \le n$ and $\sigma \in G$ such that $\sigma^{-1} g \sigma \in G_i$ [12, Theorem A'].

(2) If there are $g \in G$ and $1 \le i, j \le n$ such that $g^{-1} G_i g \subset G_j$, then $i = j$ and $g \in G_i$ [12, Theorma B'].

(3) Let G' be a subgroup of G generated by a family of subgroups $G'_i \subset G_i$, $1 \le i \le n$. Then $G' = G'_1 * \cdots * G'_n$ and $G' \cap G_i = G'_i$. The statement follows from [13, Corollary 5.4] by induction on n.

We shall next consider some natural restrictions on the family of subgroups considered in Theorem 6.6.

Definition 6. For a field F, let $G_p(H_0), G_p(H_1), \dots, G_p(H_n)$ be a family of closed subgroups of $G_p(F)$ where $G_p(H_0)$ is a free pro-p group and for each $1 \le i \le n$, (H_i, A'_i) is a locally closed extension of $(F, A'_i \cap F)$ inside $F(p)$.

We say that this family is reduced if for every $1 \le i \le n$ $G_p(H_i)$ is non-trivial, non-isomorphic to $\mathbb{Z}_p$, nor $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, when $p = 2$ and if $G_p(H_i) \cong \mathbb{Z}/2\mathbb{Z}$, then A_i is a cone.

Lemma 7.1. *Let F be a field such that $G_p(F) = G_p(H_0) * G_p(H_1) * \cdots * G_p(H_n)$ is a decomposition as in the last definition. Then there exist a family of localizer $\{B_1, \dots, B_r\}$ of F and an extension L_0 of F inside $F(p)$ which meet the following conditions:*

(i) *$G_p(L_0)$ is a free pro-p group;*

(ii) *the family of subgroups $G_p(L_0), G_p(L_1), \dots, G_p(L_r)$ is reduced, where (L_t, B'_t) is a locally closed extension of (F, B_t) inside $F(p)$, for each $1 \le t \le r$;*

(iii) *$G_p(F) = G_p(L_0) * G_p(L_1) * \cdots * G_p(L_r)$.*

Proof. For every $1 \le i \le n$, let $A_i = A'_i \cap F$.

If $G_p(H_j) \cong \mathbb{Z}_p$ for some j, then $G_p(H_0) * G_p(H_j)$ is a free pro-p group. Therefore we can remove A_j from the family $\{A_1, \dots, A_n\}$ and replace H_0 by $H'_0 = H_0 \cap H_j$ and we still get $G_p(F) \cong G_p(H'_0) * \cdots * G_p(H_{j-1}) * G_p(H_{j+1}) * \cdots * G_p(H_n)$.

In the case $p = 2$ and $G_p(H_j) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, if we denote by R_1 and R_2, respectively, the fixed fields of the components of $G_p(H_j)$, then R_1 and R_2 are ordered fields and $G_p(H_j) = G_p(R_1) * G_p(R_2)$. Since $G_p(R_t)$ has order 2, R_t is euclidean $t = 1, 2$. If there is $\sigma \in G_p(F)$ such that $\sigma G_p(R_1) \sigma^{-1} = G_p(R_2)$, then $\sigma G_p(H_j) \sigma^{-1} = G_p(H_j)$ and by Remark 3 (2), $\sigma \in G_p(H_j)$. Applying now Remark 3 (2) to σ, $G_p(R_1)$, $G_p(R_2)$ we get a contradiction. Consequently, R_1 and R_2 induce different orderings on F. Let A_{j1}, A_{j2} be these orderings. Then, for each $t = 1, 2$, $(R_t, \dot{R}_t^2)$ is a local closure of (F, A_{jt}). In this case we replace A_j in the original family by A_{j1} and A_{j2} and again we get $G_p(F) \cong G_p(H_0) * \cdots * G_p(H_{j-1}) * G_p(R_1) * G_p(R_2) * G_p(H_{j+1}) * \cdots * G_p(H_n)$.

Finally, in case $p = 2$ and $G_p(H_j)$ an order 2 group, we have that H_j is euclidean. Therefore, denoting by $B' = \dot{H}_j^2$ the unique ordering of H_j we have that (H_j, B') is also the local closure of $(F, B' \cap F)$. We then replace A_j by $B = B' \cap F$.

By repeating the above operations finitely many times, we find a decomposition of $G_p(F)$ of the desired type. □

Note that $G_p(H_0)$ is a subgroup of $G_p(L_0)$ and for each $1 \leq t \leq r$ $G_p(L_t)$ is a subgoup of some $G_p(H_i)$.

The lemma above shows that we may choose reduced families of subgroups without loss of generality. We shall next see that the choice of these families gives us a result similar to Lemma 5.5 which will be crucial to prove the converse of Theorem 6.6.

Proposition 7.2. *For a field F such that $G_p(F) = G_p(H_0) * G_p(H_1) * \cdots * G_p(H_n)$ is a decomposition as we described in Definition 6 we suppose that $G_p(H_0), G_p(H_1), \ldots, G_p(H_n)$ is a reduced family of subgroups. Let B be a localizer of F and write $A_i = A'_i \cap F$, for every $i = 1, \ldots, n$.*

(a) If B is the cone of an ordering, then there exists $1 \leq i \leq n$ such that A_i is coarser than B.

(b) If B is a valuation ring which is coarser than A_j, for some $1 \leq j \leq n$, one of the following conditions hold:

(b1) $\Gamma_B = p\Gamma_B$;

(b2) (H_j, A'_j) *is a local closure of* (F, A_j) *and there is a local closure* (L, B') *of* (F, B) *such that* $H_i = L$.

(b3) $p = 2$, *A_j is a cone,* $(\Gamma_B : 2\Gamma_B) = 2$ *and* k_B *is euclidean. Moreover, there is some* $1 \leq t \neq j \leq n$ *such that* A_t *is also a cone and* $V(A_t) = V(A_j)$.

For the proof of the statement (b) we recall a result on the subgroups of the pro-2 dihedral group $\mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Lemma 7.3. *If $G \cong \mathbb{Z}_2 \rtimes \mathbb{Z}/2\mathbb{Z}$ is the pro-2 dihedral group, then every subgroup G' of G is either cyclic or dihedral.*

Proof. Recall that the action of an order 2 element $\sigma \in G$ on the $\mathbb{Z}_2$ component is given by $\sigma^{-1}\tau\sigma = \tau^{-1}$, for every τ. Therefore every element of G which is not in the $\mathbb{Z}_2$ component has order 2.

Take a subgroup $G' \neq 1$ of G. If G' is contained in $\mathbb{Z}_2$, then G' is cyclic and torsion free. Thus $G' \cong \mathbb{Z}_2$. If $G' \not\subset \mathbb{Z}_2$, then $G = \mathbb{Z}_2 G'$ and so $\mathbb{Z}_2 \cap G'$ is a normal subroup of G' of index 2 which is either isomorphic to $\mathbb{Z}_2$ or trivial. Observe next that every $\sigma \in G' \setminus G' \cap \mathbb{Z}_2$ has order 2. Consequently $G' = (G' \cap \mathbb{Z}_2) \rtimes < \sigma >$ is also a dihedral group if $G' \cap \mathbb{Z}_2 \neq 1$. It follows also that G' has order 2 if $G' \cap \mathbb{Z}_2 = 1$. □

Proof. (Proposition 7.2) (a) Let (L, B') be a local closure of (F, B) and take $\sigma \in G_p(L)$, $\sigma \neq 1$. Then $G_p(L) = \{1, \sigma\}$. By Remark 3 (a), there are $g \in G_p(F)$ and $1 \leq i \leq n$ such that $\sigma \in gG_p(H_i)g^{-1}$. If A_i is a cone, then $gG_p(H_i)g^{-1} = G_p(L)$, which implies that $B = A_i$. If A_i is a valuation ring, then (gH_i, gA'_i) is also a local closure of (F, A_i) which is contained in L. By [20, Theorem 3.16] gA'_i is coarser than the restriction of B' to gH_i. Consequently, A_i coarser than B. ((a) is also consequence of [5, Proposition 5.4]).

(b) Since B is coarser than A_j, H_j contains a local closure (L_1, B_1') of (F, B). From valuation theory L_1 is F-isomorphic to L. Therefore, we may assume without loss of generality that $L \subset H_j$ and so $G_p(H_j) \subset G_p(L)$ $(*)$.

On the other side, by [5, Proposition 5.4], one of the following cases occur: B' has p-divisible value group, there is $1 \leq i \leq n$ such that $G_p(L) \subset G_p(H_i)$, $G_p(L) \cong \mathbb{Z}_p, \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Recall from [9, Theorem 15.8] that B' and B have the same value group and the same residue field. Thus, if the value group of B' is p-divisible so is Γ_B, and (b1) occurs.

Assume now the second case: $G_p(L) \subset G_p(H_i)$ for some $i = 1, \dots, n$. This inclusion together with the above inclusion $(*)$ yields $G_p(H_j) \subset G_p(H_i)$. Thus $j = i$, by Remark 3 (2). Hence $G_p(L) = G_p(H_j)$ and so $L = H_j$ which implies that (H_j, A_j') is a local closure of (F, A_j) and so (b2) holds.

In the other cases, the first possibility $G_p(L) \cong \mathbb{Z}_p$ cannot occur, because if $G_p(L) \cong \mathbb{Z}_p$, as a subgroup of the procyclic group, $G_p(H_j)$ is also procyclic, contrary to the assumption that the family of subgroups is reduced.

In the last case $G_p(L) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ $(p = 2)$ we apply Lemma 7.3. Since by hypothesis $G_p(H_j)$ is not isomorphic to $\mathbb{Z}_2$ nor a dihedral group, $G_p(H_j)$ has order 2. Then, as the family of subgroups is reduced, A_j is a cone.

For future use we observe that if there is and extension L of F, inside $F(2)$, such that $L \subset H_j$, for some $1 \leq j \leq n$, and $G_p(L) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, then $G_p(H_j) \cong \mathbb{Z}/2\mathbb{Z}$ and A_j is a cone (†).

Next, note that $G_p(L)$ has 2 conjugacy classes of elements of order 2 (follows from the description of $G_p(L)$ as a dihedral pro-2 group, since $(\mathbb{Z}_2 : 2\mathbb{Z}_2) = 2$). For a subgroup G' of order 2, different form $G_p(H_j)$, we have $G_p(L) = G_p(H_j) * G'$.

The fixed field H of G' is euclidean. Let $A' = \dot{H}^2$ be the unique ordering of H. Then L has exactly 2 orderings which are induced by A_j' and A', respectively (because $G_p(L)$ has 2 conjugacy classes of elements or order 2). Since B' is 2-henselian with a non-2-divisible value group it follows from [20, Corollary 3.11] that k_B, the residue field of B', has just one ordering and the value group Γ_B satisfies $(\Gamma_B : 2\Gamma_B) = 2$.

Let $A = A' \cap F$ be the restriction of A' to F. The last discussion implies that $V(A_j) = V(A) \subset B$.

By item (a) there is $1 \leq t \leq n$ such that A_t is coarser than A. If A_t is a cone, then $A = A_t$ and the proof is complete. Heading for a contradiction we assume that A_t is a valuation ring. Since B and A_t are coarser than A, they have to be comparable. If B is coarser than A_t, by valuation theory, there is some $g \in G_p(F)$ such that $G_p(H_t) \subset g^{-1} G_p(L) g$. Therefore, by the our previous remark (†), A_t is a cone, a contradiction. On the other side, since B is coarser than A_j, if A_t is coarser than B, it follows that A_t is coarser than A_j. Therefore, there is $g \in G_p(F)$ such that $g^{-1} G_p(H_j) g \subset G_p(H_t)$, which is not possible by Remark 3 (2). □

We shall see next that a decomposition of $G_p(F)$ as in the last proposition imposes some restrictions on the localizers $A_1, \dots, A_n$.

Corollary 7.4. *Keeping the condition of Proposition 7.2 we have that:*

(a) For any cone A_i, if $B = V(A_i)$, then $(\Gamma_B : 2\Gamma_B) \leq 2$ and the equality happens only if k_B is euclidean and B is exceptional.

(b) For every valuation ring A_i it follows either: (H_i, A_i') is a local closure closure of (F, A_i) or A_i has p-divisible value group.

(c) If $\mathcal{A}$ is not allowable and $1 \le i \ne j \le n$ are such that A_j is coarser than A_i, then A_j has p-divisible value group.

Proof. If (b1) or (b3) occurs, (a) is proved. In the case (b2), there is a local closure (L, B') of (F, B) such that $L = H_i$ is euclidean. Then $\dot{L} = \dot{L}^2 \cup (-1)\dot{L}^2$. Consequently, the value group of B' is 2-divisible. As B' and B have the same value group, [9, Theorem 15.8], the prove of (a) is complete.

Statement (b) follows form (b1) and (b2) of the above proposition taking $B = A_i$, because (b3) cannot occur.

(c) Observe first that if A_i is a cone and we are in the case (b3), the restrictions on the value group and on the residue field of A_j implies for a local closure (H, A) of (F, A_j) that $G_p(H)$ is the dihedral group. Recall that we can choose H in order that $G_p(H_i) \subset G_p(H)$. This inclusion together with Lemma 7.3 lead to a contradiction because we have assumed that the family $G_p(H_0), G_p(H_1), \dots, G_p(H_n)$ is a reduced.

Case (b2) cannot happens either, otherwise $G_p(H_i) = g^{-1}G_p(H_j)g$ for some $g \in G_p(F)$. Hence (c) follows from (a) of Proposition 7.2. □

Now we can prove the converse of Theorem 6.6.

Theorem 7.5. *Consider a field F and a family of locally closed extensions (H_i, A'_i), $1 \le i \le n$, of F inside $F(2)$. Assume that there is another intermediate extension $H_0 \subset F(p)$ such that $G_p(H_0)$ is a free pro-2 group and $G_p(H_0), G_p(H_1), \dots, G_p(H_n)$ is reduced. For every $1 \le i \le n$ let $A_i = A'_i \cap F$ and write $\mathcal{A} = \{A_i, \dots, A_n\}$.*

If $G_p(F) = G_p(H_0) * G_p(H_1) * \cdots * G_p(H_n)$, then there is a family $\mathcal{A}_1 \subset \mathcal{A}$ which is an allowable family of localizers of F and satisfies the conditions (P1) and (P2) of Theorem 6.6.
Moreover, if $\mathcal{A}$ is allowable, then $\mathcal{A}_1 = \mathcal{A}$.

Proof. Let us first prove that $\mathcal{A}$ satisfies (P1).

Since (H_i, A'_i) is locally closed, it follows that $\Re(A_i) \subset \dot{H}_i^p$, for every $1 \le i \le n$. Therefore, if $r \in \Re(A_1) \cap \ldots \cap \Re(A_n)$, then $D_i(r) = \dot{H}_i$, for every $i = 1, \dots, n$. Since $G_p(H_0)$ is a free pro-p group, by Lemma 3.2, $D_0(r) = \dot{H}_0$, too. Hence, for any $b \in \dot{F}$, it follows that $b \in D_i(r)$, for every $i = 0, \dots, n$. Therefore, Proposition 2.1 implies that every $b \in \dot{F}$ satisfies $b \in D_F(r)$ for $r \in \Re(A_1) \cap \ldots \cap \Re(A_n)$. Hence $r \in R(F)$ and the condition (P1) is proved.

Next let $\mathcal{A}_1 = \{A_i \in \mathcal{A} \mid A_i \text{ is not finer than any } A_j \in \mathcal{A}, j \ne i\}$. Clearly $\mathcal{A}_1$ is allowable.

We claim that $\mathcal{A}_1$ satisfies (P1) and (P2). If $A_t \notin \mathcal{A}_1$, there is $A_s \in \mathcal{A}_1$ such that A_s is coarser than A_t. Thus $\Re(A_s) \subset \Re(A_t)$. Consequently, the intersection of all $\Re(A_i)$, where A_i ranges over $\mathcal{A}_1$, satisfies

$$\bigcap \Re(A_i) = \bigcap_{i=1}^{n} \Re(A_j) \subset R(F).$$

Hence $\mathcal{A}_1$ has the property (P1).

We now prove $\mathcal{A}_1$ has (P2). Construct $\mathcal{L}$ from the family $\mathcal{A}_1$ as in Definition 4. For $B \in \mathcal{L}$, by the very definition of $\mathcal{L}$, B is coarser than 2 distinct localizers A_i, A_j of $\mathcal{A}_1$ and is the finest with this property. We shall apply Proposition 7.2 to B.

It is enough to show that case (b2) cannot occur. Assume this is not so. Then (H_i, A'_i) and (H_j, A'_j) are local closures of (F, A_i) and (F, A_j), respectively, and there are local closure (L_1, B'_1), (L_2, B'_2) of (F, B) such that $L_1 = H_i$ and $L_2 = H_j$. By valuation theory there is $g \in G_p(F)$ such that $g^{-1}G_p(L_1)g = G_p(L_2)$, contrary to Remark 3 (2).

In case (b1) we are done. In case (b3), since $V(A_i) = V(A_j)$ is coarser than both A_i and A_j, it follows that $B = V(A_i)$. Therefore B is exceptional and (P2) holds for $\mathcal{A}_1$ as claimed.

Observe now that if $\mathcal{A}$ is allowable, then $\mathcal{A}_1 = \mathcal{A}$. □

8 The localizer in the free product decomposition case

In this section we suppose that F is a field admitting an allowable family of localizers $\mathcal{A} = \{A_i, \dots, A_n\}$ which satisfies the conditions of (P1) and (P2) of Theorem 6.6. Let $G_p(F) = G_p(H_0) * G_p(H_1) * \cdots * G_p(H_n)$ be the decomposition of the Galois group where, $G_p(H_0)$ is a free pro-p group and for every $1 \le i \le n$, (H_i, A_i') is a local closure of (F, A_i). Suppose further the family of subgroups $G_p(H_0), G_p(H_1), \dots, G_p(H_n)$ is reduced.

We shall now discuss the localizers $B \notin \mathcal{A}$. In Proposition 7.2 we learned about the cones of orderings of F and also about valuation rings which are coarser than some $A_i \in \mathcal{A}$. We shall now refine this knowledge by considering valuation rings which are not comparable to any element of $\mathcal{A}$.

Proposition 8.1. *For a field F satisfying the above conditions let B be a valuation ring of F which is not comparable to A_i, for every $1 \le i \le n$. If (L, B') is a local closure of (F, B), then $G_p(L)$ is a either a free pro-p group or there are $1 \le i \le n$ and $g \in G_p(F)$ such that $g^{-1}G_p(L)g \subset G_p(H_i)$ and it is abelian and torsion free.*

Particularly, if B is independent of every $A_i \in \mathcal{A}$, then $G_p(L)$ is a free pro-p group.

Proof. Take a local closure (L, B') of (F, B). As in the proof of Proposition 7.2, by [5, Proposition 5.4], we have to consider 3 cases:

(1) there are $0 \le i \le n$ and $g \in G_p(F)$ such that $g^{-1}G_p(L)g \subset G_p(H_i)$;

(2) $G_p(L) \cong \mathbb{Z}_p, \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$;

(3) B' has p-divisible value group.

In the first case we may choose L in order that $H_i \subset L$. If $i = 0$, then $G_p(L)$ is free as a subgroup of a free pro-p group. For $1 \le i \le n$ let A be the unique extension of A_i' to L. Then A is also p-henselian [9, Theorem 15.7] and since B and A_i are not comparable, by assumption, A and B' are not comparable, too. Denote $D = AB'$ the finest valuation ring of L which is coarser than A and B' simultaneously and let $\widetilde{D}$ be its unique extension to $F(p)$. Note that the valuation rings $\pi_D(A)$ and $\pi_D(B')$ are independent valuation rings of k_D. By [2, Lemma 1.3], $\pi_D(A)$ and $\pi_D(B')$ are p-henselian and then, by [2, Proposition 1.4], the residue field $k_D = k_D(p)$ is p-closed. Hence $G_p(L)$ is the inertia group of $\widetilde{D}$ over L [9, Theorem 19.6]. Since k_B has characteristic $\neq p$ (our general assumption on localizers), the same is true for k_D. Consequently, the ramification group of $\widetilde{D}$ over F is trivial [9, Theorem 20.18]. By [9, Theorem 20.12] $G_p(L)$ is abelian. Finally, if $G_p(L)$ has torsion, then L is an ordered field [1, Theorem 3, p. 73]. It follows then from [20, Theorem 3.16] that B' is coarser than every ordering of L and has residue field k formally real. On the other side, B' and its image $\pi_D(B')$ in k_D have the same residue field. Since k_D is p-closed, so is k, contradicting k to be a formally real field. Thus $G_p(L)$ is torsion free as desired.

In the case (2) if $G_p(L) \cong \mathbb{Z}_p$, then $G_p(L)$ is free (also abelian and torsion free) and we are done.

We claim that $G_p(L) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ cannot happen. Assume, by contrary, that $G_p(L)$ is the dihedral group. Then L is an ordered field, since $\mathbb{Z}/2\mathbb{Z}$ is a subgroup of $G_p(L)$. By [20, Theorem 3.16], B' is coarser than every ordering of L. Then B is coarser than an ordering P of

F which corresponds to the restriction to F of an ordering of L. By Proposition 7.2 (a), there is $1 \le i \le n$ such that A_i is coarser than P. Consequently, B and A_i are comparable, a contradiction.

Finally, in the third case, let us first show that $\mathcal{A}_1 = \mathcal{A} \cup \{B\}$ is an allowable family that satisfies (P1) and (P2) of Theorem 6.6.

In fact, since $\mathcal{A} \subset \mathcal{A}_1$, (P1) also holds for $\mathcal{A}_1$. In order to prove that (P2) holds, let $\mathcal{L}$ and $\mathcal{L}_1$ be the sets constructed form $\mathcal{A}$ and $\mathcal{A}_1$, respectively, as the description in Definition 4. Clearly $\mathcal{L} \subset \mathcal{L}_1$. We know from Theorem 6.1 that condition (P2) holds for every $C \in \mathcal{L}$. Moreover, if $C \in \mathcal{L}_1 \setminus \mathcal{L}$, then $B \subset C$ has to occur. Therefore, since Γ_B is p-divisible so is Γ_C and so $\mathcal{A}_1$ has the property (P2), as required.

Consequently, by Theorem 6.6, $G_p(F) = G_0 * G_p(H_1) * \cdots * G_p(H_n) * G_p(L)$, for some subgroup G_0 of $G_p(F)$ which is free. Hence F and $H_1, \ldots, H_n$, L satisfy conditions (I) and (II) of Proposition 2.1. For $x \in \dot{L}$ there is $y \in \dot{F}$ such that $y \in \dot{H}_i^p$, for every $i = 0, \ldots, n$ and $y\dot{L}^p = x\dot{L}^p$. Hence, $y \in \dot{H}_0^p \cap \cdots \cap \dot{H}_n^p \cap F \subset R(F)$, by Lemma 3.3. Consequently $x \in R(F)\dot{L}^p$. Thus $\dot{L} = R(F)\dot{L}^p$ and then, by Lemma 3.1 (b), $R(L) = \dot{L}$. Finally, by Lemma 3.2, $G_p(L)$ is a free pro-p group. □

Continuing with the study of localizers of a field as proposed at the beginning of this section, note that we can reduce the study of the localizers which are finer than some A_i to the study of the localizers of the residue field of A_i. In particular, we know from Proposition 7.2 (a) that every cone Q of F is finer than some $A_i \in \mathcal{A}$. The set of orderings of F which is finer than some A_i is well known (see for example [1], [20], or [29]).Hence, the set of orderings present in $\mathcal{A}$ remains to be considered.

For the rest of this section we fix $p = 2$.

Let us denote by X_F the space of orderings of F. Write $Y = \{A_1, \ldots, A_r\}$, $r \le n$, for the set of all orderings in $\mathcal{A}$. We shall see below that Y is constituted by independent orderings.

Generalizing the notion of a positive cone of an ordering, we have a *preordering* T of F, characterized as a subgroup of $\dot{F}$ such that $\dot{F^p} \subset T$ and $T + T \subset T$. For a preordering T of F let $X(T) = \{P \in X_F \mid T \subset P\}$. It is well known that $T = \bigcap P$ where P ranges over $X(T)$ [20, Theorem 1.6].

We shall also need the notion of connected orders introduced by Marshall (see [21, § 6, p. 159] or [22, § 2]). Two orderings P_1 and P_2 are called *connected* if either $P_1 = P_2$ or there is a preordering T such that $(\dot{F} : T) = 8$ and there are 2 more orderings P_3 and P_4 for which $X(T) = \{P_1, P_2, P_3, P_4\}$. Such a preordering T is called a 4-*element fan*. We write $P_1 \sim P_2$ to say these orderings are connected. Observe that $\sim$ is an equivalence relation on X_F [22, Theorem 2.3].

The next lemma, according to Efrat [4, Lemma 2.2], gives a characterization of connected orderings by means of valuation rings.

Lemma 8.2. *For $P_1 \neq P_2 \in X_F$, $P_1 \sim P_2$, if and only if there is a valuation ring A of F coarser than both P_1 and P_2 such that $(\Gamma_A : 2\Gamma_A) \ge 4$ and k_A has just one ordering or $(\Gamma_A : 2\Gamma_A) \ge 2$ and k_A has exactly two orderings.*

We shall also need the following technical lemma.

Lemma 8.3. *For $Y = \{P_1, \ldots, P_m\} \subset X_F$ let $T = P_1 \cap \cdots \cap P_m$. For every $Q \in X_F$ such that $T \subset Q$ there is $1 \le i \le m$ such that $Q \sim P_i$.*

Proof. For every $1 \le i \le m$ let $\chi_i : \dot{F}/T \longrightarrow \{\pm 1\}$ be the character on $\dot{F}/T$ associated with P_i as well as χ_Q is associated with Q. Since $\chi_1, \ldots, \chi_m$ generated the character group $\mathrm{Hom}(\dot{F}/T, \{\pm 1\})$, by a suitable change in notation, we may assume that $\chi_Q = \chi_1\chi_2\cdots\chi_s$, $s \leq m$, and $\chi_1, \ldots, \chi_s$ are $\mathbb{F}_2$-independent in $\mathrm{Hom}(\dot{F}/T, \{\pm 1\})$ (This group is also the $\mathbb{F}_2$-dual of $\dot{F}/T$). From [21, Lemma 6.22, p. 162] $Q \sim P_i$ for every $i = 1, \ldots, s$. □

We shall also need the notion of a SAP preordering T as in [20, § 17].

Proposition 8.4. *Let F be a field as it was fixed at the begining of this section. Let $Y = \{A_1, \ldots, A_r\}$, $r \leq n$ be the set of all cones of orderings in $\mathcal{A}$ and write $T = A_1 \cap \cdots \cap A_r$. Then T is a SAP preordering and $X(T) = Y$.*

Proof. We shall first apply Proposition 7.2 (b) to a valuation ring B which is finer than some $A_i \in Y$ to get either $\Gamma_B = 2\Gamma_B$ or $(\Gamma_B : 2\Gamma_B) = 2$ and k_B with a unique ordering. Next, we show that $X(T) = \{A_1, \ldots, A_r\}$. Then it follows from [20, Theorem 16.3 and 17.12 and Theorem 16.3] that T is SAP, as desired.

For a valuation ring B, coarser than $A_i \in Y$, if (b1) or (b3) holds, we are done. In case (b2) there is a local closure (L, B') of (F, B) such that $H_i = L$. Thus L is euclidean and so $\dot{L} = \dot{L}^2 \cup (-1)\dot{L}^2$. Consequently, $\Gamma_B = 2\Gamma_B$ and the first claim is stated.

Next, as a contradiction, we assume that there is $Q \in X_F$ such that $Q \notin \{A_1, \ldots, A_r\}$ and $T \subset Q$. By Lemma 8.3, $Q \sim A_i$, for some $1 \leq i \leq r$. Since $Q \neq A_i$, there is a valuation ring B of F coarser than Q and A_i as in Lemma 8.2, which contradicts what was proved in the last paragraph. □

We end this section considering a few examples. Some application of our results will appear in a forthcoming paper.

Example 1. We can find enough examples of fields F such that $G_p(F)$ is $\mathcal{A}$-admissible, for some family $\mathcal{A}$ of localizers. For example, let F be a formally real field with finitely many orderings $\mathcal{A} = \{A_1, \ldots, A_n\}$. Denote by $\Sigma\dot{F}^2$ the set of non-trivial sums of squares. If $R(F) = \Sigma\dot{F}^2$, then $G_p(F)$ is $\mathcal{A}$-admissible. Fields verifying this condition are well-known. They can be characterized as those for which the 2-primary component of the Brauer group of F is an elementary abelian 2-group [6, Theorem 3.1]. Example of these fields include pythagorean fields, generalized Hilbert fields [18] and fields with Hasse invariant $\widetilde{u} \leq 2$ [11].

Another class of examples is provided by algebraic extensions of global fields [7, Main Theorem]. For some other examples see § 1.2 of [16].

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