Invexity Generalized and Weakly Efficient Solutions for Some Vectorial Optimization Problem in Banach Spaces

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Abstract

In this work we use the notion of vectorial critical point and Karush-Kuhn-Tucker critical point for some class of vectorial optimization problems between Banach spaces. By using these notions, we obtain a characterization for weakly efficient solutions for such optimization problems.

Key words: Multiobjective optimization, pseudoinvexity, optimality conditions.

1 Introduction and formulation of the problem

In scalar optimization, the Kuhn-Tucker (or Karush-Kuhn-Tucker) conditions are sufficient for optimality when all the functions are convex. In recent years, considerable progress has been made to weaken the convexity hypothesis and so to increase the class of functions with this property.

There is an important contribution in this direction given by Hanson in [1]. He considered invex functions. For these functions the classical Kuhn-Tucker con-

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ditions are sufficient to obtain the global optimality. Later, Martin [2] observe that in problems without constraints the invexity is a condition necessary and sufficient to obtain the global optimality. Thus, the following questions were raised: what is the bigger class of functions where the Kuhn-Tucker optimality conditions are necessary and sufficient to guarantee the global optimality? The answer to this question was given by Martin in [2] for scalar problems.

We consider the following scalar problem without contraints

$$\left.\begin{array}{l} \text{Minimize} \;\; \theta(x) \\ \text{subject to} \\ \qquad\qquad x \in S \subset \mathbb{R}^n \end{array}\right\} \qquad (P)$$

where $\theta(x)$ is a scalar function, and $S \subseteq \mathbb{R}^n$.

We recall that $\overline{x}$ is a *stationary point* if $\nabla\theta(\overline{x}) = 0$. and the optimization problem with constraints

$$\left.\begin{array}{l} \text{Minimize} \;\; \theta(x) \\ \text{subject to} \\ \qquad\qquad -g_j(x) \leq 0 \;\; j = 1, ..., m \\ \qquad\qquad x \in S \subset \mathbb{R}^n \end{array}\right\} \qquad (CP)$$

where $\theta(x)$ is a scalar function,$g = (g_1, ..., g_m) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a vectorial function, both differentiable in an open set $S \subseteq \mathbb{R}^n$.

We recall that $(\overline{x}, \overline{u}) \in S \times \mathbb{R}^m$ is a *Kuhn-Tucker stationary point* [3], if

$$\nabla\theta(\overline{x}) + \overline{u}^T \nabla g(\overline{x}) = 0,$$
$$\overline{u}^T g(\overline{x}) = 0,$$
$$\overline{u} \geq 0$$

Also, we recall that $\theta : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function at *invex* in $\overline{x}$ if there exists a vectorial function $\eta : S \times S \longrightarrow \mathbb{R}^n$ such that

$$\theta(x) - \theta(\overline{x}) \geq \eta(x, \overline{x})^T \nabla\theta(\overline{x}), \forall x \in S$$

and we say invex over $S_0 \subseteq S$, if it is invex at each point $\overline{x} \in S_0$.

Definition 1.1 *(Hanson and Mond, [1]) The function* $\theta : S \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *is* *called **pseudoinvex** at $u \in S$ if there exists a function $\eta : S \times S \to \mathbb{R}^n$ such that*

$$< \nabla\theta(u), \eta(x,u) > \;\; \geq \; 0 \Longrightarrow \theta(x) \geq \theta(u)$$

$\forall\, x \in S$. The function θ is called pseudoinvex on S if θ is pseudoinvex in each point of S.

We observe that, for the scalar case, the concepts of invexity and pseudoinvexity coincide. In [2], Martin proved the following result for problems without constraints:

Theorem 1.2 *The function θ is invex over S if and only if each stationary point is a global minimum of θ over S.*

However, for problems with constraints, the invexity only guarantee the sufficiency of optimality. Martin [2] define a class of problems where each Kuhn-Tucker critical point is in fact a global optimum, in this way, he obtain the full characterization of the solutions for the problem with constraints.

We recall that problem (CP) is *KT- invex* on S if there exists a vectorial function $\eta : S \times S \longrightarrow R^n$ such that, $\forall\; x_1, x_2 \in S$ with $g(x_1) \leq 0$ and $g(x_2) \leq 0$, then

$$\theta(x_1) - \theta(x_2) \geq \eta(x_1,x_2)^T \nabla\theta(x_2)$$
$$-\eta(x_1,x_2)^T \nabla g_j(x_2) \geq 0, \forall j \in I(x_2)$$

where $I(x_2) := \{j \in \{1,...,m\} : g_j(x_2) = 0\}$ is the index set of active restrictions in x_2. The result given by Martin [2] is:

Theorem 1.3 *Every Kuhn-Tucker stationary point for (CP) is a global minimizer if only if problem (CP) is KT-invex.*

In this work, we will show that Theorems 1.2 and 1.3 even are true for vectorial problems between Banach spaces. In what follows we will use the following notations. Let F be a Banach space, $C \subset F$ a pointed closed, convex cone, (i. e., $C \cap (-C) = \{0\}$), not equal to F and with nonempty interior. Let $x, y \in F$, we use the following notations in the rest of the paper

$$x \leqq_C y \Leftrightarrow y - x \in C$$
$$x \leq_C y \Leftrightarrow y - x \in C \setminus \{0\}$$
$$x <_C y \Leftrightarrow y - x \in \text{int } C.$$

We recall that $h : E \longrightarrow F$ is *Fréchet differentiable* at point x ([4]) (to short, differentiable at point x) if there exists a continuous linear mapping $Dh(x) : E \longrightarrow F$ such that

$$\lim_{z \to 0} \frac{\|h(x+z) - h(x) - Dh(x)h\|_F}{\|z\|_E} = 0.$$

The function h is say differentiable on $S \subset E$ if h is differentiable in each point of S. Let E and F be two Banach spaces and $f : E \longrightarrow F$ and $g : E \longrightarrow G$ two differentiable functions over the nonempty, open subset of E, $S \subset E$, and we assume that F is ordered partially by the closed, convex, pointed cone, with nonempty interior, $C \subset F$ (with $C \neq E$), and $K \subset G$ is a closed, convex cone not equal to G. The problems are going to are:

(1) Without constraints

$$\left.\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & \\ & x \in S \subset E \end{array}\right\} \qquad (VOP)$$

(2) With constraints

$$\left.\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & \\ & -g(x) \in K \\ & x \in S \subset E \end{array}\right\} \qquad (CVOP)$$

We observe that for (VOP) the feasible set is S and for (CVOP) is

$$\mathcal{F} := \{x \in S : -g(x) \in K\}.$$

The following concepts are well known

Definition 1.4 *The feasible point $\overline{x} \in S$ is called* ***efficient solution*** *if does not exist x feasible such that $f(\overline{x}) - f(x) \in C \setminus \{0\}$ (or, equivalently, $\overline{x}$ feasible is efficient if does not exist x feasible such $f(x) \leq_C f(\overline{x})$).*

Definition 1.5 *The feasible point $\overline{x} \in S$ is called* ***weakly efficient solution*** *if does not exist x feasible such that $f(\overline{x}) - f(x) \in int\ C$ (or equivalently, $\overline{x}$ is a weakly efficient solution if does not exist x feasible such that $f(x) <_C f(\overline{x})$).*

We denote by F^* the topological dual of F, and $\langle \cdot, \cdot \rangle$ the duality pairing between F^* and F. Given $C \subset F$ a convex cone, we define the *dual cone* of C,

$$C^* := \{\xi \in F^* : \langle \xi, x \rangle \geq 0, \forall x \in C\}.$$

The paper is organized as follows, in Section 2, we study the problem without constraints, we give a definition of pseudoinvex function for our problem and we will prove that, when f is pseudoinvex, the vectorial critical point is a necessary and sufficient condition for weakly efficiency. In the Section 3, we study the problem with constraints, we define the problems KT-invex, and we show that this class of problems, the Kuhn-Tucker critical point is a necessary and sufficiency condition for weakly efficiency.

2 Necessary and Sufficiency conditions for weakly efficient for the problem without constraints

Let E and F be two Banach spaces, $C \subset F$ a closed, convex cone with nonempty interior and assume that it is not the whole space F. S an open subset, nonempty, of E and $f : E \longrightarrow F$ a differentiable function on S.

Definition 2.1 *$\overline{x} \in S$ is a* ***vectorial critical point*** *of (VOP) if there exists $\lambda^* \in C^* \setminus \{0\}$ such that $\lambda^* \circ Df(\overline{x}) = 0$.*

The following result was given in [5].

Theorem 2.2 *If $\overline{x} \in S$ is a weakly efficient solution (VOP), then $\overline{x}$ is a vectorial critical point.*

Definition 2.3 *Let $f : S \subset E \longrightarrow F$ be a differentiable function in the open set S. We say that f is* ***pseudoinvex*** *in S with respect to η if there exist a vectorial function $\eta : S \times S \longrightarrow E$ such that*

$$x_1, x_2 \in S,\ f(x_1) - f(x_2) <_C 0 \Rightarrow Df(x_2)\eta(x_1, x_2) <_C 0$$

(where $Df(x_2)\eta(x_1, x_2)$ denote the value of the function $Df(x_2) \in \mathcal{L}(E, F)$ applied in the vector $\eta(x_1, x_2) \in E$ and $\mathcal{L}(E, F)$ is the space of continuous, linear operators from E into F). To prove that the vectorial critical points are coincident with the weakly efficient solutions, when the function f is pseudoinvex we recall the following result, see [6].

Lemma 2.4 *Let F be a Banach space, $C \subset E$ a closed, convex cone and $\xi \in C^* \setminus \{0\}$. Then, $\langle \xi, x \rangle > 0$ when $x \in int\ C$.*

Lemma 2.5 *If in the problem (VOP), f is pseudoinvex and $\overline{x} \in S$ is a vectorial critical point, then $\overline{x}$ is a weakly efficient solution.*

PROOF. In fact, if we assume that $\overline{x} \in S$ is vectorial critical point and not a weakly efficient solution, we prove a contradiction. In this case, there exists $\lambda^* \in C^* \setminus \{0\}$ such that

$$\lambda^* \circ Df(\overline{x}) = 0 \tag{1}$$

and exists $x \in S$ such that

$$f(x) - f(\overline{x}) \in -\text{ int } C. \tag{2}$$

On the other hand, since f is pseudoinvex, we obtain from (2) that

$$Df(\overline{x})\eta(x, \overline{x}) \in -\operatorname{int} C,$$

and (by Lemma 2.4)

$$\lambda^*(Df(\overline{x})\eta(x, \overline{x})) = [\lambda^* \circ Df(\overline{x})]\eta(x, \overline{x}) < 0,$$

this last inequality is contradictory with (1). Therefore, $\overline{x}$ is a weakly efficient solution of (VOP).

The following result is a generalization of the Farkas Theorem, see [6], pp. 59-60.

Lemma 2.6 *Let X, Y and V be normed vector spaces, $A \in L(X, V)$ and $M \in L(X, Y)$ given, $T \subseteq V$ and $Q \subseteq Y$ convex cones with $\mathit{int}\, Q \neq \varnothing$ and $b \in -T$, $s \in -Q$. Let the cone $[A, b]^T(T^*)$ be a weak-*closed. Then, the system*

$$\begin{cases} \quad Ax + b \in -T \\ Mx + s \in -\ \mathit{int}\ Q \end{cases}$$

has not solution if, and only if, there exists $\tau \in Q^* \setminus \{0\}$, $\lambda \in T^*$ such that

$$\begin{cases} \tau M + \lambda A = 0 \\ \langle \lambda, b \rangle = 0 \\ \langle \tau, s \rangle = 0 \end{cases}$$

Theorem 2.7 *The function f in (VOP) is pseudoinvex in S if, and only if, each vectorial critical point is a weakly efficient solution of (VOP).*

PROOF: From Lemma 2.5, if f is pseudoinvex, then each vectorial critical point is a weakly efficient solution of (VOP). Now, we assume that each vectorial critical point is a weakly efficient solution of (VOP). We fix $\overline{x} \in S$ and consider the following systems:

$$f(x) - f(\overline{x}) \in -\operatorname{int} C, \;\; x \in S \tag{3}$$

$$Df(\overline{x})u \in -\operatorname{int} C, \;\; u \in E. \tag{4}$$

We will prove that, the system (3) has a solution, then the system (3) has also a solution. In fact, if (3) has solution, then $\overline{x}$ is not efficient and, by hypothesis, is not a vectorial critical point, i.e., does not exist $\lambda^* \in C^* \setminus \{0\}$ such that $\lambda^* \circ Df(\overline{x}) = 0$. Taking: $A = 0 \in \mathcal{L}(E, F)$, $M = Df(\overline{x}) \in \mathcal{L}(E, F)$, $b = 0 \in E$ and $s = 0 \in F$, we deduce that does not exist $\tau \in Q^* \setminus \{0\}$ and $\lambda \in Q^*$ such that

$$\begin{cases} \tau M + \lambda A = 0 \\ \langle \lambda, b \rangle = 0 \\ \langle \tau, s \rangle = 0. \end{cases}$$

On the other hand, by Lemma 2.6, there exists $u \in E$ such that

$$\begin{cases} Au + b = 0 \in -Q \\ Mu + s = Df(\overline{x})u \in -\operatorname{int} Q. \end{cases}$$

In particular, the system (3) has solution $u \in E$. Setting: $\eta(x, \overline{x}) = u$, we obtain that f is pseudoinvex.

3 Necessary and sufficiency conditions for the weakly efficiency for the problem with constraints

In this Section, we consider the following optimization problem:

$$\left.\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & \\ & -g(x) \in -K \\ & x \in S \subset E \end{array}\right\} \qquad (CVOP)$$

where E,F and G are Banach spaces, $C \subset F$ and $K \subset G$ are closed, convex pointed cones, not equal to F and G, respectively, $\operatorname{int} C \neq \varnothing$, $S \subset E$ is an open set nonempty and the functions $f : E \longrightarrow F$ and $g : E \longrightarrow G$ are differentiable on S.

Definition 3.1 *We say that (CVOP) is* ***KT-invex*** *in $x_2 \in \mathcal{F}$ if there exists a vectorial function $\eta : S \times S \longrightarrow E$ such that for each $x_1 \in \mathcal{F}$, is satisfied:*

$$\begin{cases} f(x_1) - f(x_2) \in -\ int\ C \Rightarrow Df(x_2)\eta(x_1, x_2) \in -\ int\ C \\ -Dg(x_2)\eta(x_1, x_2) \in K. \end{cases}$$

and if the problem is KT-invex for each $x \in \mathcal{F}$, we say that (CVOP) is KT-invex.

In the finite-dimensional case, Osuna-Gómez, Rufián-Lizana and Ruíz-Canales [7] proved that to be a vector Kuhn- Tucker point is a necessary and suficient condition to weakly efficiency:

Theorem 3.2 *When $E = \mathbb{R}^n$, $F = \mathbb{R}^p$, $G = \mathbb{R}^m$, $C = \mathbb{R}^p_+$ and $K = \mathbb{R}^m_+$, every vector Kuhn-Tucker point is a weakly efficient solution if and only if problem (CVOP) is KT-invex.*

Now, we will establish the analogous infinite-dimensional results.

Theorem 3.3 *(sufficiency) If (CVOP) is a KT-invex problem, then each Kuhn-Tucker Critical point is a weakly efficient solution of (CVOP).*

PROOF. We assume that (CVOP) is KT-invex and let $\overline{x}$ be a Kuhn-Tucker critical point. In this case, there exist $\lambda^* \in C^* \setminus \{0\}$ and $\mu^* \in K^*$ such that

$$\lambda^* \circ Df(\overline{x}) + \mu^* \circ Dg(\overline{x}) = 0$$
$$\langle \mu^*, g(\overline{x}) \rangle = 0$$

and, in particular,

$$\begin{gathered} \lambda^* \circ Df(\overline{x})\eta(\overline{x}, x) + \mu^* \circ Dg(\overline{x})\eta(\overline{x}, x) = 0, \ \forall x \in \mathcal{F} \\ \langle \mu^*, g(\overline{x}) \rangle = 0. \end{gathered} \tag{3}$$

We assume the contrary, i.e., that $\overline{x}$ is not a weakly efficient solution of (CVOP). Then there exists $x \in S$, $g(x) \in -K$ such that $f(x) - f(\overline{x}) \in -\text{ int } C$ and, since $\lambda^* \in C^* \setminus \{0\}$, then by Lemma 2.4 we have

$$\lambda^*(Df(\overline{x})\eta(x, \overline{x})) < 0. \tag{4}$$

From (3) and (4), we obtain

$$\mu^*(Dg(\overline{x})\eta(x, \overline{x})) > 0. \tag{5}$$

Since (CVOP) is KT-invex, $-Dg(\overline{x})\eta(x, \overline{x}) \in K$ and $\mu^* \in K^*$, we have

$$\mu^*(Dg(\overline{x})\eta(x, \overline{x})) \leq 0,$$

this is contradictory with (5) and, therefore, $\overline{x}$ is a weakly efficient solution of (CVOP).

Theorem 3.4 *(Necessity) Asuume that each $\overline{x}$ is a Kuhn-Tucker critical point of (CVOP) the set $[Dg(\overline{x}), g(\overline{x})]^T(K^*)$ is weak-* closed. Then, if each Kuhn-Tucker critical point of (CVOP) is weakly efficient, then, (CVOP) is KT-invex.*

PROOF. Let $\overline{x} \in S$ be fixed, we consider the systems:

$$\begin{cases} Df(\overline{x})u \in -\text{ int } C \\ Dg(\overline{x})u \in -K \end{cases} \tag{6}$$

and

$$\begin{cases} f(x) - f(\overline{x}) \in - \text{ int } C \\ g(x) \in -K. \end{cases} \tag{7}$$

Then, to prove that (CVOP) is KT-invex in $\overline{x}$ is equivalently to prove that the system (6) has solution $u \in E$ when the system (7) has solution $x \in S$ (in such case, it is sufficient to take $\eta(x, \overline{x}) = u \in E$). We assume that the system (7) has solution. Then, $\overline{x}$ is not weakly efficient and, by hypothesis, $\overline{x}$ is not a Kuhn-Tucker vectorial critical point, and, therefore, does not exist $\tau \in C^* \setminus \{0\}$ and $\lambda \in K^*$ such that

$$\tau M + \lambda A = 0,$$
$$\langle \lambda, b \rangle = 0,$$
$$\langle \tau, s \rangle = 0$$

(where $A = Dg(\overline{x}) \in \mathcal{L}(E, G)$, $M = Df(\overline{x}) \in \mathcal{L}(E, F)$, $b = g(\overline{x}) \in -K$ and $s = 0 \in -C$). From Lemma 2.6, the system

$$\begin{cases} Au + b \in -K \\ Mx + s \in - \text{ int } C \end{cases}$$

has solution, or, equivalently, there exists $u \in E$ such that

$$\begin{cases} Dg(\overline{x})u + g(\overline{x}) \in -K \\ Df(\overline{x})u \in - \text{ int } C. \end{cases}$$

But

$$Dg(\overline{x})u = [Dg(\overline{x})u + g(\overline{x})] - g(\overline{x}) \in -K - K \subseteq -K$$

and, therefore, the system (7) has solution.

4 Conclusion

This work is an extension of the results obtained in [7] for the Banach space context with dominance structure given by cones. Moreover, the results given in [7] are a generalization for the vectorial case of the results obtained by Martin [2] in the scalar case. These results characterized the invex functions (that are coincident with the pseudoinevex functions in the case scalar), as those for which their stationary points are global minimum.

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