Tensor product theorems in positive characteristic

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Abstract

In this paper we study tensor products of T-prime T-ideals over infinite fields. The behaviour of these tensor products over a field of characteristic 0 was described by Kemer. First we show, using methods due to Regev, that such a description holds if one restricts oneself to multilinear polynomials only. Second, applying graded polynomial identities, we prove that the tensor product theorem fails for the T-ideals of the algebras $M_{1,1}(E)$ and $E \otimes E$ where E is the infinite dimensional Grassmann algebra; $M_{1,1}(E)$ consists of the 2×2 matrices over E having even (i.e. central) elements of E, and the other diagonal consisting of odd (anticommuting) elements of E. Then we pass to other tensor products and study the respective graded identities. We obtain new proofs of some cases of Kemer's tensor product theorem. Note that these proofs do not depend on the structure theory of T-ideals but are "elementary" ones. Finally, using graded identities once again, we show that the tensor product theorem fails in one more case when the base field is of positive characteristic. All this comes to show once more that the structure theory of T-ideals is essentially about the multilinear polynomial identities.

1 Introduction

The T-prime algebras form one of the most important classes of algebras in the PI theory. The structure theory for the T-ideals in the free associative algebra depends heavily on the identities satisfied by such algebras. This theory was developed by Kemer, see for example [11] for detailed account of this theory.

We recall some of the main definitions and notation that will be used in what follows. All prerequisites needed to follow the exposition will be given in Section 2; here we state only the necessary ones for consistency of the text. Throughout K is a fixed infinite field, $\operatorname{char} K \neq 2$; the algebras, vector space, tensor products and so on, are over K. Unless otherwise stated we consider associative and unitary algebras. If $X = \{x_1, x_2, \ldots\}$ is an infinite (countable) set we denote by $K(X)$ the free associative algebra that is freely generated over K by X. The elements of $K(X)$ are called polynomials. If A is an algebra the polynomial $f \in K(X)$ is a polynomial identity (or simply an identity) for A if $f(a_1, a_2, \ldots, a_n) = 0$ for all $a_i \in A$. The set of all identities $T(A)$ for A is an ideal of $K(X)$. This ideal is closed with respect to the endomorphisms of the algebra $K(X)$ and is called the T-ideal of A. It is easy to show that every such ideal is the T-ideal of some algebra. The variety of algebras $\operatorname{var} A = \operatorname{var} T(A)$ defined by A (or by $T(A)$) is the class of all algebras that satisfy the identities in $T(A)$.

The algebra A is T-prime (or verbally prime) if $T(A)$ is T-prime, that is $T(A)$ is prime in the class of all T-ideals of $K(X)$. This means that if I and J are T-ideals such that $IJ \subseteq T(A)$ then $I \subseteq T(A)$ or $J \subseteq T(A)$. Equivalently, $T(A)$ is T-prime if f, $g \in K(X)$, and $f(x_1, \ldots, x_m)g(x_{m+1}, \ldots, x_n) \in T(A)$ imply $f \in T(A)$ or $g \in T(A)$ (or both). Let E be the infinite dimensional Grassmann (or exterior) algebra of a vector space V. One chooses a basis e_1, e_2, $\ldots$ of V and then 1 and the monomials $e_{i_1}e_{i_2}\ldots e_{i_k}$, $i_1 < i_2 < \ldots < i_k$, $k \geq 1$ form a basis of E, and the multiplication in E is the one induced by $e_ie_j = -e_je_i$, $e_i^2 = 0$. Denote by E_i the span of all such monomials with $k \equiv i \pmod 2$, $i = 0$, 1. Then E_0 is the centre of E, the elements of E_1 anticommute and $E = E_0 \oplus E_1$ as vector spaces.

It follows from Kemer's theory that the only nontrivial T-prime T-ideals in $K(X)$, $\operatorname{char} K = 0$, are $T(M_n(K))$ for $M_n(K)$ being the matrix algebra of order $n \geq 1$; $T(M_n(E))$, $n \geq 1$, and $T(M_{a,b}(E))$, $a \geq b \geq 1$. Here $M_n(E)$ is the algebra of $n \times n$ matrices over E. The algebra $M_{a,b}(E)$ is the subalgebra of $M_{a+b}(E)$ that consists of the block matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Here $A \in M_a(E_0)$, $D \in M_b(E_0)$, and B and C are matrices $a \times b$ and $b \times a$ respectively, whose entries belong to E_1. Denote as Δ_0 the set of (i,j) such that $1 \leq i, j \leq a$ or $a+1 \leq i, j \leq a+b = n$, and as Δ_1 the set of (i,j) with $1 \leq i \leq a$, $a+1 \leq j \leq a+b$, or $1 \leq j \leq a$, $a+1 \leq i \leq a+b$. Then $M_{a,b}(E)$ consists of the matrices in $M_n(E)$ such that the (i,j)-th entry belongs to E_β when $(i,j) \in \Delta_\beta$. An important corollary of Kemer's theory is the Tensor Product Theorem (TPT).

Theorem 1 *Let* $\operatorname{char} K = 0$. *Then*

- $T(M_{a,b}(E) \otimes E) = T(M_{a+b}(E))$;
- $T(M_{a,b}(E) \otimes M_{c,d}(E)) = T(M_{ac+bd,ad+bc}(E))$;

- $T(M_{1,1}(E)) = T(E \otimes E)$.

An alternative proof of Theorem 1 was given by Regev in [18]. Regev's proof did not depend on the structure theory. Instead Regev used essentially graded polynomial identities. Some cases of the TPT were proved, again without using the structure theory of T-ideals, in [6, 7, 8, 13].

Studying the identities satisfied by T-prime algebras is, without any doubt, very important. But it is extremely difficult as well. Note that these identities are known in very few cases only, and mainly when $\operatorname{char} K = 0$. Thus $T(E)$ is described in any characteristic (even over finite fields), see [14, 4, 22]. Further on, $T(M_2(K))$ was described in [16], see also [17], and in a series of papers by Drensky, see for example [9, 10]. When $\operatorname{char} K = p > 2$ the same was done in [12, 5]. Add the description of $T(E \otimes E)$ given in [15] when $\operatorname{char} K = 0$ and in this way one gets the complete list. Hence one is led to study other types of polynomial identities such as weak identities, identities with involution, and graded ones. Naturally graded identities have attracted the attention not only as an object of independent study but also due to the broad spectrum of applications and close connection to the ordinary ones. We refer to [2, 13] and their bibliography for further reference.

In this paper we study graded identities in T-prime algebras with appropriate gradings aiming at obtaining information about the ordinary identities satisfied by them. The paper is organized as follows. In Section 3 we show that the Tensor Product Theorem holds for the multilinear parts of the corresponding T-ideals when $\operatorname{char} K = p > 2$. This is done using the methods developed by Regev in [18]. Then we prove that $T(M_{1,1}(E)) \subset T(E \otimes E)$, a proper inclusion. In order to do it we use generic constructions and some methods of [13]. We give a polynomial that is an identity for $E \otimes E$ but is not for $M_{1,1}(E)$. Of course such polynomial depends on the characteristic p of the base field. In Sections 4 and 5 we consider the G-graded identities of the algebras $M_n(E)$ and $M_{a,b}(E) \otimes E$ where $a + b = n$ and $G = \mathbb{Z}_n \times \mathbb{Z}_2$. We exhibit bases of these graded identities and show that the latter algebra satisfies some graded identities that do not hold for the former when $\operatorname{char} K = p > 2$. It is interesting to note that the difference between the respective graded T-ideals is in monomials only. In other words $M_{a,b}(E) \otimes E$ satisfies some monomial identities that do not hold for $M_n(E)$. In the last Section 6 we study in detail the case $n = 2$, $a = b = 1$. We give finite bases of the respective graded identities, and find an ordinary identity that holds for $M_{1,1}(E) \otimes E$ but does not hold for $M_2(E)$. In this case the difference between the two graded T-ideals is one monomial given in explicit form. In order to achieve these goals we use once again appropriate models for the corresponding relatively free algebras. Furthermore we make use of methods introduced in [20, 21], and further developed in [1].

We believe that the results in this paper contribute to the better understanding of the nature of the T-ideals in positive characteristic, a task whose complete solution seems somewhat beyond our knowledge at the present stage.

2 Preliminaries

We shall adopt all the notation introduced in Section 1. We repeat that K is an infinite field, $\operatorname{char} K = p \neq 2$. Unless otherwise stated all algebras A are over K, and unitary, $1 \in A$. If f, $g \in K\langle X\rangle$ the polynomial g is a consequence of f as an identity if $g \in \langle f\rangle^T$, the T-ideal generated by f, and f and g are equivalent as identities if $\langle f\rangle^T = \langle g\rangle^T$.

Let I be a T-ideal in $K\langle X\rangle$. The quotient algebra $F = K\langle X\rangle/I$ is the relatively free algebra in the variety determined by I. Denote by P_n the vector subspace of $K\langle X\rangle$ of all multilinear polynomials in x_1, x_2, ..., x_n of degree n. It is well known that when $\operatorname{char} K = 0$, the T-ideal I is generated as a T-ideal, by the set $\cup_{n\geq 1}(T \cap P_n)$. That is I is generated by its multilinear polynomials. When the field is infinite this may fail. But in this case I is generated by its multihomogeneous elements. (We refer to the first chapters of [10] for the proofs of these and of the next couple of basic facts about identities and T-ideals.) We can make a further reduction of the class of polynomials that determine a T-ideal. Let $L(X)$ be the free Lie algebra freely generated by the set X over K. If we consider $K\langle X\rangle$ as a Lie algebra with respect to the commutator $[a, b] = ab - ba$, then $L(X)$ is the subalgebra of $K\langle X\rangle$ generated by the set X. Fix an ordered basis of $L(X)$ that consists of X and of multihomogeneous elements u_1, u_2, ... of degree ≥ 2. Suppose that the variables of X are the first ones in the order. Then $K\langle X\rangle$ is the universal enveloping algebra of $L(X)$, and we may choose a basis of $K\langle X\rangle$ consisting of 1 and the monomials $x_{i_1}^{n_1} \ldots x_{i_k}^{n_k} u_{j_1} \ldots u_{j_m}$, $i_1 < \cdots < i_k$, $j_1 \leq \cdots \leq j_m$. Denote as $B(X)$ the (associative) subalgebra of $K\langle X\rangle$ generated by 1 and by all u_i. It is spanned by the products of commutators, and its elements are called proper (or commutator) polynomials. A well known theorem states that over an infinite field K, every T-ideal I is generated by $I \cap B(X)$. In other words I is generated by its proper elements. The proof of this statement can be found in [10, pp. 42–46].

Let A be an algebra and G an additive abelian group (or semigroup). Assume that $A = \oplus_{g\in G} A_g$ is a direct sum of vector subspaces and that $A_g A_h \subseteq A_{g+h}$, g, $h \in G$. Then A is G-graded. If G is the cyclic group of order n we speak about n-graded algebras. Let $X = \cup_{g\in G} X_g$ be a disjoint union of infinite sets. The free associative algebra $K\langle X\rangle$ is G-graded in a standard way. If $x \in X_g$ is a variable then $w(x) = g$ is its weight, or (homogeneous) degree. If $M = x_{i_1} \ldots x_{i_n}$ is a monomial then $w(M) = w(x_{i_1}) + \cdots + w(x_{i_n})$. Observe that the last sum is the one in G. The polynomial $f \in K\langle X\rangle$ is G-graded identity for the G-graded algebra A if $f = 0$ whenever the variables of X_g are substituted for elements of the component A_g of A. In other words if $\varphi\colon K\langle X\rangle \to A$ is a graded homomorphism then $f \in \ker\varphi$.

The Grassmann algebra $E = E_0 \oplus E_1$ is 2-graded. Let $X = Y \cup Z$, $Y \cap Z = \emptyset$ for $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$ being the even and odd variables. The polynomial $[y_1, y_2]$ is a graded identity of E, $z_1 \circ z_2$ and $[y_1, z_1]$ are other two. Here and in the sequel we denote by $[a, b] = ab - ba$ the commutator of a and b, and $a \circ b = (ab + ba)/2$. We suppose that the commutators are left normed that is $[a, b, c] = [[a, b], c]$ and so on.

One defines G-graded: T-ideals, varieties, relatively free algebras, consequences, and so on, in analogy with the case of ordinary identities. If A is G-graded then a variation of the proper polynomials is known. Since $1 \in A_0$, then every graded identity is equivalent to a finite collection of 0-proper graded identities. These are graded polynomials such that every 0-graded variable is in commutators only. The proof of this assertion repeats verbatim the one for ordinary identities, see for example [13] for a proof. We shall call such polynomials simply proper polynomials if there is no possibility of misunderstanding.

The following two facts will be useful later on.

Lemma 2 *Let A and B be two G-graded algebras with respective G-graded T-ideals $T_G(A)$ and $T_G(B)$. Then if $T_G(A) \subseteq T_G(B)$ then $T(A) \subseteq T(B)$.*

Corollary 3 *If $T_G(A) = T_G(B)$ then $T(A) = T(B)$.*

The proofs of these two assertions are obvious and left to the reader.

The generic matrix algebra $Gen(M_n(K))$ is a convenient tool in studying the identities satisfied by the matrices $n \times n$ over a field. Recall that if x_{ij}^k are commuting variables then $Gen(M_n(K))$ is the subalgebra of $M_n(K[x_{ij}^k])$ generated by the matrices $X_k = (x_{ij}^k)$. A well known result states that $Gen(M_n(K))$ is relatively free in the variety generated by $M_n(K)$.

Let $Y = \{y_1, y_2, \dots\}$ and $Z = \{z_1, z_2, \dots\}$ be variables and set $X = Y \cup Z$. Define a $\mathbb{Z}_2$-grading on $K(X)$ assuming Y even and Z odd variables. Let T be the ideal in $K(X)$ generated by all $ab - (-1)^{w(a)w(b)}ba$, a and b homogeneous monomials, and set $\Omega = K(X)/T$. Recall that $w(a)$ stands for the homogeneous degree of a, $w(a) = 0$ or 1. We shall refer to homogeneous polynomials and so on when considering homogeneity with respect to the grading, and we leave the adjective multihomogeneous for homogeneity with respect to the usual (multi-)degree of polynomials.

The algebra Ω is the free supercommutative algebra. In [3, Section 2] it was proved that $\Omega \cong K[X] \otimes_K E(Y)$ where $K[X]$ are the commutative polynomials in X and $E(Y)$ is the Grassmann algebra of the span of Y. Suppose that $Y = \{y_{ij}^k\}$, $Z = \{z_{ij}^k\}$, and set $B_k = (y_{ij}^k + z_{ij}^k)$, $C_k = (c_{ij}^k)$ where $c_{ij}^k = y_{ij}^k$ if $1 \le i, j \le a$ or $a+1 \le i, j \le a+b = n$, and $c_{ij}^k = z_{ij}^k$ otherwise. The following theorem was proved in [3, Theorem 2].

Theorem 4 *a) The matrices B_1, B_2, ..., generate a subalgebra of $M_n(\Omega)$ that is relatively free algebra in the variety generated by $M_n(E)$.*

b) The matrices C_1, C_2, ..., generate a subalgebra of $M_n(\Omega)$ that is relatively free in the variety $var\, M_{a,b}(E)$.

3 The multilinear TPT; TPT fails for $M_{1,1}(E)$ and $E \otimes E$

We start with the Tensor product theorem and its multilinear version. If I is a T-ideal in $K(X)$ we denote by $P(I)$ the set of all multilinear polynomials in I. One of the main results in this section is the following theorem.

Theorem 5 (cf. [18]) *Let K be an infinite field, $char\, K \neq 2$. Then*

1. $P(T(M_{a,b}(E) \otimes E)) = P(T(M_{a+b}(E)))$;

2. $P(T(M_{a,b}(E) \otimes M_{c,d}(E))) = P(T(M_{ac+bd,ad+bc}(E)))$;

3. $P(T(M_{1,1}(E))) = P(T(E \otimes E))$.

Proof. This theorem is the main result of Regev's paper [18]. The proof is the same as in [18]. The assumption $\operatorname{char} K = 0$ in [18] can be removed easily at the price of considering multilinear identities only. Note that in no instance in the proofs of the corresponding results in [18] there was need of dividing by integers, and the coefficients of the polynomials considered in [18] are ± 1.

Now knowing that the multilinear version of the TPT holds we show that when $\operatorname{char} K = p > 2$ the "general" TPT fails. We need some notation and definitions of [13]. First recall which gradings we shall consider. The algebra $A = M_{1,1}(E)$ is 2-graded in the following way:

$$M_{1,1}(E) = A_0 \oplus A_1, \quad A_0 = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\}, A_1 = \{\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\}$$

where a, $d \in E_0$, and b, $c \in E_1$. For $E \otimes E$ we have

$$E \otimes E = (E_0 \otimes E_0 \oplus E_1 \otimes E_1) \oplus (E_0 \otimes E_1 \oplus E_1 \otimes E_0).$$

Since these algebras are 2-graded, in order to simplify the notation we shall use the letters y_i for the 0-variables, and z_i for the 1-variables of $K(X)$. Sometimes we shall omit the indices from the respective variables. We use the notation $T_2(A)$ for the ideal of 2-graded identities of a given 2-graded algebra A.

It was proved in [13, Theorems 9, 23] that the 2-graded identities for $M_{1,1}(E)$ follow from $[y_1, y_2]$ and $z_1z_2z_3 + z_3z_2z_1$, and those for $E \otimes E$ follow from the above two plus $[y_1^p, z_1]$ where $p = \operatorname{char} K$. Thus $T_2(M_{1,1}(E)) \subset T_2(E \otimes E)$, a proper inclusion, and $T(M_{1,1}(E)) \subseteq T(E \otimes E)$. Now denote by E' the Grassmann algebra without unit, and consider the nonunitary algebra $M_{1,1}(E')$. Note that this algebra is 2-graded in a natural way, and its 1-component is the same as that of $M_{1,1}(E)$. Let $A = M_{1,1}(E') \oplus K$ be the unitary algebra obtained by $M_{1,1}(E')$ by formal adjoining of a unit.

Lemma 6 *The algebra A satisfies all 2-graded identities of $T_2(E \otimes E)$.*

Proof. The inclusion $A \subset M_{1,1}(E)$ implies that A satisfies the graded identities $[y_1, y_2]$ and $z_1z_2z_3 + z_3z_2z_1$. Observe that the algebra E' satisfies the (ordinary) identity $x^p = 0$ (see [19]). Let $a \in A_0$, then $a = \lambda I + (a_{11}e_{11} + a_{22}e_{22})$ where e_{ij} are the matrix units and $a_{ii} \in E'_0$. Now we have $(a_{11}e_{11} + a_{22}e_{22})^p = a_{11}^p e_{11} + a_{22}^p e_{22} = 0$. On the other hand λI is central and the p-th binomial coefficients are divisible by p therefore $a^p = \lambda^p I \in K$. Here we identify the field K with the scalar matrices λI, $\lambda \in K$, for I being the identity matrix. Thus A satisfies $[y^p, z]$.

Proposition 7 *The algebras A and $E \otimes E$ satisfy the same 2-graded identities.*

Proof. Due to the preceding lemma it suffices to prove that every 2-graded identity of A is satisfied by $E \otimes E$. Hence it is sufficient to show that every proper graded identity of A is satisfied by $E \otimes E$. In order to prove this fact we use ideas of [13]. As in [13, Lemma 17], choose a set $\{g_i(z_1, z_2, \ldots, z_n)\}$ of multilinear polynomials that are linearly independent modulo the T_2-ideal of A. Then the polynomials

$$y_1^{i_1} y_2^{i_2} \ldots y_k^{i_k} z_{n+1}^2 z_{n+2}^2 \ldots z_{n+r}^2 g_i(z_1, z_2, \ldots, z_n)$$

are linearly independent modulo $T_2(A)$ provided that all $i_j < p$ (see the proof in [13, Lemma 17]).

Denote by F the relatively free 2-graded algebra of A, then F is a homomorphic image of the relatively free 2-graded algebra of $M_{1,1}(E)$. Hence every proper polynomial of F can be written in the form

$$y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_m^{\alpha_m} z_{i_1}^2 z_{i_2}^2 \ldots z_{i_k}^2 g_j(z_{j_1}, z_{j_2}, \ldots, z_{j_l})$$

where $\{i_1, i_2, \ldots, i_k\} \cap \{j_1, j_2, \ldots, j_l\} = \emptyset$, $i_1 < i_2 < \cdots < i_k$, and g_j is multilinear. Furthermore we may impose the restriction $\alpha_i < p$ since if some $\alpha_i \geq p$ then the corresponding element vanishes on A. The last statement holds for 0-proper polynomials; since every variable y_i participates in commutators only we may dispense with the scalar part of the respective matrices. See [13, Proposition 12, Corollary 22] for the detailed proof of the last statement. Now it suffices to prove that the last polynomials are linearly independent in F, and in turn one has to consider the multilinear polynomials g_j only.

So we find a basis of the vector subspace of F of the multilinear polynomials in the variables z_j. But such a basis consists of the polynomials

$$z_{a_1} z_{b_1} z_{a_2} z_{b_2} \ldots z_{a_m} \widehat{z_{b_m}}$$

for $a_1 < a_2 < \ldots < a_m$, $b_1 < b_2 < \ldots < b_m$ and the hat over the variable z_{b_m} means that it may be missing. The last fact was proved in [13, Proposition 8]. Therefore we finish the proof using [13, Theorem 23].

Corollary 8 $T(A) = T(E \otimes E)$.

The algebra A satisfies the same 2-graded (and hence ordinary) identities as $E \otimes E$. Since A is subalgebra of $M_{1,1}(E)$, in order to compare the identities of $M_{1,1}(E)$ and of $E \otimes E$, it is more convenient to work in A than in $E \otimes E$.

Lemma 9 *The polynomial* $f(x_1, x_2) = [x_1^{p^2}, x_2]$ *is an identity for* A *but not for* $M_{1,1}(E)$.

Proof. If we choose $a = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in M_{1,1}(E)$ for x_1 then $a^{p^2} = a$ is not central. Hence $f \notin T(M_{1,1}(E))$. Now we prove that for every $a \in A$ the element a^{p^2} is central. Since $a = k + a'$ for $k \in K$ and $a' \in M_{1,1}(E')$ we get $a^{p^2} = k^{p^2} + (a')^{p^2}$ and k^{p^2} is central. Thus we may consider $a = a' \in M_{1,1}(E')$. Let $a = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_{1,1}(E')$; hence x, $t \in E'_0$, y, $z \in E'_1 = E_1$. Then since $p \geq 3$ we have that $p^2 \geq 4p - 3$. The entries of a^{p^2} will be linear combinations of monomials in x, y, z, t, each of them of degree p^2. Hence every monomial will contain one of the elements x, y, z, t at least p times. Now we use the fact that x and t are central in E' and y and z anticommute, and write, up to a sign, every such monomial in the form $x^\alpha y^\beta z^\gamma t^\delta$ where at least one of the powers is $\geq p$. But according to [19] the algebra E' satisfies the identity $x_1^p = 0$. Therefore $a^{p^2} = 0$ and the proof is complete.

Theorem 10 *Let* $\operatorname{char} K = p > 2$. *Then* $T(M_{1,1}(E)) \subsetneq T(E \otimes E)$. *More precisely the identity* $[x_1^{p^2}, x_2]$ *is satisfied by the latter but not by the former algebra.*

Proof. Combining the above Lemma and Proposition we obtain $T(E \otimes E) = T(A) \subsetneq T(M_{1,1})$ and we are done.

Remark 1 *The above theorem shows that the tensor product theorem fails in positive characteristic. However, it does hold if one considers the multilinear parts of the corresponding T-ideals.*

We note that the algebras $E \otimes E$ *and* $M_{1,1}(E)$ *provide an example of algebras that satisfy the same multilinear identities but have different T-ideals if* $\operatorname{char} K = p > 2$. *Another example of such a pair of algebras (and another example that shows the TPT fails in positive characteristic) will be given later.*

4 The graded identities for $M_n(E)$

In this section we fix the group G as the group $\mathbb{Z}_n \times \mathbb{Z}_2$. If n is fixed positive integer, we denote by $\bar{i}$ the residue of i modulo n, $0 \leq i \leq n - 1$.

Define a G-grading on the algebra $M_n(E)$ as follows. If $g = (\alpha, \beta) \in G$ then

$$M_n(E)_g = \{A = (a_{ij}) \in M_n(E) \mid a_{ij} \in E_\beta \text{ if } \overline{j - i} = \alpha \text{ and } 0 \text{ otherwise}\}.$$

Thus this grading is the natural extension of the usual $\mathbb{Z}_n$-grading on the matrix algebra $M_n(K)$ and of the $\mathbb{Z}_2$-grading on E. We leave to the reader the easy exercise that the above is indeed a G-grading. For a homogeneous element h of degree $(\alpha, \beta) \in G$ in a G-graded algebra A we write $\alpha(h) = \alpha \in \mathbb{Z}_n$ and $\beta(h) = \beta \in \mathbb{Z}_2$.

For the computations that follow it will be useful to note that the n-grading on the algebra $M_n(K)$ is the following. Its i-th component consists of the matrices

$$\sum k_{rs} e_{rs}, \quad k_{rs} \in K, \quad s - r \equiv i \pmod{n},$$

for $i = 0, 1, \ldots, n - 1$. These are the matrices whose nonzero entries belong to the $(i-1)$-st diagonal, and when this diagonal reaches the "border" of the matrix (at the position $(n - i, n)$), it "continues" from the beginning $(n - i + 1, 1)$ of the next row. See the matrix from 1 below in order to visualise the picture of the grading.

This interpretation shows at once that the gradings in consideration are well defined.

One may define analogously a G-grading on the generic algebra F in $\operatorname{var} M_n(E)$ (see Theorem 4). But such a grading is not the most convenient for our goals since the notation would be rather complicated and clumsy. Instead we introduce another model for the relatively free algebra in $\operatorname{var} M_n(E)$. In some sense it corresponds to appropriate change of rows and columns in the respective matrices.

First we "rename" the variables. Consider the set $X = Y \cup Z$ where

$$Y = \{y_i^d \mid d \in \mathbb{Z}_n, i \geq 1\}, \qquad Z = \{z_i^d \mid d \in \mathbb{Z}_n, i \geq 1\}.$$

Of course we impose all the restrictions of Theorem 4. Let $X = \{x_1, x_2, \dots\}$. Then we define $A_k = (a_{rs}^k) \in M_n(\Omega)$ to be the matrix with entries

$$a_{rs}^k = \begin{cases} y_k^{\overline{r-1}} & \text{if } \alpha(x_k) = \overline{s-r} \text{ and } \beta(x_k) = 0, \\ z_k^{\overline{r-1}} & \text{if } \alpha(x_k) = \overline{s-r} \text{ and } \beta(x_k) = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad k = 1, 2, \dots$$

Now denote as H the G-graded subalgebra of $M_n(\Omega)$ generated by the matrices A_k, $k \geq 1$.

Proposition 11 *The G-graded algebra H is relatively free in the variety of G-graded algebras generated by $M_n(E)$.*

Proof. The proof repeats verbatim the one for the generic matrices. Let $\varphi \colon K(X) \to H$ be the homomorphism defined by $x_k \mapsto A_k$, $k \geq 1$. Then it is obviously graded homomorphism onto H. An easy computation shows that its kernel is exactly $T_G(M_n(E))$, and the proposition is proved.

Remark 2 *The algebra H is isomorphic to the relatively free (ungraded) algebra in $\operatorname{var} M_n(E)$ since it satisfies the same G-graded identities as $M_n(E)$ does.*

Now we write down a list of G-graded polynomials that turn out to be a basis of the G-graded identities for $M_n(E)$. Let S be the set of the identities:

Identity	$(\alpha(x_1), \beta(x_1))$	$(\alpha(x_2), \beta(x_2))$	$(\alpha(x_3), \beta(x_3))$
$x_1x_2 - x_2x_1$	$(0,0)$	$(0,0)$	
	$(0,0)$	$(0,1)$	
$x_1x_2 + x_2x_1$	$(0,1)$	$(0,1)$	
$x_1x_2x_3 - x_3x_2x_1$	$(\alpha,0)$	$(-\alpha,0)$	$(\alpha,0)$
	$(\alpha,1)$	$(-\alpha,0)$	$(\alpha,0)$
	$(\alpha,0)$	$(-\alpha,1)$	$(\alpha,0)$
$x_1x_2x_3 + x_3x_2x_1$	$(\alpha,1)$	$(-\alpha,1)$	$(\alpha,1)$
	$(\alpha,0)$	$(-\alpha,1)$	$(\alpha,1)$
	$(\alpha,1)$	$(-\alpha,0)$	$(\alpha,1)$

Denote by I the ideal of G-graded identities in $K(X)$ generated by the set S.

Lemma 12 $I \subseteq T_G(M_n(E))$.

Proof. The proof consists of a direct (and easy) computation based on the multiplication rules in the matrix and the Grassmann algebra, according to the definition of the grading.

Remark 3 *Let A and B be $n \times n$ matrices over a ring R, $A = \sum_{\kappa=1}^{n} a_\kappa e_{\kappa,i-1+\kappa}$, $B = \sum_{\kappa=1}^{n} b_\kappa e_{\kappa,j-1+\kappa}$ where the second indices in the matrix units e_{ij} are taken modulo n i.e., $n+1=1$, $n+2=2$ and so on. Then $AB = \sum_{\kappa=1}^{n} a_\kappa b_{i_\kappa} e_{\kappa,i+j-2+\kappa}$. Hence the entries of AB are zeros except for the ones on the diagonal that is parallel to the main diagonal, and starts at position $i+j-1$ on the first row. When this diagonal reaches the "border" of the matrix it continues at the beginning of the next row of AB, again parallel to the diagonal. This assertion follows directly from the definition of the product. Note that when $R = E$ is the Grassmann algebra the above rule for the product shows once again that the definition of the G-grading on $M_n(E)$ is correct.*

Lemma 13 *Let $M = M(x_1, \ldots, x_m) \in K(X)$ be a nonzero monomial of length r. Suppose that $M \in K(X)_{(\alpha,\beta)}$. Then*

$$M(A_1, \ldots, A_m) = \pm \begin{pmatrix} 0 & \ldots & 0 & M_0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & M_1 & \ldots & 0 \\ & \ldots & & & & \ddots & \\ 0 & \ldots & 0 & 0 & 0 & \ldots & M_{-\alpha-1} \\ M_{-\alpha} & \ldots & 0 & 0 & 0 & \ldots & 0 \\ & \ddots & & & & \ldots & \\ 0 & \ldots & M_{\overline{n-1}} & 0 & 0 & \ldots & 0 \end{pmatrix} \quad (1)$$

where $M_\gamma = y_{i_1}^{\alpha_1+\gamma} \ldots y_{i_t}^{\alpha_t+\gamma} z_{j_1}^{\beta_1+\gamma} z_{j_s}^{\beta_s+\gamma}$ for suitable $1 \le i_1 \le \ldots \le i_t \le m$, $1 \le j_1 \le \ldots \le j_s \le m$, and α_κ, $\beta_\kappa \in \mathbb{Z}_n$, $\gamma = 0, 1, \ldots, n-1$. Note that in the upper indices in the monomials M_γ, the summation is the one in $\mathbb{Z}_n$.

Proof. As it often happens the proof of an extensive assertion is short and straightforward. The present is not an exception. One inducts on $r = t+s$, the base $r = 1$ being obvious. If $M(x_1, \ldots, x_m) = N(x_1, \ldots, x_m)x_q$ for some q then one applies the induction to N and uses Remark 3.

Proposition 14 *Suppose that the monomials M, $N \in K(X)$. Assume that $M(A_1, \ldots, A_m)$ and $\pm N(A_1, \ldots, A_m)$ have at the same position, the same nonzero entry (belonging to the free anticommutative algebra). Then $M \equiv \pm N \pmod I$.*

Proof. We induct on the length q of M. The proposition is evidently true if $q = 1$, so suppose $q > 1$. Assume that the common nonzero entry is at position (h, k). Then write $M = M_1 x_t M_2$ for some t where $\alpha(M_1) = r$ and $\alpha(M_2) = s$ for r, $s \in \mathbb{Z}_n$.

First we consider the case $\beta(x_t)=0$. The (h,k)-th entry of the monomial $M(A_1,\ldots,A_m)$ will be the following:

$$M'e_{h,h+r}\times y_t^{\overline{h+r-1}}e_{h+r,k-s}\times M''e_{k-s,k},\qquad \text{if } \alpha(x_t)=k-s-h-r\in\mathbb{Z}_n.$$

Recall that the bar $\overline{u}$ means that we take the residue of u modulo n. On the other hand, one may write $N=N'x_tN''$. When we evaluate N on the matrices A_κ, the contribution of A_t to the (h,k)-th entry of $N(A_1,\ldots,A_m)$ will be exactly $y_t^{\overline{h+r-1}}e_{h+r,k-s}$. But then according to Remark 3, N can be written as $N=N_1x_tN_2$ where the $(h,h+r)$-th entry of the evaluation of N_1 on the A_i's is some $N'e_{h,h+r}\neq 0$; therefore $\alpha(N_1)=\alpha(M_1)=r\in\mathbb{Z}_n$. In this way we conclude that if $M=M_1x_tM_2x_tM_3\ldots M_{l-1}x_tM_l$ then $N=N_1x_tN_2x_tN_3\ldots N_{l-1}x_tN_l$ for suitable monomials N_κ. Furthermore for some permutation σ of $\{1,2,\ldots,l\}$ one has

$$\alpha(M_1x_tM_2x_t\ldots x_tM_\kappa)=\alpha(N_1x_tN_2x_t\ldots x_tN_{\sigma(\kappa)}).$$

Now let the monomial M start with x_1. Then $N=N_1x_1N_2$ for $\alpha(N_1)=0$. We consider three cases in this situation.

Case 1. $M=x_1M_1x_1M_2$ and $\alpha(x_1M_1)=0$. Then $N=N_3x_1N_4x_1N_5$ such that $\alpha(N_3)=\alpha(N_3x_1N_4)=0$. But then $\alpha(x_1N_4)=0$ and $N\equiv\pm x_1N_4N_3x_1N_5 \pmod I$ according to the defining identities of I, see the table for the set S.

Case 2. $M=M_1x_ax_bM_2$, $N=N_3x_aN_4x_1N_5x_bN_6$, and $N_1=N_3x_aN_4$, such that $\alpha(M_1)=\alpha(N_3)$, $\alpha(M_1x_a)=\alpha(N_3x_aN_4x_1N_5)$. Then $\alpha(N_4x_1N_5)=0$. But $\alpha(N_3x_aN_4)=\alpha(N_1)=0$ and therefore $\alpha(N_3x_a)=-\alpha(N_4)=\alpha(x_1N_5)$. So we conclude, according to the identities of S, that $N\equiv\pm x_1N_5N_4N_3x_ax_bN_6 \pmod I$.

Case 3. Neither of the former two cases holds. Then write $M=x_{i_1}x_{i_2}\ldots x_{i_q}$, and choose a variable, say x_1, that appears in N_1, $N_1=N_3x_1N_4$. Then for some r, $1\leq r\leq q$, we will have $i_r=1$. Therefore $\alpha(N_3)=\alpha(x_{i_1}x_{i_2}\ldots x_{i_{r-1}})$. If $r<q$ then suppose $N=N_5x_{i_{r+1}}N_6$, and hence $\alpha(N_5)=\alpha(x_{i_1}x_{i_2}\ldots x_{i_r})$. But the degree (the length) $\deg N_1>\deg N_5$ since if $\deg N_1=\deg N_5$ then we would have Case 1; if $\deg N_1<\deg N_5$ then this would be Case 2. So the variable $x_{i_{r+1}}$ participates in N_1 as well. Therefore for some r_0, the monomials N_1 and $x_{i_{r_0}}x_{i_{r_0+1}}\ldots x_{i_q}$ will be homogeneous of the same G-degree.

Now if N starts with x_j then $M=M_3M_4x_jM_5$ for some monomials M_i such that $\alpha(M_3M_4)=0$, $M_4x_jM_5=x_{i_{r_0}}x_{i_{r_0+1}}\ldots x_{i_q}$. As a result we get that

$$\alpha(M_4x_jM_5)=\alpha(x_{i_{r_0}}x_{i_{r_0+1}}\ldots x_{i_q})=\alpha(N_1)=0\in\mathbb{Z}_n.$$

But then we have $\alpha(M_3)=-\alpha(M_4)=\alpha(x_jM_5)$ hence $M\equiv x_j(M_5M_4M_3) \pmod I$. But the last monomial starts with x_j.

All three cases considered, we have shown that $M\equiv U \pmod I$, $N\equiv V \pmod I$, and U and V start with the same letter say x. Write $U=xU'$, $V=xV'$. Since $I\subseteq T_G(M_n(E))$ we have that

$$M(A_1,\ldots,A_m)=U(A_1,\ldots,A_m),\quad N(A_1,\ldots,A_m)=V(A_1,\ldots,A_m).$$

Now $U'(A_1,\ldots,A_m)$ and $\pm V'(A_1,\ldots,A_m)$ will satisfy the assumption of the proposition. Namely they have the same nonzero entry at the same position, and we apply the induction.

Then $U' \equiv \pm V' \pmod I$, and $U \equiv \pm V \pmod I$, and we are done with the proof when $\beta(x_t) = 0$. If $\beta(x_t) = 1$ one follows verbatim the above proofs.

Lemma 15 *Suppose that $M(x_1, \ldots, x_m) \in K(X)$ is a monomial and let $M(A_1, \ldots, A_m) = 0$, then $M \in I$.*

Proof. The equality $M(A_1, \ldots, A_m) = 0$ implies that some variable, say x, appears at least twice in M. (If M were linear in each variable then the explicit form of the product from Lemma 13 would imply that a multilinear monomial in the free supercommutative algebra must vanish which is impossible.) Hence $M = M_1 x M_2 x M_3$ for some monomials M_1, M_2, M_3, some of them possibly empty. Furthermore $\alpha(M_1) = \alpha(M_1 x M_2)$ and $\beta(x) = 1$ since according to Lemma 13 all entries of the product must be 0. But this cannot happen if the entries are linear in the variables z. Therefore $\alpha(x) = -\alpha(M_2)$, and using the identity $x_1x_2x_3 + x_3x_2x_1 = 0$ with $x_1 = x_3 = x$, $x_2 = M_2$, we get that $xM_2x \in I$. Thus $M = M_1xM_2xM_3 \in I$.

The main result in this section is the following theorem. When $\operatorname{char} K = 0$ it was obtained in [7].

Theorem 16 *Let K be an infinite field, $\operatorname{char} K = p \neq 2$ and let $G = \mathbb{Z}_n \times \mathbb{Z}_2$. The G-graded identities of the algebra $M_n(E)$ follow from the set of graded identities S.*

Proof. We already saw (Lemma 12) that $I \subseteq T$ where $T = T_G(M_n(E))$. Hence let $f \in T$, we have to show $f \in I$. Write $f = \sum_{i=1}^r a_i f_i \pmod I$ where f_i are G-homogeneous monomials. Suppose further that the integer r is the least possible with this property thus all $a_i \neq 0$ in K. If $r \geq 1$ then some $f_i \neq 0$. Take $f_1 \neq 0 \pmod I$ and then $f_1(A_1, \ldots, A_m) \neq 0$ according to Lemma 15. But $a_1 f_1(A_1, \ldots, A_m) = -\sum_{i=2}^r a_i f_i(A_1, \ldots, A_m)$, and for some q, $2 \leq q \leq r$, say $q = 2$, the matrices $f_1(A_1, \ldots, A_m)$ and $\pm f_2(A_1, \ldots, A_m)$ have, at the same position, the same nonzero entry. Then Proposition 14 yields $f_1 \equiv \pm f_2 \pmod I$. We write $f \equiv (a_1 \pm a_2) f_1 + \sum_{i=3}^r a_i f_i$ and we get a combination with $r-1$ (or $r-2$) terms, a contradiction. Therefore $r = 0$ and the theorem is proved.

5 The graded identities for $M_{a,b}(E) \otimes E$

In this section we describe the graded identities of the algebra $M_{a,b}(E)$, $a + b = n$. Most of the statements and of the proofs are quite similar to their analogues in the previous section, and in order not to be (too) boring we shall sketch or even omit some of them. Recall that G stands for the group $\mathbb{Z}_n \times \mathbb{Z}_2$. First we define a G-grading on the algebra $M_{a,b}(E) \otimes E$. Let $(\alpha, \beta) \in G$ and set $M_{a,b}(E)_{(\alpha,\beta)}$ to be the subspace of the matrices in $M_{a,b}(E)$ having as (i,j)-th entry an element of E_β when $\overline{j-i} = \alpha$ in $\mathbb{Z}_n$ and $(i,j) \in \Delta_\beta$, and 0 otherwise. (The sets Δ_β were defined at the beginning of the paper, just before Theorem 1.) Then we define G-grading on the algebra $M_{a,b}(E) \otimes E = P$ as follows:

$$P_{(\alpha,\beta)} = (M_{a,b}(E))_{(\alpha,\beta)} \otimes E_0 + (M_{a,b}(E))_{(\alpha,\beta+1)} \otimes E_1.$$

Now we proceed in constructing a model for the relatively free graded algebra. Note that we could use the construction of Theorem 4 but it would not be the most appropriate for our purposes. Consider the $n \times n$ matrices $A_i = (a_{ij})$ and $B_i = (b_{ij})$ defined as follows. The entries a_{ij} of A equal $y_i^{\overline{r-1}}$ when $\alpha(x_i) = \overline{s-r}$ and $(r,s) \in \Delta_0$, and 0 otherwise. Analogously $b_{ij} = z_i^{\overline{r-1}}$ when $\alpha(x_i) = \overline{s-r}$ and $(r,s) \in \Delta_1$, and 0 otherwise. Denote by SC the free supercommutative algebra freely generated by $X = Y \cup Z$, $Y = \{y_i\}$, $Z = \{z_i\}$. Then consider the elements $C_i \in M_n(\Omega) \otimes SC$ defined as follows. If $\beta(x_i) = 0$ then $C_i = A_i \otimes y_i + B_i \otimes z_i$; if $\beta(x_i) = 1$ then $C_i = B_i \otimes y_i + A_i \otimes z_i$. Let F be the algebra generated by C_i, $i \geq 1$. The algebra F is G-graded in a natural way.

Lemma 17 *$T_G(P) = T_G(F)$.*

Proof. The proof consists in showing the two inclusions which are immediate.

Remark 4 *Observe that if we consider $\varphi\colon K(X) \to F$ defined by $x_i \mapsto C_i$ then φ is a homomorphism, and it is onto. But this does not mean that $\ker\varphi = T_G(P)$. On the contrary, choose $a = b = 1$ and $f(x_1) = x_1^2$ for $(\alpha(x_1), \beta(x_1)) = (1,0)$. Then $\varphi(f) = 0$. On the other hand if we substitute x_1 for $d = \begin{pmatrix} 0 & e_1 \\ e_2 & 0 \end{pmatrix} \otimes e_5 + \begin{pmatrix} 0 & e_3 \\ e_4 & 0 \end{pmatrix} \otimes e_6$ then $f(d) = d^2 \neq 0$.*

This is due to the fact that φ is not a G-graded homomorphism and consequently $\ker\varphi$ is not G-graded ideal. For an example, choose x_1 and x_2 of the component $(1,0)$ in the grading, and see that $f(x_1 + x_2) = (x_1 + x_2)^2 \notin \ker\varphi$. But on the other hand it is immediate that $T_G(F) \subset \ker\varphi$.

Set I the ideal of the G-graded identities generated by the polynomials from the set S defined in Section 4 and by all monomials in $T_G(P)$.

Lemma 18 *$I \subseteq T_G(P)$.*

Proof. See the corresponding proof of the previous section, that of Lemma 12.

Corollary 19 *1. $T_G(M_n(E)) \subseteq T_G(M_{a,b}(E) \otimes E)$;*
2. $T(M_n(E)) \subseteq T(M_{a,b}(E) \otimes E)$.

Proof. The proof follows immediately from Lemma 18 and from Theorem 16.

Proposition 20 *Suppose $M(x_1, \ldots, x_m)$, $N(x_1, \ldots, x_m) \in K(X)$ are monomials. Let the matrices $M(H_1, \ldots, H_m)$ and $\pm N(H_1, \ldots, H_m)$ where $H_i = A_i$ or $H_i = B_i$ for every i, have at some position the same nonzero entry. Then $M \equiv \pm N \pmod{I}$.*

Proof. The proof follows almost word by word that of Proposition 14, only minor changes are needed. That is why we do not give it.

Theorem 21 *The G-graded identities of $M_{a,b}(E) \otimes E$, $a+b = n$, follow from the polynomials of the set S plus all monomials that lie in $T_G(M_{a,b}(E) \otimes E)$.*

Proof. Since in this proof there are several "deviations" of the known scheme (see the proof of Theorem 16) we give it in detail. It suffices to prove that if f is multihomogeneous and is G-graded identity for $M_{a,b}(E)\otimes E$ then $f\in I$. Choose, as in Theorem 16, the least $r\geq 0$ such that $f\equiv\sum_{i=1}^{r}a_if_i \pmod I$ for f_i being monomials in $K(X)$ and $0\neq a_i\in K$. Let us suppose that $r>0$.

But f_1 is not a graded identity of $M_{a,b}(E)\otimes E$ due to the minimality of r. Therefore $f_1\neq 0$ on $M_{a,b}(E)\otimes E$, and $f_1(w_1,\ldots,w_m)\neq 0$ for suitable $w_k=C_{t_{k,1}}+\ldots+C_{t_{k,n_k}}$, and $t_{u,v}\neq t_{r,s}$ if $(u,v)\neq(r,s)$. Write $f_1=x_{i_1}x_{i_2}\ldots x_{i_q}$.

Let $J=\{t_{i,j}\mid 1\leq i\leq m, 1\leq j\leq n_i\}\subset\mathbb{N}$, and define the function $g\colon J\to\{1,2,\ldots,m\}$ by $g(t_{u,v})=u$. Note that g is well defined. Since $f_1(w_1,\ldots,w_m)\neq 0$ then we obtain a term $H_{j_1}\ldots H_{j_q}\otimes e\neq 0$ from it that will not cancel. Here $H_{j_k}=A_{j_k}$ or B_{j_k} for all k, and $e\in E$. Observe that $g(j_k)=i_k$.

Now from the equality $a_1f_1=-\sum_{i=2}^{r}a_if_i$ we obtain that the right-hand side contains a term $H_{j_{\sigma(1)}}\ldots H_{j_{\sigma(q)}}\otimes e$ for some permutation σ of $\{1,2,\ldots,q\}$, and furthermore $H_{j_1}\ldots H_{j_q}$ and $\pm H_{j_{\sigma(1)}}\ldots H_{j_{\sigma(q)}}$ share a nonzero entry at some position (the same for both). Assume that this term comes from $f_2=x_{s_1}\ldots x_{s_q}$, then $g(j_{\sigma(k)})=s_k$. Now let $h_1(x_{j_1},\ldots,x_{j_q})=x_{j_1}\ldots x_{j_q}$ and $h_2(x_{j_1},\ldots,x_{j_q})=x_{j_{\sigma(1)}}\ldots x_{j_{\sigma(q)}}$, then $h_1\equiv\pm h_2 \pmod I$ according to Proposition 20. But $i_{\sigma(k)}=g(j_{\sigma(k)})=s_k$ hence

$$f_1(x_1,\ldots,x_m)=h_1(x_{j_1},\ldots,x_{j_q}),\quad f_2(x_1,\ldots,x_m)=h_2(x_{i_1},\ldots,x_{i_q}).$$

Therefore we finish the proof in the same manner as that of Theorem 16.

As an immediate consequence we obtain one of the main results of [8], and another proof of a statement of Kemer's TPT.

Corollary 22 *Let* $\operatorname{char} K=0$*, then the* G*-graded identities of* $M_{a,b}(E)\otimes E$ *follow from those of the set* S*. Furthermore* $M_{a,b}(E)\otimes E$ *and* $M_n(E)$*,* $n=a+b$*, satisfy the same ordinary identities.*

Remark 5 *Note that when* $\operatorname{char} K>0$ *we cannot claim the validity of the above Corollary. As we shall see in the next section, there exist monomials that are* G*-graded identities for* $M_{a,b}(E)\otimes E$ *but that do not follow from the graded identities of* S*. Such a monomial for the algebra* $M_{a,a}(E)\otimes E$ *is for example*

$$c_a(x_1,\ldots,x_p)=x_1x_2x_1x_3x_1\ldots x_1x_px_1$$

with $\alpha(x_i)=a\in\mathbb{Z}_n$ *and* $\beta(x_1)=0$*,* $n=2a$*. Then* c_a *is a graded identity for* $M_{a,a}(E)\otimes E$*. On the other hand, let* $d=\sum_{i=1}^{a}(e_{i,a-1+i}+e_{a+i,i})\in M_{2a}(E)$*, then* $c_a(d,d,\ldots,d)=d\neq 0$*. Therefore we have the strict inclusion*

$$T_G(M_{2a}(E))\subsetneq T_G(M_{a,a}(E)\otimes E)),\quad G=\mathbb{Z}_{2a}\times\mathbb{Z}_2.$$

We consider in detail the case $a=1$ *in the next section.*

6 The identities for $M_{1,1}(E) \otimes E$ and $M_2(E)$

Here we fix $n = 2$, $a = b = 1$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, and p is the characteristic of the base field K. Denote by I the ideal of G-graded identities generated by the set S and set I_1 the ideal of G-graded identities generated by I and by the polynomial

$$c(x_1, x_2, \ldots, x_p) = x_1 x_2 x_1 x_3 x_1 \ldots x_1 x_p x_1$$

with $\alpha(x_i) = 1$ for all i and $\beta(x_1) = 0$.

Lemma 23 *If* $\operatorname{char} K = p \neq 0$ *then* $I_1 \subseteq T_G(M_{1,1}(E) \otimes E)$.

Proof. We already know from Lemma 18 that $I \subseteq T_G(M_{1,1}(E) \otimes E)$ and thus it remains to show that c is indeed a graded identity for $M_{1,1}(E)\otimes E$. The polynomial c is linear in the variables $x_2, \ldots, x_p$. Therefore in order to check whether it vanishes on $M_{1,1}(E)\otimes E$ it suffices to substitute x_1 for $a_1 = \sum_{j=1}^{r} \begin{pmatrix} 0 & \alpha_j \\ \beta_j & 0 \end{pmatrix} \otimes \gamma_j$, and x_i, $i > 1$, for $a_i = \begin{pmatrix} 0 & \delta_i \\ \varepsilon_i & 0 \end{pmatrix} \otimes \varphi_i$. Here α_i, β_i, γ_i, δ_i, $\varepsilon_i \in E_1$ and $\varphi_i \in E_0 \cup E_1$. But we have $a_1 a_i = \sum_{j=1}^{r} \begin{pmatrix} \alpha_j \varepsilon_i & 0 \\ 0 & \beta_j \delta_i \end{pmatrix} \otimes \gamma_j \varphi_i$ whenever $i > 1$. Using this fact and $t^2 = 0$ for $t \in E_1$ we obtain immediately that $c(a_1, \ldots, a_p) = 0$ when $r < p$. So consider $r = p$. Then $c(a_1, \ldots, a_p) = \sum \begin{pmatrix} 0 & u_\sigma \\ v_\sigma & 0 \end{pmatrix} \otimes w_\sigma$ for σ running over the permutations of $\{1, 2, \ldots, p\}$, and $u_\sigma = \alpha_{\sigma(1)} \varepsilon_2 \alpha_{\sigma(2)} \varepsilon_3 \ldots \alpha_{\sigma(p-1)} \varepsilon_p \alpha_{\sigma(p)}$, and similar expressions for v_σ and w_σ. But the α_i and ε_j anticommute, hence

$$c(a_1, \ldots, a_p) = p! \begin{pmatrix} 0 & u_1 \\ v_1 & 0 \end{pmatrix} \otimes w_1 = 0$$

in K. The case $r > p$ is easily reduced to the former one by means of choosing subsets of p elements each.

Remark 6 *We compute* $c(e_{12} + e_{21}, \ldots, e_{12} + e_{21}) = e_{12} + e_{21}$ *in* $M_2(E)$ *hence* c *is not a* G*-graded identity for* $M_2(E)$, *and* $T_G(M_2(E)) \subsetneq T_G(M_{1,1}(E) \otimes E)$.

Lemma 24 *1. If* $f \in K\langle X\rangle$ *is a monomial and* $x \in X$ *is such that* $\beta(x) = 1$ *and* $\alpha(x) = \alpha(f)$ *then the monomial* $xfx \in I_1$.

2. If f_1, $f_2 \in K\langle X\rangle$ *are monomials and* $x \in X$ *is a variable,* $\beta(x) = 1$, *then* $x f_1 x f_2 x \in I_1$.

Proof. 1. When $\alpha(x) = 1$, using the graded identity $x_1 x_2 x_3 + x_3 x_2 x_1 = 0$ one obtains $xfx \equiv -xfx \pmod{I_1}$, and therefore $xfx \in I_1$. If $\alpha(x) = 0$, we may consider $\deg f \geq 1$ since, when $f = 1$, $x_1 x_2 = -x_2 x_1$ and $x^2 \in I_1$. So let $\alpha(x) = 0$ and $\deg f > 0$. If $\beta(f) = 0$ then use $x_1 x_2 = x_2 x_1$ and get $xfx \equiv xxf \in I_1$. Analogously if $\beta(f) = 1$ we use $x_1 x_2 = -x_2 x_1$ and get $xfx \equiv -xxf \in I_1$.

2. Suppose first $\alpha(x) = 0$, then according to the first statement of the Lemma we may consider $\alpha(f_1) = \alpha(f_2) = 1$ only. Hence $\alpha(f_1 x f_2) = 0$, and $x f_1 x f_2 x \in I_1$. Similarly one deals with the case when $\alpha(x) = 1$.

Proposition 25 *If the monomial* $f(x_1,\dots,x_m)\in K(X)$ *is a* G*-graded identity for* $M_{1,1}(E)\otimes E$ *then* $f\in I_1$.

Proof. Suppose that $f\notin I_1$, we shall find $a_i\in M_{1,1}(E)\otimes E$ such that $f(a_1,\dots,a_m)\neq 0$. We induct on $q=\deg f$, the length of the monomial f. The base $q=1$ is trivial; suppose now $q>1$. Then we write $f=hx_i$ for some monomial h and some i. Now if $h\in I_1$ the assertion of the proposition is true. So we assume $h\notin I_1$. Assume further that $i=1$, so $f=hx_1$. Then for some $b_i\in M_{1,1}(E)\otimes E$, $h(b_1,\dots,b_m)\neq 0$. Suppose that $\deg_{x_1} f=d$ that is there are d entries of x_1 in f. We can choose the b_i's in such a way that the generators e_1, e_2, e_3, e_4 of the Grassmann algebra E appear in neither of them. Write $f=f_1x_1f_2x_1\dots x_1f_dx_1$ where f_i are monomials that do not contain x_1. Hence $f_i=f_i(x_2,\dots,x_m)$. We shall write $f(x)$, $h(b)$ and so on, to indicate $f(x_1,\dots,x_m)$, $h(b_1,\dots,b_m)$ respectively. We divide the remaining proof in four cases depending on the homogeneous degree $(\alpha(x_1),\beta(x_1))$ of x_1.

Case 1. $(\alpha(x_1),\beta(x_1))=(0,0)$. In this case, since $f_1(b)b_1f_2(b)b_1\dots b_1f_d(b)\neq 0$ we have $f_1(b)\dots f_d(b)\neq 0$. Then for $a=\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes 1$ we get that

$$f(a,b_2,\dots,b_m)=f_1(b)af_2(b)a\dots f_d(b)a=f_1(b)\dots f_d(b)\neq 0.$$

Case 2. $(\alpha(x_1),\beta(x_1))=(1,1)$; choose $a=\begin{pmatrix}0&e_1\\e_2&0\end{pmatrix}\otimes 1$. But according to Lemma 24 in this case $d\leq 2$. When $d=1$ we get $f(a,b_2,\dots,b_m)=h(b)a\neq 0$. If $d=2$ then again by Lemma 24, we obtain $\alpha(f_2)=0$. But $h(b)=f_1(b)b_1f_2(b)\neq 0$ and hence $f_1(b)af_2(b)\neq 0$ as well. On the other hand, for every α, $\beta\in E_0$,

$$\begin{pmatrix}0&e_1\\e_2&0\end{pmatrix}\begin{pmatrix}\alpha&0\\0&\beta\end{pmatrix}\begin{pmatrix}0&e_1\\e_2&0\end{pmatrix}=\begin{pmatrix}\beta e_1e_2&0\\0&\alpha e_2e_1\end{pmatrix}.$$

Therefore $f(a,b_2,\dots,b_m)=f_1(b)af_2(b)a\neq 0$.

Case 3. $(\alpha(x_1),\beta(x_1))=(0,1)$; let $a=\begin{pmatrix}1&0\\0&1\end{pmatrix}\otimes e_1$, $b=\begin{pmatrix}1&0\\0&e_2e_3\end{pmatrix}\otimes e_4$. Once again by Lemma 24 we have $d\leq 2$. If $d=1$ then $f(a,b_2,\dots,b_m)=h(b)a\neq 0$. Let $d=2$, then $\alpha(f_2)=1$. But $h(b)=f_1(b)b_1f_2(b)\neq 0$ and hence $f_1(b)af_2(b)\neq 0$ as well. If we write $f_2(b)=\sum_j\begin{pmatrix}0&\alpha_j\\\beta_j&0\end{pmatrix}\otimes\gamma_j$ then

$$\begin{aligned}af_2(b)b &= \sum_j\begin{pmatrix}0&\alpha_je_2e_3\\\beta_j&0\end{pmatrix}\otimes e_1\gamma_je_4,\\ bf_2(b)a &= \sum_j\begin{pmatrix}0&\alpha_j\\e_2e_3\beta_j&0\end{pmatrix}\otimes e_4\gamma_je_1.\end{aligned}$$

So we get $f(a+b,b_2,\dots,b_m)=f_1(b)(a+b)f_2(b)(a+b)$ and by Lemma 24, it equals $f_1(b)af_2(b)b+f_1(b)bf_2(b)a=f_1(b)af_2(b)\begin{pmatrix}1-e_2e_3&0\\0&e_2e_3-1\end{pmatrix}\otimes e_4\neq 0$.

Case 4. $(\alpha(x_1), \beta(x_1)) = (1, 0)$. Set $a = \begin{pmatrix} 0 & e_1 \\ e_2 & 0 \end{pmatrix} \otimes e_3$, then computing the element $f(a + b_1, b_2, \ldots, b_m)$ we obtain that it equals its homogeneous component that is linear in a, namely

$$\sum_{i=1}^{d} f_1(b)b_1 \ldots f_i(b)af_{i+1}(b)b_1 \ldots f_d(b)b_1.$$

First let $\alpha(f_i) = 1$ for all $i \geq 2$. Then if $d \geq p$ we apply the graded identity c, and get $f \in I_1$. Hence $d < p$, and we apply $x_1x_2x_3 = x_3x_2x_1$ and get

$$f(a + b_1, b_2, \ldots, b_m) = df_1(b)b_1f_2(b)b_1 \ldots b_1f_d(b)a = dh(b)a \neq 0.$$

Thus we are to consider $\alpha(f_i) = 0$ for some $i \geq 2$. Let t be the number of all j, $2 \leq j \leq d$ such that $\alpha(f_jx_1f_{j+1} \ldots x_1f_d) = 0$, and let r be the largest j with this property. We set further $u = f_1(b)b_1f_2(b)b_1 \ldots b_1f_d(b)a$ and

$$v = f_1(b)b_1f_2(b)b_1 \ldots b_1f_{r-1}(b)af_r(b)b_1 \ldots b_1f_d(b)b_1.$$

Then by $x_1x_2x_3 = x_3x_2x_1$ we get $f(a + b_1, b_2, \ldots, b_m) = (d - t)u + tv$. Furthermore, if $d - t \geq p$ and/or $t \geq p$ we apply the graded identity c and obtain $f \in I_1$. Now taking into account the equalities

$$\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}\begin{pmatrix} 0 & e_1 \\ e_2 & 0 \end{pmatrix} = \begin{pmatrix} \alpha\delta e_2 & 0 \\ 0 & \beta\gamma e_1 \end{pmatrix},$$
$$\begin{pmatrix} 0 & e_1 \\ e_2 & 0 \end{pmatrix}\begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} = \begin{pmatrix} e_1\delta\beta & 0 \\ 0 & e_2\gamma\alpha \end{pmatrix},$$

we see that e_1 and e_2 are in different rows of the respective products for u and v. Therefore u and v turn out linearly independent, and we conclude that $f(a + b_1, b_2, \ldots, b_m) \neq 0$.

Now combining Theorem 21 together with Proposition 25, we obtain the following theorem.

Theorem 26 *Let char* $K = p \neq 2$. *The* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-graded identities of the graded algebra* $M_{1,1}(E) \otimes E$ *follow from those of the set* S *(with* $n = 2$, $a = b = 1$*) and the identity* $c(x_1, \ldots, x_p) = 0$.

Now assume that $E' \subset E$ is the Grassmann algebra without unit. We recall that E' satisfies the (ordinary) identity $x^p = 0$ (see for example [19]). Define the algebra A as the subalgebra of $M_2(E)$ consisting of the matrices whose entries on the second diagonal belong to E'. Then A is G-graded with the grading inherited by the one on $M_2(E)$.

In the next couple of statements we describe the G-graded identities of A for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. We shall use the matrices A_i defined at the beginning of Section 4, with $n = 2$. In order to adapt them to the algebra A we introduce additional relations in the algebra Ω namely $(y_i^\alpha)^p = 0$ when $(\alpha(x_i), \beta(x_i)) = (1, 0)$. The following lemma is the variant of Lemma 15 for the present situation.

Lemma 27 *If* $f(x_1,\dots,x_m)\in K(X)$ *is a monomial such that* $f(A_1,\dots,A_m)=0$ *then* $f\in I_1$.

Proof. Suppose first that, in the notation of Lemma 13 and Eq. 1, the variable z_j^β appears twice (or more) in the monomial M_0. Here $j\in\{j_1,\dots,j_s\}$, $\beta\in\{\beta_1,\dots,\beta_s\}$. Then $f=f_1x_jf_2x_jf_3$ for some monomials f_1, f_2, f_3. But we have $\alpha(f_1)=\alpha(f_1x_jf_2)$ and $\beta(x_j)=1$ therefore $\alpha(x_j)=-\alpha(f_2)$. Now by the graded identity $x_1x_2x_3+x_3x_2x_1=0$ with $x_1=x_3=x_j$, $x_2=f_2$, we obtain $x_jf_2x_j\in I_1$.

If the above is not the case then according to Eq. 1, necessarily some variable y_i^α appears p times (or more) in M_0, for $(\alpha(x_i),\beta(x_i))=(1,0)$. We write $f=f_1x_if_2x_i\dots x_if_px_if_{p+1}$, then

$$\alpha(f_1)=\alpha(f_1x_if_2)=\ldots=\alpha(f_1x_if_2x_i\dots x_if_p).$$

Thus $\alpha(x_i)=-\alpha(f_r)$ for all $r=2,\ldots,p$. But then the identity $c=0$ can be applied and we obtain $f\in I_1$ as well.

Theorem 28 *The* $\mathbb{Z}_2\times\mathbb{Z}_2$*-graded identities of the algebra* A *admit a basis consisting of the identities of the set* S *(with* $n=2$*) and the identity* $c=0$.

Proof. The proof repeats word by word the one of Theorem 16.

Now using graded identities we draw conclusions about the ordinary ones.

Corollary 29 *Let* $\operatorname{char}K=p\neq 2$. *Then the algebras* $M_{1,1}(E)\otimes E$ *and* A *satisfy the same ordinary polynomial identities.*

Proof. According to Theorems 26 and 28 these two algebras satisfy the same G-graded identities.

Now we give one more example where the Tensor product theorem fails in positive characteristic.

Theorem 30 *Let* K *be an infinite field,* $\operatorname{char}K=p>2$. *Then* $T(M_2(E))\subsetneq T(M_{1,1}(E)\otimes E)$.

Proof. Comparing the bases of the G-graded identities satisfied by these two algebras we obtain immediately that $T(M_2(E))\subseteq T(M_{1,1}(E)\otimes E))$. In order to show that the inclusion is proper one, we exhibit a polynomial that is an identity for $M_{1,1}(E)\otimes E$ but not for $M_2(E)$. Note that such a polynomial cannot be multilinear due to Theorem 5. Choose $f=[x_1,x_2]^{4p-3}$.

Then we compute $f(e_{21}-e_{12},e_{11})=e_{21}+e_{12}\neq 0$ in $M_2(E)$ hence $f\notin T(M_2(E))$.

On the other hand, if a, $b\in A$ then a direct verification shows that $[a,b]\in M_2(E')$. But for every $c\in M_2(E')$ we have that $c^{4p-3}=0$ due to the identity $x^p=0$ in E' (and the pigeonhole principle). Therefore $f\in T(A)$. Now according to Corollary 29, $T(A)=T(M_{1,1}(E)\otimes E)$ and $f\in T(M_{1,1}(E)\otimes E)$. Hence the inclusion of the theorem is proper.

We state some open problems whose solution would help to much better understanding of the polynomial identities satisfied by the algebras $M_n(E)$ and $M_{a,b}(E)$ over fields of positive characteristic.

1. Let $n > 2$ and $G = \mathbb{Z}_n \times \mathbb{Z}_2$; find a finite set of monomials in $K(X)$ that is a basis for the graded identities of $M_{a,b}(E) \otimes E$ modulo the identities of S.

2. Let $A_{a,b}$ be the subalgebra of $M_n(E)$ of all matrices of the form (a_{ij}) where $a_{ij} \in E$ if $(i,j) \in \Delta_0$ and $a_{ij} \in E'$ if $(i,j) \in \Delta_1$. Is it true that $T(M_{a,b}(E) \otimes E) = T(A_{a,b})$? Observe that $c_a = 0$ is a graded identity for the algebra $A_{a,a}$. (See Remark 5 for the definition of c_a.)

3. Find an ordinary identity satisfied by the algebra $A_{a,b}$ but not by the algebra $M_{a+b}(E)$.

4. We know that $T(M_{k,l}(E) \otimes E) = T(M_{r,s}(E) \otimes E)$ whenever $k+l = r+s$ and $\operatorname{char} K = 0$. Do these T-ideals coincide when K is an infinite field, $\operatorname{char} K = p > 2$? Observe that $c_k = 0$ is not a graded identity for $M_{r,s}(E) \otimes E$ when $r > s$ and $r + s = 2k$, because if $d = (e_{1,k+1} + e_{k+1,1}) \otimes 1$ then $c_k(d, \ldots, d) = d \neq 0$.

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