Some relations between variational-like inequality problem and the efficient solutions of the vectorial optimization problem

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Abstract

In this work, we study the equivalence between the solutions of variational-like inequality problem and the solutions of some nonsmooth, non-convex vectorial optimization problem.

Key words: Variational-like vectorial inequality, vectorial optimization problem, Clarke generalized gradient, efficient solution.

1 INTRODUCTION

The connection between variational inequalities and optimization problems have long been known (e.g. [14], [9], [1]). One the main work in this direction was done by Gianessi [11]. The paper [11] was done in the finite-dimensional

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context. This problem has been extensively investigated in recent years. Various results on the existence of solutions have been obtained for much variational inequalities (e.g. [5], [17], [19], [20]). Also, connections between variational inequalities and vectorial optimization problems have been studied in [12], [21] and [26], for instance.

By using a variational-like inequality, Lee et. al. [19] obtain some results of existence of solutions for nonsmooth invex problems, which are generalizations those obtained by Chen and Craven [6] for differentiable convex problems. Recently, Gianessi [12] showed the equivalence between efficient solutions of differentiable, convex optimization problem and the solutions of a variational inequality of Minty type. Also, he proved the equivalence between weak efficient solutions of a differentiable, convex optimization problem and solutions of a variational inequality of weak-Minty type.

Following this way, Lee [18] established the equivalence between the solutions of the inequalities of Minty and Stampacchia type for subdifferential (in the analysis convex sense) and the efficient solutions and weakly efficient solutions, respectively, of vectorial, nonsmooth, convex optimization problems. Moreover, using these characterizations, he proved a theorem on the existence of the weakly efficient solutions of vectorial, nonsmooth, convex optimization problem, under weak hypothesis of compactness.

In this work, we extend the results obtained early by Lee [18] for the nonsmooth invex context.

This paper is divided as follows: In Section 2 we fix some basic notation and terminology. In Section 3 we prove some connections between efficient solutions and vectorial optimization problems. In Section 4, we consider the case of weakly efficient solutions. Finally, in Section 5 we use the results of the above sections to show a existence result of the weakly efficient solutions of nonsmooth invex vectorial optimization problem, under weak hypothesis of compactness.

2 PRELIMINARIES

In this section we recall some notions of nonsmooth analysis. For more details see, for instance, Clarke [7]. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative octant. In the sequel Ω will be a nonempty open subset of $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}$ is said to be *Lipschitz* near $x \in \Omega$ if, for some $K > 0$,

$$|f(y) - f(z)| \leq K\|y - z\|,$$

for all y, z within a neighbourhood of x. We say that f is *locally Lipschitz* on Ω if f is Lipschitz near any given point of Ω. The *generalized directional derivative* of f at x in the direction v, denoted by $f^o(x; v)$, is defined as follows:

$$f^o(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{1}{t}[f(y + tv) - f(y)].$$

The *generalized gradient* of f at x, denoted by $\partial f(x)$, is the subset of $\mathbb{R}^n$ given by

$$\partial f(x) = \{\xi \in \mathbb{R}^n : f^o(x; v) \geq \langle \xi, v \rangle, \ \ \forall v \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$. The set $\partial f(x)$ is nonempty when f is Lipschitz near $x \in \Omega$.

Let X be a nonempty subset of $\mathbb{R}^n$. The *distance function* related to X, is the function $d_X(\cdot) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_X(x) := \inf\{\|x - c\| : c \in X\}.$$

The distance function is not everywhere differentiable but is globally Lipschitz. Let $x \in X$. A vector $v \in \mathbb{R}^n$ is said to be *tangent* to X at x if $d^o_X(x; v) = 0$. The set of tangent vectors to X at x is a closed convex cone in $\mathbb{R}^n$, called *Clarke tangent cone* and denoted by $T_X(x)$:

$$T_X(x) = \{v \in \mathbb{R}^n : d^o_X(x, v) = 0\}.$$

The *Clarke normal cone* to X at x can be defined by polarity with $T_X(x)$:

$$N_X(x) = \{\xi \in \mathbb{R}^n : \langle \xi, v \rangle \leq 0, \forall v \in T_X(x)\}.$$

Suppose that f is a locally Lipschitz function on Ω and attains a minimum over X at $\overline{x}$. Then

$$0 \in \partial f(\overline{x}) + N_X(\overline{x}). \tag{1}$$

We say that $\overline{x} \in X$ is a *Clarke stationary* point of f over X if (1) holds. Hanson [15] considered differentiable functions, $f : \mathbb{R}^n \to \mathbb{R}$ for which, for all $x, y \in \mathbb{R}^n$, there exists $\eta(y, x) \in \mathbb{R}^n$ such that

$$f(y) - f(x) \geq \langle \nabla f(x), \eta(y, x) \rangle.$$

Nowadays, such functions are generally known as *invex functions* due to Craven [8], who first called them by this term. This invexity notion for functions generalizes the concept of convexity and allows to extend sufficient conditions of optimality and duality results to nonconvex optimization problems (see, for instance, [2], [13], [24]). Invexity has now been extended to nondifferentiable locally Lipschitz functions, see, for example, Craven [8], Reiland [25] and Phuong, Sach and Yen [23]. We use the definition provided in [23]: let X be a nonempty subset of Ω and suppose that $f : \Omega \to \mathbb{R}$ is a locally Lipschitz function on Ω. We say that f is *invex* on X if, for every $x, y \in X$, there is $\eta(y, x) \in T_X(x)$, such that,

$$f(y) - f(x) \geq f^o(x; \eta(y, x)).$$

The above notion of invexity is very powerful because it allows to treat smooth and nonsmooth constrained optimization problems in the presence of an abstract constraint set X. This is pursued in later sections. An important result obtained by Phuong, Sach and Yen is the following invexity characterization:

Theorem 2.1 *([23], p. 590)Let X be a nonempty subset of Ω. A locally Lipschitz function f is invex on X if and only if every Clarke stationary point of f over X is a global minimum.*

3 RELATIONS BETWEEN EFFICIENT SOLUTIONS OF VECTORIAL OPTIMIZATION PROBLEM AND VARIATIONAL-LIKE INEQUALITIES

Let X be a nonempty subset of $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., p$ given functions. We consider the following vectorial optimization problem:

$$\left.\begin{array}{l} \text{minimize } f(x) := (f_1(x), ..., f_p(x)) \\ \text{subject to: } x \in X. \end{array}\right\} \qquad \text{(P)}$$

The following notions of efficiency are so known:

Definition 3.1 • *$y \in X$ is a **efficient solution** of (P) if for each $x \in X$,*

$$(f_1(x) - f_1(y), ..., f_p(x) - f_p(y)) \notin -\mathbb{R}^p_+ \setminus \{0\};$$

• *$y \in X$ is a **property efficient solution** of (P) if $y \in X$ is a efficient* *solution of (P) and if there exists $M > 0$ such that for each $i = 1, ..., p$*

$$\frac{f_i(x) - f_i(y)}{f_j(y) - f_j(x)} \leq M$$

for some j such that $f_j(x) > f_j(y)$, when $f_i(x) < f_i(y)$ and $x \in X$;

- *$y \in X$ is a **weakly efficient solution** of (P) if for each $x \in X$,*

$$(f_1(x) - f_1(y), ..., f_p(x) - f_p(y)) \notin -int\mathbb{R}^p_+.$$

Now, we assume that f_i are locally Lipschitz and invex functions on X respect to η. We consider in this Section the following variational-like inequalities:

Minty type vectorial variational-like inequality $(MVLI)$

$(MVLI)$: To find $y \in X$ such that, for each $x \in X$ and any $\xi_i \in \partial f_i(x)$, $i = 1, ..., p$

$$(\xi_1^T \eta(x, y), ..., \xi_p^T \eta(x, y)) \notin \mathbb{R}^p_+ \setminus \{0\}$$

(where $\partial f_i(x)$ is the Clarke generalized gradient of f_i at x).

Stampacchia type vectorial variational-like inequality $(SVLI)$

$(SVLI)$ To find $y \in Y$ such that for each $x \in X$, there exist $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ such that

$$(\xi_1^T \eta(x, y), ..., \xi_p^T \eta(x, y)) \notin \mathbb{R}^p_+ \setminus \{0\}.$$

Proposition 3.2 *Assume that X is a nonempty subset of $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., p$ are locally Lipschitz functions and invex on X respect to η. If $y \in X$ is a weak efficient solution of (P), then, is a solution of $(MVLI)$.*

PROOF. Let $y \in X$ be a efficient solution of (P). Then, for each $x \in X$, we have

$$(f_1(x) - f_1(y), ..., f_p(x) - f_p(y)) \notin -\mathbb{R}^p_+ \setminus \{0\}. \tag{2}$$

Since f_i is invex respect to η, $f_i^0(x; \eta(y, x)) \leq f_i(y) - f_i(x)$ and, therefore

$$\xi_i^T \eta(y, x) \leq f_i^0(x; \eta(y, x)) \leq f_i(y) - f_i(x), \forall \xi_i \in \partial f_i(x). \tag{3}$$

From (2) and (3), we obtain that $y \in X$ is a solution of $(MVLI)$. ■

Theorem 3.3 *Let X be a nonempty subset of $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., p$ are invex locally Lipschitz functions on X respect to η. If $y \in X$ is solution of $(SVLI)$, then y is a efficient solution of (P).*

PROOF. Let $y \in X$ a solution of $(SVLI)$. Then, for each $x \in X$, there exist $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ such that

$$(\xi_1^T \eta(x, y), ..., \xi_p^T \eta(x, y)) \notin -\mathbb{R}_+^p \setminus \{0\}. \tag{4}$$

The functions f_i are invex and, for each $x \in X$,

$$f_i(x) - f_i(y) \geq f_i^0(y, \eta(x, y)) \geq \xi_i^T \eta(x, y) \tag{5}$$

so, from (4) and (5) we obtain:

$$(f_1(x) - f_1(y), ..., f_p(x) - f_p(y)) \notin -\mathbb{R}_+^p \setminus \{0\}$$

and, therefore, $y \in X$ is an efficient solution of (P). ■

Proposition 3.2 and Theorem 3.3 imply:

Corollary 3.4 *We assume that X is a nonempty subset of $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., p$ are invex locally Lipschitz functions on X with respect to η. If $y \in X$ is a solution of $(SVLI)$, then $y \in X$ is a solution of $(MVLI)$.*

Thus, (SVI) is a sufficient condition for the efficiency in (P). However, this condition is not necessary (see for example [18], p. 172). We will show that the proper efficiency in (P) is a necessary condition. To prove the last affirmation we will make use the following result (see [16]).

Lemma 3.5 *Let Γ be an arbitrary set, Y a Hausdorff vectorial space, D a compact subset of Y, $F : \Gamma \times D \longrightarrow \mathbb{R}$ a function such that $F(x, \cdot)$ is concave and upper semicontinuous on D for each $x \in \Gamma$ fix and $F(\cdot, y)$ is convex, for each $y \in D$ fix. Then,*

$$\inf_{x \in \Gamma} \max_{y \in D} F(x, y) \geq 0 \Leftrightarrow \sup_{y \in D} \inf_{x \in \Gamma} F(x, y) \geq 0.$$

Proposition 3.6 *Let X be a compact subset of $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., p$ invex locally Lipschitz functions respect to η on X. We assume that for each $y \in X$, the function $\eta(\cdot, y)$ is linear. Then, the following affirmations are equivalent:*

a) *$y \in X$ is a property efficient solution of (P).*

b) *There exist* $\lambda_i > 0$, $i = 1, ..., p$ *such that* y *is solution of the following scalar variational-like inequality: to find* $y \in X$ *such that there exist* $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ *such that for each* $x \in X$,

$$(\lambda_1 \xi_i + ... + \lambda_p \xi_p)^T \eta(x, y) \geq 0.$$

c) *There exist* $\lambda_i > 0$, $i = 1, ..., p$ *such that* y *is solution of the following scalar variational-like inequality: to find* $y \in X$ *such that for each* $x \in X$, *there exist* $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ *such that*

$$(\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x, y) \geq 0.$$

PROOF. a) $\Rightarrow$ b): If $y \in X$ is a property efficient solution of (P), since the functions f_i are invex and X is compact, we have (see [3]) there exist $\lambda_i > 0$, $i = 1, ..., p$ such that $y \in X$ is a solution of the scalar optimization problem:

$$\left.\begin{aligned} &\text{minimize } \lambda_1 f_1(x) + ... + \lambda_p f_p(x) \\ &\text{subject to: } x \in X. \end{aligned}\right\} \qquad \text{(SP)}$$

We observe that, the function $\lambda_1 f_1 + ... + \lambda_p f_p$ is invex and, by using Theorem 2.1, we obtain

$$0 \in \partial(\sum_{i=1}^{p} \lambda_i f_i)(y) + N_X(y) \subset \sum_{i=1}^{p} \lambda_i \partial f_i(y) + N_X(y) \qquad (7)$$

Then, there exist $\mu \in N_X(y)$ and $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ such that

$$0 = \mu + \sum_{i=1}^{p} \lambda_i \xi_i. \qquad (8)$$

By other hand, $\eta(x, y) \in T_X(y), \forall x \in X$ furthermore,

$$\langle \mu, \eta(x, y) \rangle \leq 0, \forall x \in X. \qquad (9)$$

From (8) and (9) follows $\sum_{i=1}^{p} \lambda_i \langle \xi_i, \eta(x, y) \rangle \geq 0, \forall x \in X$, that is,

$$(\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x, y) \geq 0$$

Hence, b) is proved.

b) $\Rightarrow$ a): We assume that there exist $y \in X$ such that $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ where for each $x \in X$, $(\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x, y) \geq 0$. We shall assume that y

is not a properly efficient solution of (P) and exhibit a contradiction. Then y is not a solution of the following scalar minimization problem (see [10]):

$$\begin{aligned} &\text{minimize } \lambda_1 f_1(x) + ... + \lambda_p f_p(x) \\ &\text{subject to: } x \in X. \end{aligned} \tag{10}$$

that is, there exists $x \in X$ such that $\sum_{i=1}^{p} \lambda_i f_i(x) < \sum_{i=1}^{p} \lambda_i f_i(y)$, and furthermore,

$$\sum_{i=1}^{p} \lambda_i (f_i(x) - f_i(y)) < 0. \tag{11}$$

By other hand, using the invexity of the functions f_i, we obtain

$$\sum_{i=1}^{p} \lambda_i (f_i(x) - f_i(y)) \geq \sum_{i=1}^{p} \lambda_i f_i^0(y, \eta(x,y)) \geq \sum_{i=1}^{p} \lambda_i \xi_i^T \eta(x,y) \geq 0, \tag{12}$$

which contradicts (11). Hence, y is properly efficient solution of (P).

b) $\Leftrightarrow$ c): We suppose that $y \in X$ and that there exist $\lambda_i > 0$ such that there are $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ such that, for each $x \in X$, $(\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x,y) \geq 0$. Or equivalently,

$$\max_{\xi_i \in \partial f_i(y)} \inf_{x \in X} (\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x,y) \geq 0. \tag{13}$$

Let

$$\begin{aligned} &D := \prod_{i=1}^{p} \partial f_i(y) \\ &\Gamma := X \\ &F(x,\xi) := (\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x,y). \end{aligned} \tag{14}$$

The function F holds the hypotheses of Lemma 3.5. In fact, for $x \in \Gamma$ fix, $F(x,\cdot)$ is continuous on D (because is a form linear define between finite-dimensional spaces and, in particular, is upper semicontinuous). Moreover, $F(x,\cdot)$ is simultaneously concave and convex. Consequently, (13) is equivalently to

$$\inf_{x \in X} \max_{\xi_i \in \partial f_i(y)} (\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x,y) \geq 0 \tag{15}$$

and this last inequality is exactly the statement b). Furthermore, b) and c) are equivalently. ■

Remark 3.7 *We observe that in the proof of the Proposition 3.6, we not use the fact that $\eta(\cdot, y)$ is linear for the la equivalence between a) and b).*

From Proposition 3.6 and Remark 3.7, it follows easily the following theorem:

Theorem 3.8 *Let X a compact, nonempty subset of $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., p$ invex locally Lipschitz functions on X respect to η. If y is a properly efficient solution of (P), then y is solution of $(SVLI)$.*

PROOF. If y is a properly efficient solution of (P), then using Proposition 3.6, there exist $\lambda_i > 0$, $\xi_i \in \partial f_i(y)$, $\forall i = 1, ..., p$ such that

$$(\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x, y) \geq 0, \ \forall x \in X. \tag{16}$$

We shall assume that exists $x \in X$ such that for each $\xi_i \in \partial f_i(y)$ holds

$$(\xi_1^T \eta(x, y), ..., \xi_p^T \eta(x, y)) \in -\mathbb{R}_+^p \setminus \{0\}$$

and exhibit a contradiction. Since, λ_i are all strictly positives we have

$$\sum_{i=1}^{p} (\lambda_i \xi_i)^T \eta(x, y) < 0,$$

which contradicts (16). Therefore, y is solution of $(SVLI)$. ■

4 RELATIONS BETWEEN WEAKLY EFFICIENT SOLUTIONS OF VECTORIAL OPTIMIZATION PROBLEM AND VARIATIONAL-LIKE INEQUALITIES

In this Section, we will consider variational-like vectorial inequalities of weak-Minty and weak-Stampacchia type, which we formulate as follows:

Weak-Minty variational-like inequality $(WMVLI)$**:**

$(WMVLI)$: To find $y \in X$ such that, for each $x \in X$ and each $\xi_i \in \partial f_i(x)$, $i = 1, ..., p$,

$$(\xi_1^T \eta(x, y), ..., \xi_p^T \eta(x, y)) \notin -\text{int}\mathbb{R}_+^p.$$

Weak-Stampacchia variational-like inequality $(WSVLI)$

$(WSVLI)$: To find $y \in X$ such that for each $x \in X$, there exist $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ such that

$$(\xi_1^T \eta(x,y), ..., \xi_p^T \eta(x,y)) \notin -\text{int}\mathbb{R}^p_+.$$

Under some hypotheses, it is possible to show that the solutions of $(WMVLI)$ and $(WSVLI)$ are coincident. This will be done now. Before, we will recall the following definition: Given the function $\eta : S \times S \longrightarrow \mathbb{R}^n$ where S is a nonempty subset of $\mathbb{R}^n$, we say that a set S is *invex* respect to η at $x \in S$ if for each $y \in S$ and each $t \in [0,1]$, $x + t\eta(y,x) \in S$. also, we say that S is invex, if it is invex for each $x \in S$.

Theorem 4.1 *Let X be a nonempty subset of $\mathbb{R}^n$, invex respect to η and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., p$ invex locally Lipschitz functions respect to η.*

(1) *If y is a solution of $(WMVLI)$. Then, y is a solution of $(WSVLI)$.*
(2) *We assume that the function η is anti-symmetric (i.e., $\eta(x,y) = -\eta(y,x), \forall x, y \in X$) and that $y \in X$ is a solution of $(WSVLI)$. Then y is a solution of $(WMVLI)$.*

PROOF. (1). We suppose that $y \in X$ is solution of $(WSVLI)$. Then, for each $x \in X$, there exist $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ and such that

$$(\xi_1^T \eta(x,y), ..., \xi_p^T \eta(x,y)) \notin -\text{int}\mathbb{R}^p_+.$$

Let $\widehat{\xi}_i \in \partial f_i(x)$, $i = 1, ..., p$. We claim that $(\xi_i - \widehat{\xi}_i)^T \eta(x,y) \leq 0, i = 1, ..., p$. In fact, from the anti-symmetry of η and (17), we have

$$(\xi_i - \widehat{\xi}_i)^T \eta(x,y) = \xi_i \eta(x,y) - \widehat{\xi}_i \eta(x,y) = \xi_i \eta(x,y) + \widehat{\xi}_i \eta(y,x) \qquad (17)$$

Also, since f_i is invex,

$$\xi_i^T \eta(x,y) \leq f_i^0(y, \eta(x,y)) \leq f_i(x) - f_i(y),\ \ i = 1, ..., p \qquad (18)$$

and, moreover,

$$\widehat{\xi}_i^T \eta(y,x) \leq f_i^0(x, \eta(y,x)) \leq f_i(y) - f_i(x),\ \ i = 1, ..., p \qquad (19)$$

and, adding (18)-(19), we obtain

$$(\xi_i - \widehat{\xi}_i)^T \eta(x,y) \leq 0, i = 1, ..., p$$

that is, for each $x \in X$ and each $\widehat{\xi}_i \in \partial f_i(x)$, $i = 1, ..., p$, we have

$$(\widehat{\xi}_1^T \eta(x,y), ..., \widehat{\xi}_p^T \eta(x,y)) \geq (\xi_1^T \eta(x,y), ...\xi_p^T \eta(x,y)) \notin -\text{int}\mathbb{R}_+^p \qquad (20)$$

consequently,

$$(\widehat{\xi}_1^T \eta(x,y), ..., \widehat{\xi}_p^T \eta(x,y)) \notin -\text{int}\mathbb{R}_+^p \qquad (21)$$

that is, y is solution of $(WMVLI)$.

We will prove the statement (2). To done this, we suppose that $y \in X$ is solution of $(WMVLI)$. In this case, for each $x \in X$ and each $\xi_i \in \partial f_i(x)$, $i = 1, ..., p$, $(\xi_1^T \eta(x,y), ..., \xi_p^T \eta(x,y)) \notin -\text{int}\mathbb{R}_+^p$. For $z \in X$ fix, we consider the sequence $(\alpha_k) \in (0,1]$, with $\alpha_k \longrightarrow 0$ when $k \longrightarrow \infty$ and we define $z_k := y + \alpha_k \eta(z,y)$. Since X is invex respect to η, then the sequence (z_k) belong to X. The set $\partial f_i(z_k)$ is nonempty and therefore we can to take $\xi_i^k \in \partial f_i(z_k)$, $i = 1, ..., p$ for each $k \in \mathbb{N}$. But y is a solution of $(WMVLI)$ and therefore,

$$(\xi_1^{kT} \eta(x,y), ..., \xi_p^{kT} \eta(x,y)) \notin -\text{int}\mathbb{R}_+^p. \qquad (22)$$

We can assume that all the functions f_i have the same Lipschitz constant K. Since $\xi_i^k \in \partial f_i(z_k)$, for each k, we have $\left\|\xi_i^k\right\| \leq K$, for $i = 1, ..., p$. For each i, (ξ_i^k) is a bounded subsequence in $\mathbb{R}^n$, and, without loss of generality, we can suppose that $\xi_i^k \longrightarrow \widehat{\xi}_i$, $i = 1, ..., p$ when $k \longrightarrow \infty$. By other hand, $\xi_i^k \in \partial f_i(z_k)$ for each k and $z_k \longrightarrow y$ when $k \longrightarrow \infty$ and since the set-valued mapping ∂f_i is closed (see [7] p. 29), we obtain $\widehat{\xi}_i \in \partial f_i(y)$, $i = 1, ..., p$. Taking $k \longrightarrow \infty$ in (22) and observing that the set $(-\text{int}\mathbb{R}_+^p)^c$ is closed in $\mathbb{R}^p$, we obtain

$$(\widehat{\xi}_i^T \eta(x,y), ..., \widehat{\xi}_p^T \eta(x,y)) \notin -\text{int}\mathbb{R}_+^p. \qquad (23)$$

Therefore, $y \in X$ is solution of $(WSVLI)$. ■

Theorem 4.2 *Let X a nonempty subset of $\mathbb{R}^n$, invex respect to η and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., p$ invex locally Lipschitz functions respect to η on X. We suppose that η is anti-symmetric. Then, $y \in X$ is a weakly solution of (P) iff $y \in X$ is solution of $(WSVLI)$.*

PROOF. Firstly, we suppose that $y \in X$ is not weakly efficient solution of (P). Then, there exists $z \in X$ such that

$$f_i(y) > f_i(z), \; i = 1, ..., p. \qquad (24)$$

Let $\xi_i \in \partial f_i(y), i = 1, .., p$. Since f_i are invex, we have

$$\xi_i^T \eta(z, y) \leq f_i^0(y, \eta(z, y)) \leq f_i(z) - f_i(y) \tag{25}$$

therefore,

$$f_i(z) - f_i(y) \geq \xi_i^T \eta(z, y),\ \forall \xi_i \in \partial f_i(y) \tag{26}$$

and, from (24) and (26), we have

$$\xi_i \in \partial f_i(y), (\xi_i^T \eta(z, y), ..., \xi_p^T \eta(z, y)) \in -\text{int}\mathbb{R}_+^p \tag{27}$$

for each $z \in X$ and, consequently, $y \in X$ is not solution of $(WSVLI)$.

Now, we suppose that $y \in X$ is not solution of $(WSVLI)$. In this case, there exists $\overline{x} \in X$ such that, for each $\xi_i \in \partial f_i(\overline{x})$, $i = 1, ..., p$

$$(\xi_1^T \eta(\overline{x}, y), ..., \xi_p^T \eta(\overline{x}, y)) \in -\text{int}\mathbb{R}_+^p. \tag{28}$$

Let $\xi_i \in \partial f_i(\overline{x})$. Since f_i are invex,

$$f_i(y) - f_i(\overline{x}) \geq f_i^0(\overline{x}; \eta(y, \overline{x})) \geq \xi_i^T \eta(y, \overline{x}),\ i = 1, ..., p \tag{29}$$

consequently,

$$\xi_i^T \eta(y, \overline{x}) \leq f_i(y) - f_i(\overline{x}),\ i = 1, .., p. \tag{30}$$

The function η is anti-symmetric and from (30) follows

$$\xi_i^T \eta(\overline{x}, y) \geq f_i(\overline{x}) - f_i(y),\ i = 1, .., p \tag{31}$$

From (28) and (31), we obtain

$$0 > f_i(\overline{x}) - f_i(y),\ i = 1, ..., p \tag{32}$$

and, therefore, $y \in X$ is not a weakly efficient solution of (P). ■

The following result is an alternative theorem for invex functions, the proof can be to see in [3]:

Lemma 4.3 *(Invex Gordan's Theorem) Let C be a nonempty closed subset of Ω. Suppose that $f_i : \Omega \to \mathbb{R}$, $i \in I = \{1, \ldots, m\}$ are locally Lipschitz* *functions and invex on C, for a common η. If $f(x) = \max\{f_i(x)\ i \in I\}$ attains a minimum on C, then either*

(i) *there exists* $x \in C$ *such that* $f_i(x) < 0 \quad i \in I$, *or*
(ii) $\exists \lambda_i \geq 0, \quad i \in I,$ *not all zero, such that* $\sum_{i\in I} \lambda_i f_i(x) \geq 0 \quad \forall x \in C$, *but never both.*

Now, we give an analogous result of the Proposition 3.6 for weakly efficient solutions.

Proposition 4.4 *Let X be a compact, nonempty subset of $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ invex locally Lipschitz functions respect to η on X. We assume that for each $y \in X$ the function $\eta(\cdot, y)$ is linear. Then, are equivalently:*

(a) *$y \in X$ is a weakly efficient solution of (P).*
(b) *There exist $\lambda_i \geq 0$, $i = 1, ..., p$ not all zeros such that $y \in X$ is a solution of the following scalar variational inequality: to find $y \in X$ such that there exist $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$such that, for each $x \in X$*

$$(\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x, y) \geq 0.$$

(c) *There exist $\lambda_i \geq 0$, $i = 1, ..., p$, not all zero and such that $y \in X$ is a solution of the following scalar variational inequality: to find $y \in X$ such that for each $x \in X$, there exist $\xi_i \in \partial f_i(y)$ such that*

$$(\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x, y) \geq 0.$$

PROOF. The equivalence between (b) and (c) is proved analogously as in Proposition 3.6.

(a) $\Rightarrow$ (b): We assume that $y \in X$ is a weakly efficient solution of (P). Since f_i are invex, the functions ϕ_i,

$$\phi_i(x) = f_i(x) - f_i(y)\ (x \in X)$$

are invex respect to η on X. By hypotheses, there not exists $x \in X$ such that $\phi_i(x) < 0$, $i = 1, .., p$. Consequently, from Lemma 4.3 there exist $\overline{\lambda}_i \geq 0$, $i = 1, ..., p$ not all zero and such that $\sum_{i=1}^{p} \lambda_i \phi_i(x) \geq 0, \forall x \in X$. Or, equivalently, x is solution of the scalar problem (SP)

$$\left.\begin{array}{l} \text{minimize } \sum_{i=1}^{p} \lambda_i f_i(x) \\ \text{subject to: } x \in X \end{array}\right\} \qquad \text{(SP)}$$

and therefore,

$$0 \in \partial(\sum_{i=1}^{p} \lambda_i f_i)(y) + N_X(y).$$

Thus, there exist $\mu \in N_X(y)$, $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ such that

$$\mu + \sum_{i=1}^{p} \lambda_i \xi_i.$$

Since $\eta(x, y) \in T_X(y), \forall x \in X$, we obtain

$$0 \geq \langle \mu, \eta(x, y) \rangle = -\sum_{i=1}^{p} \lambda_i \langle \xi_i, \eta(x, y) \rangle, \ \forall x \in X$$

that is,

$$(\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x, y), \forall x \in X.$$

That is exactly the affirmation given in (b).

Now, (b) $\Rightarrow$ (a): We assume that there exist $\lambda_i \geq 0$, $i = 1, ..., p$ not all zero, $\xi_i \in \partial f_i(y)$ such that

$$(\lambda_1 \xi_i + ... \lambda_p \xi_p)^T \eta(x, y) \geq 0, \forall x \in X. \tag{34}$$

If y no is a weakly efficient, then y is not solution of the pondered scalar problem associate to λ, that is, there exists $x \in X$ such that $\sum_{i=1}^{p} \lambda_i (f_i(x) - f_i(y)) < 0$. By other hand,

$$\sum_{i=1}^{p} \lambda_i (f_i(x) - f_i(y)) \geq \sum_{i=1}^{p} \lambda_i f_i^0(y, \eta(x, y)) \geq \sum_{i=1}^{p} \lambda_i \xi_i^T \eta(x, y) \tag{35}$$
$$= (\lambda_1 \xi_i + ... \lambda_p \xi_p)^T \eta(x, y) \geq 0 \tag{36}$$

this contradict (34). ■

Easily, from Theorems 4.1, 4.2 and Proposition 4.4, we obtain:

Theorem 4.5 *Let X be a compact, nonempty subset of $\mathbb{R}^n$, invex respect to η and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., p$ are invex locally Lipschitz functions respect to η*

on X. We assume that the function η is anti-symmetric and such that $\eta(\cdot, y)$ is linear for each $y \in Y$. Then, the following affirmations are equivalently :

(a) *$y \in X$ is a weakly efficient solution of (P).*
(b) *$y \in X$ is solution of $(WMVLI)$.*
(c) *$y \in X$ is solution of $(WSVLI)$.*
(d) *There exist $\lambda_i \geq 0$, $i = 1, ..., p$, not all zero and such that $y \in X$ and is solution of the following scalar inequality: to find $y \in X$ such that there exist $\xi_i \in \partial f_i(y)$, $i = 1, ..., p$ such that, for each $x \in X$*

$$(\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x, y) \geq 0.$$

(e) *There exist $\lambda_i \geq 0$, $i = 1, ..., p$, not all zero and such that $y \in X$ is a solution of the following scalar inequality $y \in X$ such that for each $x \in X$, there exist $\xi_i \in \partial f_i(y)$such that*

$$(\lambda_1 \xi_1 + ... + \lambda_p \xi_p)^T \eta(x, y) \geq 0.$$

5 EXISTENCE OF WEAKLY EFFICIENT SOLUTIONS

We begin this Section recalling a fixed point theorem for set-valued mappings, which is a generalization of the classical fixed point theorem of Fan- Browder, and whose proof can be to find in Park [22].

Lemma 5.1 *Let X be a convex, nonempty subset of a Hausdorff topological vectorial space E and K a compact, nonempty subset of X. Let $A, B : X \rightrightarrows X$ be two set-valued mappings satisfying the following conditions:*

(1) *$Ax \subset Bx$, $\forall x \in X$.*
(2) *Bx is a convex set, $\forall x \in X$.*
(3) *$Ax \neq \varnothing$, $\forall x \in K$.*
(4) *$A^{-1}y$ is an open set, $\forall y \in X$.*
(5) *For each N, finite subset of$X$, there exist L_N compact, convex, nonempty subset of X such that $L_N \supseteq N$ and for each $x \in L_N \setminus K$, $Ax \cap L_N \neq \varnothing$.*

Then, there exists $\overline{x} \in X$ such that $\overline{x} \in B\overline{x}$.

We will use the results of the previous section, together with Lemma 5.1 to establish the existence of weakly efficient solution for the nonsmooth invex vectorial optimization problem under weak compactness hypothesis on the feasible set X.

Theorem 5.2 *Let X be a nonempty subset of $\mathbb{R}^n$, invex respect to η and $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$, $i = 1, ..., p$ invex locally Lipschitz functions respect to η. We* *assume that η is an anti-symmetric function and such that $\eta(\cdot, y)$ is convex and continuous, for each $y \in X$. Also, we assume that there exists a compact, nonempty subset, K, of X such that, for each finite subset N of X there exists a compact, convex and nonempty subset, L_N, of X, such that $L_N \supseteq N$ and for each $x \in L_N \setminus K$, there exists $z \in L_N$ such that there exist $\xi_i \in \partial f_i(z)$, $i = 1, ..., p$ satisfying*

$$(\xi_1^T \eta(z,x), ..., \xi_p^T \eta(z,x)) \in -int\mathbb{R}^p_+.$$

Then, (P) has a weakly efficient solution.

PROOF. We wrote the proof using a concise notation to make it clearer. We denote by $\partial f(x)$ the set $\partial f_1(x) \times ... \times \partial f_p(x)$ $(x \in X)$. Let $s = (s_1, ..., s_p)$ where $s_i \in \mathbb{R}^n$, $i = 1, ..., p$. We denote by $s^T \eta(x,y)$ the vector $(s_1^T \eta(x,y), ..., s_p^T \eta(x,y)) \in \mathbb{R}^p$. Let $A, B : X \rightrightarrows X$ be two set-valued mappings given by:

$$Ax := \{z \in X : \exists t \in \partial f(z),\ t^T \eta(z,x) \in -\text{int}\mathbb{R}^p_+\}$$
$$Bx := \{z \in X : \forall s \in \partial f(x),\ s^T \eta(z,x) \in -\text{int}\mathbb{R}^p_+\}.$$

We will prove (using Lemma 5.1) that there exists $y \in K$ such that $Ay = \varnothing$ or, equivalently, that y is solution of $(WMVLI)$ and by Theorems 4.1 and 4.2 it is sufficient to prove our result. In first time, we will prove that the set-valued mappings A and B satisfy the conditions (1), (2), (4) and (5) of the Lemma 5.1 and that B has not fixed point. So, Lemma 5.1 will be implies the existence of $y \in K$ such that $Ay = \varnothing$. We will show that the condition (1) of the Lemma 5.1 holds: Let $x \in X$ and $z \in Ax$. Then, there exist $t = (\xi_1, ..., \xi_p) \in \partial f(z)$ such that

$$(\xi_1^T \eta(z,x), ..., \xi_p^T \eta(z,x)) \in -\text{int}\mathbb{R}^p_+. \tag{37}$$

Let $s = (\widehat{\xi}_1, ..., \widehat{\xi}_p) \in \partial f(x)$. Using the invexity of functions f_i and the anti-symmetry of η, we have for each $i = 1, ..., p$

$$\widehat{\xi}_i^T \eta(z,x) \leq f_i^0(x; \eta(z,x)) \leq f_i(z) - f_i(x) = -(f_i(x) - f_i(z)) \tag{38}$$
$$\leq -\xi_i^T \eta(x,z) = \xi_i^T \eta(z,x). \tag{39}$$

From (37) and (38) follows

$$(\widehat{\xi}_1^T \eta(z,x), ..., \widehat{\xi}_p^T \eta(z,x)) \in -\text{int}\mathbb{R}^p_+$$

and, therefore, $z \in Bx$. Now, we will see that the second condition of Lemma 5.1 holds: let $x \in X$, $z_1, z_2 \in Bx$ and $\lambda \in [0,1]$. Then, for each $s = (\xi_1, ..., \xi_p) \in \partial f(x)$, we have

$$(\xi_1^T \eta(z_1, x), ..., \xi_p^T \eta(z_1, x)), (\xi_1^T \eta(z_2, x), ..., \xi_p^T \eta(z_2, x)) \in -\text{int}\mathbb{R}_+^p. \quad (40)$$

For each $j = 1, ..., p$, we consider $\xi_j = (\xi_j^{(1)}, ..., \xi_j^{(n)})$, $\xi_j^{(k)} \in \mathbb{R}$, $\eta(x,y) = (\eta_1(x,y), ..., \eta_n(x,k))$, $\eta_i(x,y) \in \mathbb{R}$. Then, from the convexity of η_i and (40), we obtain

$$\begin{aligned}
\xi_j^T \eta(\lambda z_1 + (1-\lambda) z_2, x) &= \sum_{i=1}^n \xi_j^{(i)} \eta_i(\lambda z_1 + (1-\lambda) z_2, x) && (41)\\
&\leq \sum_{i=1}^n \xi_j^{(i)} [\lambda \eta_i(z_1, x) + (1-\lambda) \eta_i(z_2, x)] && (42)\\
&= \lambda \xi_j^T \eta(z_1, x) + (1-\lambda) \xi_j^T \eta(z_2, x) < 0, \ j = 1, ..., p && (43)
\end{aligned}$$

Furthermore, $\lambda z_1 + (1-\lambda) z_2 \in Bx$. The fourth condition of Lemma 5.1 is proved as follows: We prove that, for each $z \in X$, the set $(A^{-1}z)^c$ is closed. To done this, we consider a sequence $(x_n) \in (A^{-1}z)^c$ and such that x_n converge to x. Then, $x_n \notin A^{-1}z$ that is, $z \in Ax_n$, $\forall n \in N$. Let $t = (\xi_1, ..., \xi_p) \in \partial f(z)$ such that

$$(\xi_1^T \eta(z, x_n), ..., \xi_p^T \eta(z, x_n)) \notin -\text{int}\mathbb{R}_+^p. \quad (44)$$

Since $\eta(\cdot, z)$ is continuous and anti-symmetric, we have that $\eta(z, \cdot)$ also is continuous and being $(-\text{int}\mathbb{R}_+^p)^c$ closed, making $n \longrightarrow \infty$ in (44)) we obtain

$$(\xi_1^T \eta(z, x), ..., \xi_p^T \eta(z, x)) \notin -\text{int}\mathbb{R}_+^p.$$

and, furthermore, $x \in (A^{-1}z)^c$. By our hypotheses, the condition (5) of the Lemma 5.1 holds. However, B has not fixed point, because if there exist a point fixed, it would exist $x \in X$ such that for each $s \in \partial f(x)$, $s^T \eta(x,x) = 0 \in -\text{int}\mathbb{R}^p$, which is absurd. Consequently, from Lemma 5.1, there exists $y \in K$ such that $Ay = \varnothing$. ■

Corollary 5.3 *Let X be a nonempty subset of $\mathbb{R}^n$, invex respect to η and η anti-symmetric such that $\eta(\cdot, y)$ is convex and continuous. We assume that*

$$K = \{x \in X : (f_1^0(z_0, \eta(z_0, x)), ..., f_p^0(z_0, \eta(z_0, x)) \notin -int\mathbb{R}_+^n\}$$

is compact for some $z_0 \in X$. Then, (P) has weakly efficient solutions.

PROOF. Let N be a nonempty and finite subset of X. We define: $L_N := \overline{\text{co}}(N \cup K)$ (where $\overline{\text{co}}A$ denote the closed convex hull of A). Then, for each $x \in L_N \setminus K \subset X \setminus K$, we have

$$(f_1^0(z_0, \eta(z_0, x)), ..., f_p^0(z_0, \eta(z_0, x))) \in -\text{int}\mathbb{R}_+^p. \tag{45}$$

Let $z := z_0 \in K \subset L_N$, $\xi_i \in \partial f_i(z)$. We have:

$$\xi_i^T \eta(z_0, x) \leq f_i^0(z_0, \eta(z_0, x)), \ i = 1, ..., p \tag{46}$$

and, from (45) and (46), we obtain

$$(\xi_1^T \eta(z, x), ..., \xi_p^T \eta(z, x)) \in -\text{int}\mathbb{R}_+^p.$$

Thus, the hypotheses of Theorem 5.2 are verified and consequently (P) has weakly efficient solutions. ■

CONCLUSIONS: In this paper, we study the equivalence between solutions of the vectorial variational-like inequalities of Minty and Stampacchia type and the efficient solutions and weakly efficient solutions of the nonsmooth, invex, vectorial optimization problem. The results were obtained by using an approach analogous those used by Gianessi [12] . Also, these results are a generalization those proved by Lee [18].

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